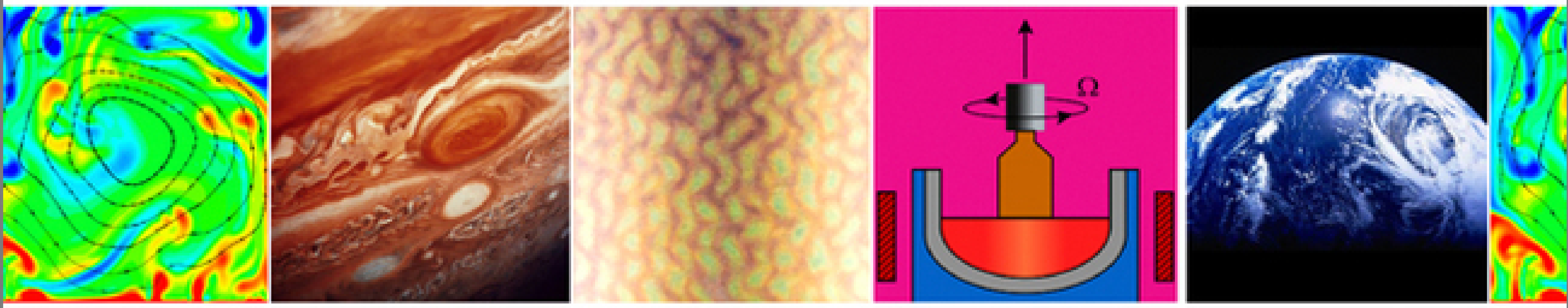
Marcello Lappa

Rotating Thermal Flows

in Natural and Industrial Processes



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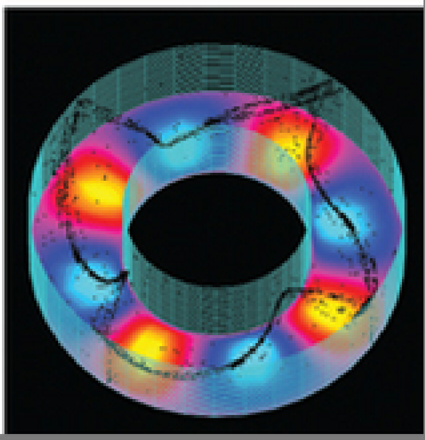

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To a red rose

Contents

Preface

The relevance of self-organization, pattern formation, nonlinear phenomena and non-equilibrium behaviour in a wide range of fluid-dynamics problems in *rotating systems*, somehow related to the science of materials, crystal growth, thermal engineering, meteorology, oceanography, geophysics and astrophysics, calls for a concerted approach using the tools of thermodynamics, fluid-dynamics, statistical physics, nonlinear dynamics, mathematical modelling and numerical simulation, in synergy with experimentally oriented work.

The reason behind such a need, of which the present book may be regarded as a natural consequence, is that in many instances pertaining to such fields one witnesses remarkable affinities between *large-scale-level processes* and the same entities on the *smaller* (laboratory) *scale*; despite the common origin (they are related to 'rotational effects'), such similarities (and the important related implications) are often ignored in typical analyses related to one or the other category of studies.

With the specific intent to extend the treatment given in an earlier Wiley text (*Thermal Convection: Patterns, Evolution and Stability*, Chichester, 2010, which was conceived in a similar spirit), the present book is entirely focused on *hybrid* regimes of convection in which one of the involved forces is represented by standard gravity or surface tension gradients (under various heating conditions: from below, from the side, etc.), while the other arises by virtue *of rotation*.

The analogies and kinships between the two fundamental classes of models mentioned above, one dealing with issues of complex behaviour at the laboratory (technological application) level and the second referring to the strong nonlinear nature of large-scale (terrestrial atmosphere, oceans and more) evolution, are defined and discussed in detail.

The starting point for such a development is the recognition that such phenomena share an important fundamental feature, a group of equations, strictly related, from a mathematical point of view, to model mass, momentum and energy transfer, and the mathematical expressions used therein for the 'driving forces'.

Although other excellent monographs that have appeared in the literature (e.g. to cite the most recent ones: Marshall and Plumb, 2007, *Atmosphere, Ocean, and Climate Dynamics*, Academic Press; Vallis, 2006, *Atmospheric and Oceanic Fluid Dynamics*, Cambridge University Press) have some aspects in common with the present book, they were expressly conceived for an audience consisting of meteorologists.

Here the use of jargon is avoided, this being done under the declared intent to increase the book's readability and, in particular, make it understandable also for those individuals who are not 'pure' meteorologists (or 'pure' professionals/researchers working in the field of materials science), thereby promoting the exchange of ideas and knowledge integration.

In this context, it is expressly shown how the aforementioned isomorphism between small and large scale phenomena becomes beneficial to the definition and ensuing development of an integrated comprehensive framework, allowing the reader to understand and assimilate the underlying quintessential mechanisms without requiring familiarity with specific literature on the subject.

A Survey of the Contents

In Chapter 1 the main book topics are placed in a precise theoretical context by introducing some necessary notions and definitions, such a melange of equations and nondimensional numbers being propaedeutical to the subsequent elaboration of more complex concepts and theories.

Chapter 2 deals with *Rayleigh–Bénard convection* in simplified (infinite and finite) geometrical models, which is generally regarded as the simplest possible laboratory system incorporating the essential forces that occur in natural phenomena (such as circulations in the atmosphere and ocean currents) and many technological applications (too numerous to list).

The astonishing richness of possible convective modes for this case is presented with an increasing level of complexity as the discussion progresses, starting from the ideal case of a system of infinite (in the horizontal direction) extent in which the role of centrifugal force is neglected (with related phenomena including the Küppers–Lortz instability, domain chaos, the puzzling appearance of patterns with square symmetry, spiral defect chaos and associated dynamics of chiral symmetry breaking), passing through the consideration of finite-sized geometries and the reintroduction of the centrifugal force, up to a presentation of the myriad of possible solutions and bifurcations in cylindrical containers under the combined effects of vertical (gravity), radial (centrifugal) and azimuthal (Coriolis) forces.

