# QUADRATIC FORMS, LINEAR ALGEBRAIC GROUPS, AND COHOMOLOGY

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# QUADRATIC FORMS, LINEAR ALGEBRAIC GROUPS, AND COHOMOLOGY

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# Preface

We dedicate this volume to Professor Parimala on the occasion of her 60th birthday. It contains a variety of papers related to the themes of her research. Parimala's first striking result was a counterexample to a quadratic analogue of Serre's conjecture (*Bulletin of the American Mathematical Society*, 1976). Her influence has continued through her tenure at the Tata Institute of Fundamental Research in Mumbai (1976–2006), and now her time at Emory University in Atlanta (2005–present).

A conference was held from 30 December 2008 to 4 January 2009, at the University of Hyderabad, India, to celebrate Parimala's 60th birthday (see the conference's Web site at http://mathstat.uohyd.ernet.in/conf/quadforms2008). The organizing committee consisted of J.-L. Colliot-Thélène, Skip Garibaldi, R. Sujatha, and V. Suresh. The present volume is an outcome of this event.

We would like to thank all the participants of the conference, the authors who have contributed to this volume, and the referees who carefully examined the submitted papers. We would also like to thank Springer-Verlag for readily accepting to publish the volume. In addition, the other three editors of the volume would like to place on record their deep appreciation of Skip Garibaldi's untiring efforts toward the final publication.

We are grateful for the support and the hospitality of the University of Hyderabad, especially the members of the Department of Mathematics and Statistics. We would like to thank the office staff of the Department of Mathematics and Statistics and the other staff of the University responsible for providing administrative and logistical support.

We are also extremely grateful to the University Grants Commission, India and the National Board for Higher Mathematics for financial support.

Paris, France — *J.-L. Colliot-Thélène*
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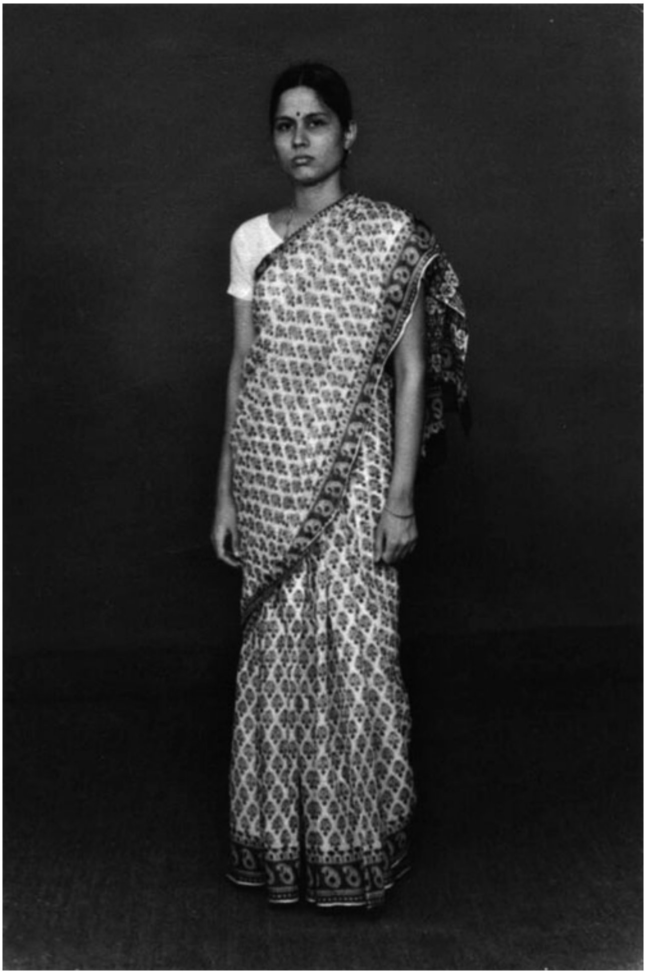