Similar concepts apply to the case of convection driven by internal heating in rotating self-gravitating spherical shells (Chapter 3), whose typical manifestation under the effect of radial buoyancy is represented by an unsteady *columnar mode* able to generate differential rotation under given circumstances. Exotic modes of convection (such as hexagons, oblique rolls, hexarolls, knot convection and so on) are also reviewed and linked to specific regions of the parameter space.

Then, attention is switched from rotating systems with bottom (or internal) heating to laterally heated configurations (temperature gradient directed horizontally, gravity directed vertically), which leads in a more or less natural way to the treatment of so-called *sloping convection* (Chapter 4), known to be the dominant mechanism producing large-scale spiralling eddy structures in Earth's atmosphere, but also eddy structures and wavy patterns in typical problems of crystal growth from the melt.

Apart from providing a general overview of so-called *quasigeostrophic theory*, Chapter 5 also gives some insights into the fundamental difference between the two main categories of fluid-dynamic instabilities in rotating fluids: one associated with problems for which the unstable modes essentially involve *mass and temperature redistribution* (e.g. Rayleigh–Bénard or Marangoni–Bénard convection considered in Chapters 2 and 7, respectively); and the other including problems such as stably stratified and unstratified shear instabilities, *barotropic and baroclinic instabilities*, which appear to be connected to the *self-excitation of waves* rather than to the direct redistribution of mass and temperature.

A number of works are reviewed, which focus on the mechanism by which mechanical and wave signals interplay to control how individual convective structures decide whether to grow, differentiate, move or die, and thereby promote pattern formation during the related process. Moreover, starting from the cardinal concept of the *Rossby wave*, some modern approaches, such as the so-called CRW (counter-propagating-Rossby-wave) perspective, an ingenious application of what has become known as 'potential vorticity thinking', are also invoked and used to elaborate a specific mathematical formalism and some associated important microphysical reasoning.

As a natural continuation of preceding chapters, Chapter 6 develops the important topic of *geostrophic turbulence*.

The basic ideas of inertial range theory are illustrated and extended phenomenologically by incorporating ideas of vortex–vortex and vortex–strain interactions that are normally present in physical and not spectral space. Then, a critical analysis of the distinctive marks of geostrophic turbulence (and its relationship with other classical models of turbulence) is developed. The main theories for *jet formation* and stability are discussed, starting from the fundamental concept of turbulent 'decascade' of energy. Subsequent arguments deal with the role played in maintaining turbulence by baroclinic effects and/or other types of 3D instabilities and on the so-called *baroclinic life cycle*. An overview of the main characteristic wavenumbers and scales relating to distinct effects is also elaborated.

Similarities between Earth's phenomena and typical features of *outer planet* (Jupiter and Saturn) dynamics are discussed as well. After the exposition of the general theory for vortex–vortex coalescence, a similar treatment is also given for phenomena of wave–wave and wave–mean-flow interference.

The remaining chapters are entirely devoted to phenomena occurring on the lab scale, thereby allowing most of the arguments introduced in earlier chapters *to spread from their traditional heartlands of meteorology and geophysics to the industrial field* (and related applications).

Along these lines, Chapter 7 is concerned with the interplay between rotation and flows induced by surface tension gradients (more specifically, *Marangoni–Bénard convection* and so-called *hydrothermal waves*, considered as typical manifestations of surface-tension-driven flows in configurations of technological interest subjected to temperature gradient *perpendicular or parallel*, respectively, to the liquid/gas interface).

The modification of the classical hydrothermal mechanism due to rotation, in particular, is discussed on the basis of concepts of system invariance breaking (due to rotation) and of the fundamental processes allowing waves to derive energy from the basic flow (an interpretation is given as well for still unexplained observations appeared in the literature).

Chapter 8 provides specific information on cases with important background applications in the realm of crystal growth from the melt, for example the Bridgman, floating zone and Czochralski (CZ) techniques, considering, among other things, the interesting subject of *interacting baroclinic and hydrothermal waves*, together with an exposition of the most recent theories about the origin of the so-called *spoke patterns*.

The CZ configuration is used as a classical example of situations in which fluid motion is brought about by different coexisting mechanisms: Marangoni convection, generated by the interfacial stresses due to horizontal temperature gradients along the free surface and gravitational convection driven by the volumetric buoyancy forces caused by thermally and/or solutally generated density variations in the bulk of the fluid, without forgetting the presence of phenomena of a rotational nature (baroclinic instability) and those deriving from temperature contrasts induced in the vertical direction by radiative or other (localized) effects.

The exposition of turbulence given in Chapter 6 about typical planetary dynamics is *extended* in this chapter to topics of crystal growth showing commonalities and differences due to 'contamination' exerted on the geostrophic flow by effects of surface-tension or gravitational nature (thermal plumes and jets).

Then a survey is given of very classical problems in rotating fluids which come under the general heading of *differential-rotation-driven flows*. This subject includes a variety of prototypical laboratory-scale models of industrial devices (among them: centrifugal pumps, rotating compressors, turbine disks, computer storage drives, turbo-machinery, cyclone separators, rotational viscometers, pumping of liquid metals at high melting point, cooling of rotating electrical motors, rotating heat exchangers, etc.).

Rotating magnetic fields are also considered (Chapter 9) as a potential technological means for counteracting undesired flow instabilities. Some attention is also devoted to so-called *swirling flow* and related higher modes of convection (Taylor-vortex flow, Görtler vortices, instabilities of the Bodewadt layer, etc.).

Last, but not least, a synthetic account is elaborated for flows produced by *angular vibrations* (i.e. situations in which the constant rotation rate considered in earlier chapters is replaced by an angular displacement varying sinusoidally with time with respect to an initial reference position) and rocking motions (Chapter 10), which complements, from a theoretical point of view, the analogous treatment given in Wiley's earlier book on *Thermal Convection* (2010) of purely translational vibrations, and may be of interest for researchers and scientists who are now coordinating their efforts to conceive new strategies for flow control.