**Fig. 1** Parimala during her high school years

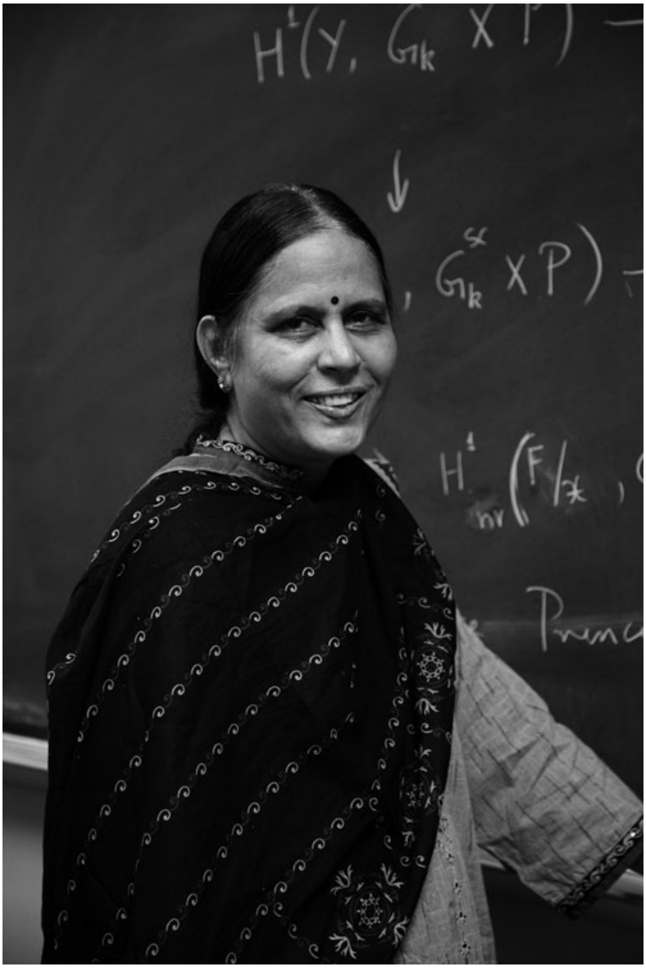

**Fig. 2** Parimala at Emory in 2009

# Contents

## Part II Invited Articles

# Part I
# Surveys

# Multiples of forms

Eva Bayer-Fluckiger

*To my friend Parimala*

**Summary** The aim of this paper is to survey and extend some results concerning multiples of (quadratic, hermitian, bilinear...) forms.

## Introduction

Let $k$ be a field of characteristic $\neq 2$. Let $k_{\mathrm{s}}$ be a separable closure of $k$, and set $\Gamma_k = \mathrm{Gal}(k_{\mathrm{s}}/k)$. Let $\mathrm{cd}_2(\Gamma_k)$ be the 2-cohomological dimension of $\Gamma_k$. It is a classical question whether it is possible to characterize quadratic forms over $k$ up to isomorphism via some cohomological invariants. For instance, it is well-known that if $\mathrm{cd}_2(\Gamma_k) = 1$, then two quadratic forms are isomorphic if and only if they have the same dimension and discriminant.

The same problem is relevant for hermitian forms over division algebras, $G$-forms (where $G$ is a finite group), systems of quadratic or hermitian forms, etc.

Such a direct comparison does not seem to be possible for fields of arbitrary cohomological dimension. For this reason, the following weaker question was proposed in [2, 8.5] (in the context of trace forms of Galois algebras). Let $I$ be the fundamental ideal of the Witt ring of $k$, let $d$ be a positive integer, and let $\phi \in I^d$. Can we compare $\phi \otimes q$ and $\phi \otimes q'$ in terms of some cohomological invariants ?