Acknowledgements

The present book should be regarded as a natural and due extension of my earlier monograph *Thermal Convection: Patterns, Evolution and Stability* (published by Wiley at the beginning of 2010) in which I presented a critical, focused and 'comparative' study of different types of thermal convection typically encountered in natural or technological contexts (thermogravitational, thermocapillary and thermovibrational), including the effect of magnetic fields and other means of flow control. That book attracted much attention and comments, as witnessed by the many reviews that have appeared in distinct important scientific journals (R.D. Simitev (2011) *Geophys. Astrophys. Fluid Dyn*., **105** (1), 109–111; A. Nepomnyashchy (2011) *Eur. J. Mech. – B/Fluids*, **30** (1), 135; A. Gelfgat (2011) *Cryst. Res. Technol*., **46** (8), 891–892; J. A. Reizes (2011) *Comput. Therm. Sci*., **3** (4), 343–344).

The success of the 2010 book and the express requests of many referees to 'complete' the treatment of thermal convection, including the influence of Coriolis and centrifugal forces, as well as the development of turbulence, led me to undertake the present new work, for which I gratefully acknowledge also the many unknown reviewers selected by John Wiley & Sons, who initially examined the new book project, for their critical reading and valuable comments.

Following the same spirit of the earlier 2010 monograph, I envisaged to consider both natural and industrial processes, and develop a common framework so to promote the exchange of ideas between researchers and professionals working in distinct fields (in particular between the materials science and geophysical communities).

Along these lines, deep gratitude goes to many colleagues around the world pertaining to both such categories for generously sharing with me their precious recent experimental and numerical data (in alphabetical order): Prof. R. Bessaïh, Prof. F.H. Busse, Prof. R.E. Ecke, Prof. A.Yu. Gelfgat, Prof. N. Imaishi, Prof. A. Ivanova, Prof. V. Kozlov, Dr. R.P.J. Kunnen, Prof. I. Mutabazi, Prof. P.B. Rhines, Prof. P. Read, Prof. V. Shevtsova, Prof. I. Ueno.

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Last but not least, I am also indebted to my wife Paola, to whom this book is dedicated. Writing is a solitary activity; nevertheless, her good humour, sensitivity and vitality have made this project a pleasure, especially in the evening and during weekends. This new treatise is largely due to her optimism, encouragement, and patience.

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1

Equations, General Concepts and Nondimensional Numbers

Prior to expanding on the subject of convection in rotating fluids and related myriad manifestations, some propaedeutical concepts and accompanying fundamental mathematics must be provided to help the reader in the understanding of the descriptions and elaborations given later.

Along these lines, the goal of this introductory chapter is to stake out some common ground by providing a survey of *overarching principles, characteristic nondimensional parameters and governing equations*.

Such a theoretical framework, in its broadest sense, attempts to classify and characterize all forces potentially involved in the class of phenomena considered in the present book.

As the chapter progresses, in particular, balance equations are first introduced assuming an inertial frame of reference, hence providing the reader with fundamental information about the nature and properties of forces of nonrotational origin (Sections 1.1–1.5); then such equations are reformulated in a rotating coordinate system (Section 1.6) in which the so-called centrifugal and Coriolis forces emerge naturally as 'noninertial' effects.

While such a practical approach justifies the use of continuum mechanics and of macrophysical differential equations for the modelling of the underlying processes, it is insufficient, however, for the understanding/introduction of a microscopic phenomenological theory. Such development requires some microphysical reasoning. The cross-link between macro- and micro-scales is, in general, a challenging problem. Due to page limits, here we limit ourselves to presenting the Navier–Stokes and energy equations directly in their macroscopic (continuum) form, the reader being referred to other texts (e.g. Lappa, 2010) for a complete elaboration of the approach leading from a microscopic phenomenological model to the continuum formalism.

1.1 The Navier-Stokes and Energy Equations

The Navier–Stokes equations, named after Claude-Louis Navier and George Gabriel Stokes (Navier, 1822; Stokes, 1845), describe the motion of a variety of fluid substances, including gases, liquids and even solids of geological sizes and time-scales. Thereby, they can be used to model flows of

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technological interest (too many to mention; e.g. fluid motion inside a crucible used for crystal growth or for the processing of metal alloys), but also weather, ocean currents and even motions of cosmological interest.

In their macroscopic (continuum) form these equations establish that the overall mass must be conserved and that changes in momentum can be simply expressed as the sum of dissipative viscous forces, changes in pressure, gravity, surface tension (in the presence of a free surface) and other forces (electric, magnetic, etc.) acting on the fluid.

1.1.1 The Continuity Equation

The mass balance equation (generally referred to in the literature as the continuity equation) reads:

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot \left(\rho \underline{V}\right) = 0 \tag{1.1a}$$

that, in terms of the substantial derivative $D/Dt = \partial/\partial \mathrm{t} + \underline{\mathrm{V}} \cdot \underline{\nabla}$ (also known as 'material' or 'total' derivative), can be rewritten as

$$\frac{D\rho}{Dt} + \rho \underline{\nabla} \cdot \underline{V} = 0 \tag{1.1b}$$

where ρ and $\underline{V}$ are, respectively, the fluid density and velocity.

1.1.2 The Momentum Equation

The momentum equation reads:

$$\frac{\partial}{\partial t}\left(\rho \underline{V}\right) + \underline{\nabla} \cdot \underline{\underline{\Phi}}_{mt} = \underline{F}_b \tag{1.2}$$

where $\underline{F}_b$ is the generic body force acting on the fluid and $\underline{\underline{\Phi}}_{mt}$ the flux of momentum, which can be written as

$$\underline{\underline{\Phi}}_{mt} = \rho \underline{V}\,\underline{V} - \underline{\underline{\tau}} \tag{1.3}$$

where $\underline{\underline{\tau}}$ is known as the *stress tensor*. Such a tensor can be regarded as a stochastic measure of the exchange of microscopic momentum induced at molecular level by particle random motion (it provides clear evidence of the fact that viscous forces originate in molecular interactions; we shall come back to this concept later).

Substituting Equation 1.3 into Equation 1.2, it follows that:

$$\frac{\partial}{\partial t}\left(\rho \underline{V}\right) + \underline{\nabla} \cdot \left(\rho \underline{V}\,\underline{V} - \underline{\underline{\tau}}\right) = \underline{F}_b \tag{1.4a}$$

$$\rho \frac{D\underline{V}}{Dt} - \underline{\nabla} \cdot \underline{\underline{\tau}} = \underline{F}_b \tag{1.4b}$$

1.1.3 The Total Energy Equation

Introducing the total energy as:

$$E = \frac{1}{2}\rho V^2 + \rho u_{\mathrm{int}} \tag{1.5}$$

the total energy balance equation can be cast in condensed form as:

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{1}{2}V^2 + u_{\mathrm{int}}\right)\right] + \underline{\nabla} \cdot \left[\rho\left(\frac{1}{2}V^2 + u_{\mathrm{int}}\right)\underline{V} + \underline{J}_u - \underline{V} \cdot \underline{\underline{\tau}}\right] = \underline{F}_b \cdot \underline{V} \tag{1.6a}$$

or in terms of the substantial derivative:

$$\rho\frac{D}{Dt}\left(\frac{1}{2}V^2+u_{\text{int}}\right)+\underline{\nabla}\cdot\left[\underline{J}_u-\underline{V}\cdot\underline{\underline{\tau}}\right]=\underline{F}_b\cdot\underline{V} \tag{1.6b}$$

where

$$\underline{\Phi}_e=\left(\frac{1}{2}\rho V^2+\rho u_{\text{int}}\right)\underline{V}+\underline{J}_u-\underline{V}\cdot\underline{\underline{\tau}} \tag{1.7}$$

$\underline{J}_u$ being the diffusive flux of internal energy (it measures transport at the microscopic level of molecular kinetic energy due to molecular random motion).

1.1.4 The Budget of Internal Energy

A specific balance equation for the single internal energy can be obtained from subtracting the kinetic energy balance equation from the total energy balance equation (Equation 1.6).

Obviously, a balance equation for the pure kinetic energy can be introduced by taking the product of the momentum balance equation with $\underline{V}$:

$$\rho\frac{D}{Dt}\left(\frac{V^2}{2}\right)-\left(\underline{\nabla}\cdot\underline{\underline{\tau}}\right)\cdot\underline{V}=\underline{F}_b\cdot\underline{V} \tag{1.8a}$$

this equation, using the well-known vector identity $\underline{\nabla}\cdot\left(\underline{V}\cdot\underline{\underline{\tau}}\right)=\left(\underline{\nabla}\cdot\underline{\underline{\tau}}\right)\cdot\underline{V}+\underline{\underline{\tau}}:\underline{\nabla V}$ can be rewritten as

$$\rho\frac{D}{Dt}\left(\frac{V^2}{2}\right)-\underline{\nabla}\cdot\left(\underline{V}\cdot\underline{\underline{\tau}}\right)=\underline{F}_b\cdot\underline{V}-\underline{\underline{\tau}}:\underline{\nabla V} \tag{1.8b}$$

from which, among other things, it is evident that the diffusive flux of kinetic energy can be simply expressed as the scalar product between $\underline{V}$ and the stress tensor. Subtracting, as explained before, Equation 1.8b from Equation 1.6b, one obtains:

$$\rho\frac{Du_{\text{int}}}{Dt}+\underline{\nabla}\cdot\underline{J}_u=\underline{\underline{\tau}}:\underline{\nabla V} \tag{1.9a}$$

or

$$\frac{\partial\rho u_{\text{int}}}{\partial t}+\underline{\nabla}\cdot\left[\rho u_{\text{int}}\underline{V}+\underline{J}_u\right]=\underline{\underline{\tau}}:\underline{\nabla V} \tag{1.9b}$$

that is the aforementioned balance equation for the internal energy.

1.1.5 Closure Models

In general, the ‘closure’ of the thermofluid–dynamic balance equations given in the preceding sections, i.e. the determination of a precise mathematical formalism relating the diffusive fluxes (stress tensor and the diffusive flux of internal energy) to the macroscopic variables involved in the process, is not as straightforward as many would assume.

For a particular but fundamental category of fluids, known as ‘newtonian fluids’ the treatment of this problem, however, becomes relatively simple.

For the case considered in the present book (nonpolar fluids and absence of torque forces), the stress tensor can be taken symmetric, i.e. $\tau_{\text{ij}}=\tau_{\text{ji}}$ (conversely, a typical example of fluids for which the stress tensor is *not* symmetric is given by ‘micropolar fluids’, which represent fluids consisting of rigid, randomly oriented particles suspended in a viscous medium).

If the considered fluid is in quiescent conditions (i.e. there is no macroscopic motion) the stress tensor is diagonal and simply reads

$$\underline{\underline{\tau}} = -p\underline{\underline{I}} \tag{1.10a}$$

where $\underline{\underline{I}}$ is the unity tensor and p is the pressure.

In the presence of bulk convection the above expression must be enriched with the contributions induced by macroscopic fluid motion.

In the most general case such a contribution should be related to the gradient of velocity $\underline{\nabla V}$ via a tensor having tensorial order 4 (from a mathematical point of view a relationship between two tensors having order two has to be established using a four-dimension tensor). According to some simple considerations based on the isotropy of fluids (i.e. their property of not being dependent upon a specific direction) and other arguments provided over the years by various authors (Isaac Newton's landmark observations in liquids; later, the so-called Chapman–Enskog expansion elaborated for gases by Grad (1963) and Rosenau (1989)), the four-dimensional tensor relating the stress tensor to the gradients of mass velocity simply reduces to a proportionality (scalar) constant that does not depend on such gradients:

$$\underline{\underline{\tau}} = -p\underline{\underline{I}} + 2\mu\left(\underline{\nabla V}\right)_o^s \tag{1.10b}$$

where the constant of proportionality μ is known as the dynamic viscosity (it may be regarded as a macroscopic measure of the intermolecular attraction forces) and the tensor $\left(\underline{\nabla V}\right)_o^s$ (known as the viscous stress tensor or the dissipative part of the stress tensor) comes from the following decomposition of $\underline{\nabla V}$:

$$\underline{\nabla V} = \frac{1}{3}\left(\underline{\nabla}\cdot\underline{V}\right)\underline{\underline{I}} + \left(\underline{\nabla V}\right)_o^s + \left(\underline{\nabla V}\right)^a \tag{1.11}$$

where

$$\left(\underline{\nabla V}\right)_o^s = \left(\underline{\nabla V}\right)^s - \frac{1}{3}\left(\underline{\nabla}\cdot\underline{V}\right)\underline{\underline{I}}, \qquad \left(\underline{\nabla V}\right)^s = \frac{\underline{\nabla V} + \underline{\nabla V}^T}{2} \tag{1.12a}$$

$$\left(\underline{\nabla V}\right)^a = \frac{\underline{\nabla V} - \underline{\nabla V}^T}{2} \tag{1.12b}$$

The three contributions in Equation 1.11 are known to be responsible for volume changes, deformation and rotation, respectively, of a generic (infinitesimal) parcel of fluid (moving under the effect of the velocity field $\underline{V}$; the reader being referred to Section 1.2 for additional details about the meaning of $\left(\underline{\nabla V}\right)^a$ and its kinship with the concept of vorticity).

Moreover, in general, the diffusive flux of internal energy can be written as (Fourier law):

$$\underline{J}_u = -\lambda\underline{\nabla}T \tag{1.13}$$

where λ is the thermal conductivity and T the fluid temperature.

Using such closure relationships, and taking into account the following vector and tensor identities:

$$\underline{\nabla}\cdot\left(p\underline{\underline{I}}\right) = \underline{\nabla}p \tag{1.14}$$

$$\underline{\nabla}\cdot\left(p\underline{V}\right) = p\underline{\nabla}\cdot\underline{V} + \underline{V}\cdot\underline{\nabla}p \tag{1.15}$$

$$p\underline{\underline{I}} : \underline{\nabla V} = p\underline{\nabla}\cdot\underline{V} \tag{1.16}$$

$$\left(\underline{\nabla V}\right)_o^s : \underline{\nabla V} = \left(\underline{\nabla V}\right)_o^s : \left[\frac{1}{3}\left(\underline{\nabla}\cdot\underline{V}\right)\underline{\underline{I}} + \left(\underline{\nabla V}\right)_o^s + \left(\underline{\nabla V}\right)^a\right] = \left(\underline{\nabla V}\right)_o^s : \left(\underline{\nabla V}\right)_o^s \tag{1.17}$$

the balance equations can be therefore rewritten in compact form as follows:

Momentum:

$$\frac{\partial}{\partial t}\left(\rho \underline{V}\right) + \underline{\nabla} \cdot \left(\rho \underline{V}\,\underline{V}\right) + \underline{\nabla} p = \underline{\nabla} \cdot \left[2\mu \left(\underline{\nabla V}\right)_o^s\right] + \underline{F}_b \tag{1.18a}$$

$$\rho \frac{D\underline{V}}{Dt} + \underline{\nabla} p = \underline{\nabla} \cdot \left[2\mu \left(\underline{\nabla V}\right)_o^s\right] + \underline{F}_b \tag{1.18b}$$

Kinetic energy:

$$\frac{\partial}{\partial t}\left(\rho \frac{V^2}{2}\right) + \underline{\nabla} \cdot \left(\rho \frac{V^2}{2} \underline{V}\right) + \underline{V} \cdot \underline{\nabla} p = \underline{\nabla} \cdot \left[2\mu \underline{V} \cdot \left(\underline{\nabla V}\right)_o^s\right] - 2\mu \left(\underline{\nabla V}\right)_o^s : \left(\underline{\nabla V}\right)_o^s + \underline{F}_b \cdot \underline{V} \tag{1.19a}$$

$$\rho \frac{D}{Dt}\left(\frac{V^2}{2}\right) + \underline{V} \cdot \underline{\nabla} p = \underline{\nabla} \cdot \left[2\mu \underline{V} \cdot \left(\underline{\nabla V}\right)_o^s\right] - 2\mu \left(\underline{\nabla V}\right)_o^s : \left(\underline{\nabla V}\right)_o^s + \underline{F}_b \cdot \underline{V} \tag{1.19b}$$

Internal energy:

$$\frac{\partial \rho u_{\text{int}}}{\partial t} + \underline{\nabla} \cdot \left[\rho u_{\text{int}} \underline{V}\right] = \underline{\nabla} \cdot \left(\lambda \underline{\nabla} T\right) - p \underline{\nabla} \cdot \underline{V} + 2\mu \left(\underline{\nabla V}\right)_o^s : \left(\underline{\nabla V}\right)_o^s \tag{1.20a}$$

$$\rho \frac{D u_{\text{int}}}{Dt} = \underline{\nabla} \cdot \left(\lambda \underline{\nabla} T\right) - p \underline{\nabla} \cdot \underline{V} + 2\mu \left(\underline{\nabla V}\right)_o^s : \left(\underline{\nabla V}\right)_o^s \tag{1.20b}$$

Total (Internal+Kinetic) energy

$$\frac{\partial}{\partial t}\left[\rho \left(\frac{1}{2} V^2 + u_{\text{int}}\right)\right] + \underline{\nabla} \cdot \left[\rho \left(\frac{1}{2} V^2 + u_{\text{int}}\right) \underline{V}\right] = \underline{\nabla} \cdot \left(\lambda \underline{\nabla} T - p \underline{V} + 2\mu \underline{V} \cdot \left(\underline{\nabla V}\right)_o^s\right) + \underline{F}_b \cdot \underline{V} \tag{1.21a}$$

$$\rho \frac{D}{Dt}\left(\frac{1}{2} V^2 + u_{\text{int}}\right) = \underline{\nabla} \cdot \left(\lambda \underline{\nabla} T - p \underline{V} + 2\mu \underline{V} \cdot \left(\underline{\nabla V}\right)_o^s\right) + \underline{F}_b \cdot \underline{V} \tag{1.21b}$$

1.2 Some Considerations about the Dynamics of Vorticity

1.2.1 Vorticity and Circulation

Apart from the classical fluid-dynamic variables such as mass, momentum, (kinetic, internal or total) energy, whose balance equations have been shortly presented in the preceding section, 'vorticity' should be regarded as an additional useful mathematical concept for a better characterization of certain types of flow. Generally used in synergetic combination with the other classical concepts, this quantity has been found to play a fundamental role in the physics of vortex-dominated flows, *its dynamics being the primary tool to understand the time evolution of dissipative vortical structures*.

In the following we provide some related fundamental notions, together with a short illustration of the related interdependencies with other variables, as well as a derivation of the related balance equation.

Along these lines, it is worth starting the discussion with the observation that, in general, vorticity can be related to the amount of 'circulation' or 'rotation' (or more strictly, the local angular rate of rotation) in a fluid (it is intimately linked to the moment of momentum of a generic small fluid particle about its own centre of mass). The average vorticity in a small region of fluid flow, in fact, can be

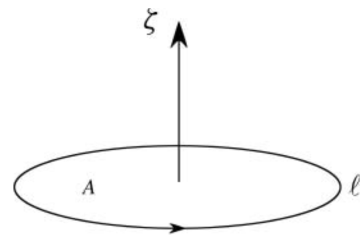

Figure 1.1 *Vorticity as a measure of the rate of rotational spin in a fluid.*

defined as the circulation Γ around the boundary of the small region, divided by the area A of the small region.

$$\bar{\zeta} = \frac{\Gamma}{A} \tag{1.22a}$$

where the fluid circulation Γ is defined as the line integral of the velocity $\underline{V}$ around the closed curve ℓ in Figure 1.1.

$$\Gamma = \oint_{\ell} \underline{V} \cdot \hat{\underline{t}} d\ell \tag{1.22b}$$

$\hat{\underline{t}}$ being the unit vector tangent to ℓ.

In practice, the vorticity at a point in a fluid can be regarded as the limit of Equation 1.22a as the area of the small region of fluid approaches zero at the point:

$$\zeta = \frac{d\Gamma}{dA} \tag{1.22c}$$

In addition to the previous modelling, using the Stokes theorem (purely geometrical in nature), which equates the circulation Γ around ℓ to the flux of the curl of $\underline{V}$ through any surface area bounded by ℓ:

$$\Gamma = \oint_{\ell} \underline{V} \cdot \hat{\underline{t}} d\ell = \int_{A} \left(\underline{\nabla} \wedge \underline{V}\right) \cdot \hat{\underline{n}}\ dS \tag{1.23}$$

where $\hat{\underline{n}}$ is the unit vector perpendicular to the surface A bounded by the closed curve ℓ (it is implicitly assumed that ℓ is smooth enough, i.e. it is locally lipschitzian; this implies that the existence of the unit vector perpendicular to the surface is guaranteed), it becomes evident from a mathematical point of view that the vorticity at a point can be defined as the curl of the velocity:

$$\underline{\zeta} = \underline{\nabla} \wedge \underline{V} \tag{1.24}$$

Therefore, it is a vector quantity, whose direction is along the axis of rotation of the fluid.

Notably, ζ has the same components as the anti-symmetric part of the tensor $\underline{\nabla V}$, that is in line with the explanation given in Section 1.1.5 about the physical meaning of $\left(\underline{\nabla V}\right)^{a}$.

Related concepts are the vortex line, which is a line that is at any point tangential to the local vorticity; and a vortex tube which is the surface in the fluid formed by all vortex lines passing through a given (reducible) closed curve in the fluid. The ‘strength’ of a vortex tube is the integral of the vorticity across a cross-section of the tube, and is the same everywhere along the tube (because vorticity has zero divergence).

In general, it is possible to associate a vector vorticity with each point in the fluid; thus the whole fluid space may be thought of as being threaded by vortex lines which are everywhere tangental to the local vorticity vector. These vortex lines represent the local axis of spin of the fluid particle at each point.

The related scalar quantity:

$$Z = \left(\underline{\nabla V}\right)^a : \left(\underline{\nabla V}\right)^a = \frac{\zeta^2}{2} \tag{1.25}$$

is generally referred to in the literature as the 'density of enstrophy'. It plays a significant role in some theories and models for the characterization of turbulence (as will be discussed in Chapter 6) and in some problems related to the uniqueness of solutions of the Navier–Stokes equations (Lappa, 2010). By simple mathematical manipulations it can also appear in global budgets of kinetic energy.

1.2.2 Vorticity in Two Dimensions

Apart from the point of view provided by mathematics (illustrated in Section 1.2.1 and related mathematical developments), there is another interesting way to introduce the notion of vorticity and to obtain insights into its properties, which is more adherent to the 'way of thinking' of experimentalists.

Due to intrinsic properties of the curl operator, for two-dimensional flows, vorticity reduces to a vector perpendicular to the plane.

Experimentalists have shown that, for such conditions, as an alternative to the classical definition, the strength of this vector at a generic point at any instant may be defined as *the sum of the angular velocities of any pair of mutually perpendicular, infinitesimal fluid lines* (contained in the plane of the 2D flow) *passing through such a point* (see Figure 1.2).

Equivalently (under a more physics-related perspective), Shapiro (1969) defined the vorticity of a generic fluid particle *as exactly twice the angular velocity of the solid particle at the instant of its birth originating ('by magic', e.g. by suddenly freezing) from the considered fluid particle* (resorting to this definition, one may therefore regard $\zeta/2$ as the average angular velocity of the considered fluid element; it is in this precise sense that the vorticity acts as a measure of the local rotation, or spin, of fluid elements, as mentioned before).

At this stage it is also worth highlighting how, as a natural consequence of such arguments, it becomes obvious that for a fluid having locally a 'rigid rotation' around an axis, i.e. moving like a solid rotating cylinder, vorticity will be simply *twice the system angular velocity*.

Another remarkable consequence of this observation is that for a fluid contained in a cylindrical tank rotating around its symmetry axis and being in relative motion with respect to the tank walls, the vorticity of any generic fluid particle will be given by the sum of two contributions, one related to the overall rotation of the container, as discussed above (rigid-rotation contribution), and the other (relative

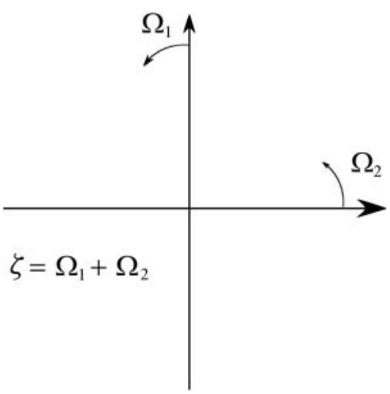

Figure 1.2 *Vorticity as the sum of the angular velocity of two short fluid line elements that happen, at that instant, to be mutually perpendicular (Shapiro, 1969).*

contribution) due to the motion displayed by the particle with respect to the rotating frame of reference (a coordinate system rotating at the same angular velocity of the container). These two components of vorticity are known as the solid-body vorticity (2Ω) and the relative vorticity (ζ), their sum being generally referred to the absolute vorticity ($\zeta + 2\Omega$).

1.2.3 Vorticity Over a Spherical Surface

Several useful generalizations of the concepts provided in the preceding Section 1.2.2 can be made. As a relevant example, it is worth providing some fundamental information about the related concept of *solid-body vorticity over the surface of a sphere*, which, among other things, will also prove to be very useful in the context of the topics treated in the present book (in particular, this notion has extensive background applications to planetary atmosphere dynamics).

By simple geometrical arguments, the component of vorticity perpendicular to a spherical surface due to solid rotation can be written as:

$$f = 2\Omega \sin(\varphi) \tag{1.26}$$

where φ is the latitude shown in Figure 1.3.

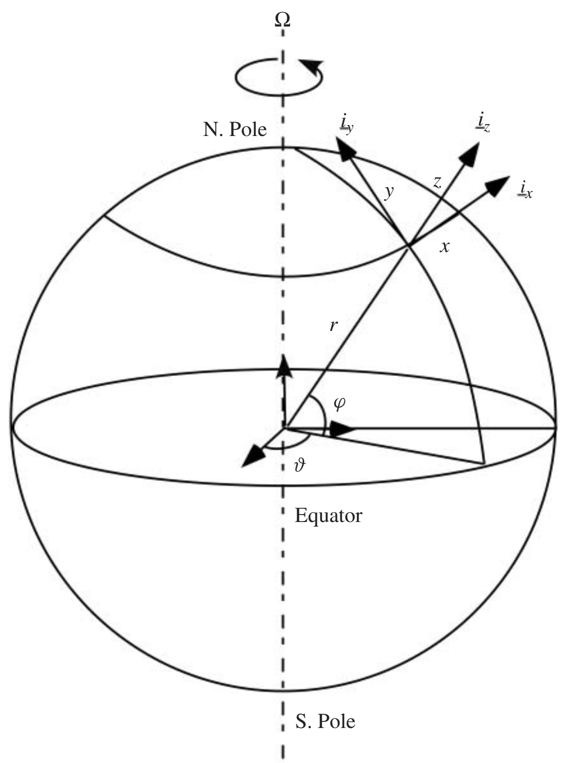

Figure 1.3 *Typical global and local coordinate systems used for a planet. The orthogonal unit vectors* $\underline{i}_x$, $\underline{i}_y$ *and* $\underline{i}_z$ *point in the direction of increasing longitude* ϑ, *latitude* φ *and altitude* z. *Locally, however, the mean motion can be considered planar and a rectangular reference system (x,y,z) can conveniently be introduced with coordinates measuring distance along* $\underline{i}_x$, $\underline{i}_y$ *and* $\underline{i}_z$, *respectively, i.e.* x *increasing eastward,* y *northward and* z *vertically upward.*

The parameter f above is generally referred to as the *Coriolis parameter* (or *frequency*). As evident in Equation 1.26, it accounts for the local intensity of the contribution brought to local fluid vorticity by planetary rotation and depends on the latitude through the sine function.

A useful simplification, however, can be introduced considering an observation made originally by Rossby (1939). Rossby's point (which can be justified formally by a Taylor series expansion) is that the 'sphericity' of Earth can be accommodated in a relatively simple way if the local planetary vorticity is properly interpreted and allowed a simplified variation with the latitude. In practice, with such a model (generally referred to as the 'β-plane approximation') the Coriolis parameter, f, is set to vary linearly in space as

$$f = f_{\mathrm{REF}} + \beta y \tag{1.27}$$

where f_{REF} is the value of f at a given latitude (a reference value) and $\beta = \partial f/\partial y$ is the rate at which the Coriolis parameter increases northward (the local y-axis being assumed to be directed from the equator towards the north pole; the reader being referred to Figure 1.3 and its caption for additional details):

$$\beta = \frac{\partial f}{\partial y} = \frac{1}{R_{\mathrm{Earth}}}\frac{\partial f}{\partial \varphi} = \frac{2\Omega\cos(\varphi)}{R_{\mathrm{Earth}}} \tag{1.28}$$

R_{Earth} being the average radius of Earth.

The component of the absolute fluid vorticity perpendicular to the Earth's surface, therefore, will simply read

$$\zeta + f_{\mathrm{REF}} + \beta y \tag{1.29}$$

The name 'beta' for this approximation derives from the convention to denote the linear coefficient of variation by the Greek letter β. The associated reference to a 'plane', however, must not be confused with the idea of a tangent plane touching the surface of the sphere at the considered latitude; the β-plane model, in fact, does describe the dynamics on a hypothetical tangent plane. Rather, such an approach is employed to take into proper account the latitudinal gradient of the planetary vorticity, while retaining a relatively simple form of the dynamical equations (indeed, it can be shown in a relatively easy way that the linear variation of f does not contribute nonlinear terms to the balance equations).

The reader will easily realize that the practical essence of such a simplification is that it only retains the effect of the Earth's curvature on the meridional (along a meridian) variations of the Coriolis parameter, while discarding all other curvature effects.

This approximation is generally valid in midlatitudes. Interestingly, however, it also works as an exact reference model for laboratory experiments in which the gradient of planetary vorticity is simulated using cylindrical containers with 'inclined' planar ('sloping') end walls (for which the vertical distance between the top and bottom boundaries varies linearly with the distance from the rotation axis).

A more restrictive simplification for a planetary atmosphere is the so-called f-plane approach in which the latitudinal variation of f is ignored, and a constant value of f appropriate for a particular latitude is considered throughout the domain. This is typically used in latitudes where f is large, and for scales that do not feel the curvature of the Earth. The closest thing on Earth to an f-plane is the Arctic Ocean ($\beta = 0$ at the Pole). Continuing the analogy with laboratory experiments, moreover, this model would correspond to a classical rotating tank with *purely horizontal top and bottom boundaries* (for which f would reduce to twice the angular velocity, i.e. $f = 2\Omega$).

1.2.4 The Curl of the Momentum Equation

In general, for any flow (2D or 3D) a specific balance equation for vorticity can be derived by simply taking the curl of the momentum equations (Equation 1.18) and taking into account the following identities:

$$\underline{\nabla} \cdot \underline{\zeta} = \underline{\nabla} \cdot \left(\underline{\nabla} \wedge \underline{V}\right) = 0 \tag{1.30}$$

$$\underline{V} \cdot \underline{\nabla V} = \underline{\nabla}\left(\frac{V^2}{2}\right) + \underline{\zeta} \wedge \underline{V} \tag{1.31}$$

$$\underline{\nabla} \wedge \underline{\nabla}\left(\frac{V^2}{2}\right) = 0 \tag{1.32}$$

$$\underline{\nabla} \wedge \left(\underline{V} \wedge \underline{\zeta}\right) = \underline{V}\left(\underline{\nabla} \cdot \underline{\zeta}\right) - \underline{\zeta}\left(\underline{\nabla} \cdot \underline{V}\right) + \underline{\zeta} \cdot \underline{\nabla V} - \underline{V} \cdot \underline{\nabla \zeta} \tag{1.33}$$

$$\underline{\nabla} \wedge \left(\frac{1}{\rho}\underline{\nabla} p\right) = \frac{1}{\rho}\underline{\nabla} \wedge \underline{\nabla} p - \frac{1}{\rho^2}\underline{\nabla}\rho \wedge \underline{\nabla} p = -\frac{1}{\rho^2}\underline{\nabla}\rho \wedge \underline{\nabla} p \tag{1.34}$$

This leads to

$$\frac{D\underline{\zeta}}{Dt} = \frac{\partial \underline{\zeta}}{\partial t} + \underline{V} \cdot \underline{\nabla \zeta} = \underline{\zeta} \cdot \underline{\nabla V} - \underline{\zeta}\left(\underline{\nabla} \cdot \underline{V}\right) + \frac{1}{\rho^2}\underline{\nabla}\rho \wedge \underline{\nabla} p + \underline{\nabla} \wedge \left[\frac{\underline{\nabla} \cdot 2\mu\left(\underline{\nabla V}\right)_o^s}{\rho}\right] + \underline{\nabla} \wedge \left(\frac{1}{\rho}\underline{F}_b\right) \tag{1.35}$$

The first term on the right member of this equation $\underline{\zeta} \cdot \underline{\nabla V}$, is known to be responsible for possible stretching of vortex filaments along their axial direction; this leads to contraction of the cross-sectional area of filaments and, *as a consequence of the conservation of angular momentum*, to an increase in vorticity (this term is absent in the case of 2D flows). The second term $\underline{\zeta}\left(\underline{\nabla} \cdot \underline{V}\right)$ describes possible stretching of vorticity due to flow compressibility. The third term is generally known as the baroclinic term (it accounts for changes in vorticity due to interaction of density and pressure gradients acting inside the fluid). The fourth term shows that vorticity can be produced or damped by the action of viscous stresses. The last term accounts for possible production of vorticity due to other body forces.

1.3 Incompressible Formulation

A cardinal simplification traditionally used (this monograph is not an exception to this common rule) in the context of studies dealing with thermal convection in both natural and industrial processes is to consider the density constant ($\rho = \text{const} = \rho_o$). Resorting to such approximation, all the governing equations derived in the preceding subsection can be rewritten in a simpler form (in general, the approximation of constant density is considered together with that of constant transport coefficients, μ and λ, which leads to additional useful simplifications).

Indeed, the continuity equations can be simplified as:

$$\underline{\nabla} \cdot \underline{V} = 0 \tag{1.36}$$

as a consequence, in Equation 1.18a

$$\underline{\nabla} \cdot \left[2\mu\left(\underline{\nabla V}\right)_o^s\right] = \mu\underline{\nabla} \cdot \left[\underline{\nabla V} + \underline{\nabla V}^T\right] = \mu\left[\nabla^2\underline{V} + \underline{\nabla}\left(\underline{\nabla} \cdot \underline{V}\right)\right] = \mu\nabla^2\underline{V} \tag{1.37}$$