This question was mostly studied for hermitian forms over division rings with involution and for trace forms of Galois algebras (see for instance [2–4, 7]). This paper will survey and slightly extend these results. For instance, we will show the following

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**Theorem.** *Suppose that* $\mathrm{cd}_2(\Gamma_k) \leq d$, *and let G be a finite group that has no quotient of order* 2. *Let q and* $q'$ *be two G-quadratic forms defined on the same* $k[G]$*-module, and let* $\phi \in I^{d-1}$. *Then*

$$\phi \otimes q \simeq \phi \otimes q'.$$

# 1 Multiples of Quadratic Forms

## 1.1 Galois Cohomology

Let $k_s$ be a separable closure of $k$, and set $\Gamma_k = \mathrm{Gal}(k_s/k)$. For any discrete $\Gamma_k$-module $C$, set $H^i(k,C) = H^i(\Gamma_k,C)$. We say that the *2-cohomological dimension* of $\Gamma_k$ is at most $d$, denoted by $\mathrm{cd}_2(\Gamma_k) \leq d$, if $H^i(k,C) = 0$ for all $i > d$ and for every finite 2-primary $\Gamma_k$-module $C$.

Set $H^i(k) = H^i(k,\mathbf{Z}/2\mathbf{Z})$, and recall that $H^1(k) \simeq k^*/k^{*2}$. For all $a \in k^*$, let us denote by $(a) \in H^1(k)$ the corresponding cohomology class. We use the additive notation for $H^1(k)$. If $a_1,\ldots,a_n \in k^*$, we denote by $(a_1) \cup \cdots \cup (a_n) \in H^n(k)$ their cup product.

If $U$ is a linear algebraic group defined over $k$, let $H^1(k,U)$ be the pointed set $H^1(\Gamma_k,U(k_s))$ (cf. [11], [12, Chap. 10]).

## 1.2 Quadratic Forms

All quadratic forms are supposed to be nondegenerate. We denote by $W(k)$ the Witt ring of $k$, and by $I = I(k)$ the fundamental ideal of $W(k)$. For all $a_1,\ldots,a_n \in k^*$, let us denote by $\langle\langle a_1,\ldots,a_n\rangle\rangle = \langle 1,-a_1\rangle \otimes \cdots \otimes \langle 1,-a_n\rangle$ the associated $n$-fold Pfister form. It is well known that $I^n$ is generated by the $n$-fold Pfister forms. The following has been conjectured by Milnor

**Theorem 1.2.1 (cf. Orlov–Vishik–Voevodsky [8]).** *For every positive integer n, there exists an isomorphism*

$$e_n : I^n/I^{n+1} \to H^n(k)$$

*such that*

$$e_n(\langle\langle a_1,\ldots,a_n\rangle\rangle) = (a_1) \cup \cdots \cup (a_n)$$

*for all* $a_1,\ldots,a_n \in k^*$.

It is easy to see that the above theorem has the following consequences (cf. [3]):

**Corollary 1.2.2.** *Suppose that* $\mathrm{cd}_2(\Gamma_k) \leq d$. *Let q and* $q'$ *be two quadratic forms with* $\dim(q) = \dim(q')$, *and let* $\phi \in I^d$. *Then* $\phi \otimes q \simeq \phi \otimes q'$.

For every quadratic form $q$, let us denote by $\mathrm{disc}(q) \in H^1(k)$ its discriminant. Recall that if $n = \dim(q)$, then $\mathrm{disc}(q) = (-1)^{n(n-1)/2}\det(q)$.

**Corollary 1.2.3.** *Suppose that* $\mathrm{cd}_2(\Gamma_k) \leq d$*. Let* $q$ *and* $q'$ *be two quadratic forms with* $\dim(q) = \dim(q')$*, and let* $\phi \in I^{d-1}$*. Then*

$$\phi \otimes q \simeq \phi \otimes q' \text{ if and only if } e_{d-1}(\phi) \cup (\mathrm{disc}(q)) = e_{d-1}(\phi) \cup (\mathrm{disc}(q')) \in H^d(k).$$

For any quadratic form $q$, let us denote by $w_2(q) \in Br_2(k)$ the Hasse–Witt invariant of $q$. Recall that if $q \simeq \langle a_1, \ldots, a_n \rangle$, then $w_2(q) = \prod_{i<j}(a_i, a_j)$, where $(a_i, a_j)$ is the quaternion algebra over $k$ determined by $a_i$ and $a_j$. We can extend the previous results as follows:

**Corollary 1.2.4.** *Suppose that* $\mathrm{cd}_2(\Gamma_k) \leq d$*. Let* $q$ *and* $q'$ *be two quadratic forms. Suppose that* $\dim(q) = \dim(q')$ *and* $\det(q) = \det(q')$*. Let* $\phi \in I^{d-2}$*. Then*

$$\phi \otimes q \simeq \phi \otimes q' \text{ if and only if } e_{d-2}(\phi) \cup w_2(q) = e_{d-2}(\phi) \cup w_2(q') \in H^d(k).$$

*Proof.* Let $Q = q \oplus (-q')$, and let $\dim(q) = \dim(q') = m$, $\det(q) = \det(q') = d$, Then $\det(Q) = (-1)^m d^2 = (-1)^m$ in $k^*/k^{*2}$, hence $\mathrm{disc}(Q) = (-1)^m(-1)^m = 1$. This implies that $Q \in I^2$.

As $Q \in I^2$, $e_2(Q)$ is defined. We have $e_2(Q) = w_2(q) + w_2(q')$. Indeed, we have $w_2(-q') = w_2(q') + (-1, (-1)^{m(m-1)/2}d^{m-1})$, and $w_2(Q) = w_2(q) + w_2(q') + (d, (-1)^m d)$. Using this, a computation shows that $w_2(Q) = w_2(q) + w_2(q') + (-1, (-1)^{m(m-1)/2})$. On the other hand, we have $e_2(Q) = w_2(Q)$ if $m \equiv 0, 1 \pmod 4$, $e_2(Q) = w_2(Q) + (-1,-1)$ if $m \equiv 2, 3 \pmod 4$. Therefore, we get $e_2(Q) = w_2(q) + w_2(q')$. □

Let $\phi \in I^{d-2}$. Then $\phi \otimes q \simeq \phi \otimes q'$ if and only if $\phi \otimes Q$ is hyperbolic. This is equivalent to $e_{d-2}(\phi) \cup e_2(Q) = 0$, hence to $e_{d-2}(\phi) \cup w_2(q) = e_{d-2}(\phi) \cup w_2(q')$.

## 2 Hermitian forms over Division Algebras with Involution

Let $D$ be a division algebra over $k$. An *involution* of $D$ is a $k$-linear antiautomorphism $\sigma : D \to D$ of order 2. Let $K$ be the center of $D$. We say that $(D, \sigma)$ is a *division algebra with involution over* $k$ if the fixed field of $\sigma$ in $K$ is equal to $k$. If $K = k$, then $\sigma$ is said to be of the *first kind*. After extension to $k_s$, the involution $\sigma$ is determined by a symmetric or a skew-symmetric form. In the first case, $\sigma$ is said to be of *orthogonal type*, in the second one, of *symplectic type*. If $K \neq k$, then $K$ is a quadratic extension of $k$ and the restriction of $\sigma$ to $K$ is the nontrivial automorphism of $K$ over $k$. In that case, the involution is said to be of the *second kind*, or a *unitary involution*, or a $K/k$-involution. Details on algebras with involution are in Chap. 8 of Scharlau's book [10].

Let $(D, \sigma)$ be a division algebra with involution over $k$. A *hermitian form* over $(D, \sigma)$ is by definition a pair $(V, h)$, where $V$ is a finite dimensional $D$-vector space, and $h : V \times V \to D$ is hermitian with respect to $\sigma$. We say that $(V,h)$ is *hyperbolic* if there exists a sub $D$-vector space $W$ of $V$ with $\dim(V) = 2\dim(W)$ and such that $h(x,y) = 0$ for all $x,y \in W$. This leads to a notion of Witt group $W(D,\sigma)$ (see for instance [10, Chap. 7, Sect. 2]). Note that the tensor product of a quadratic form over $k$ with a hermitian form over $(D,\sigma)$ is a hermitian form over $(D,\sigma)$, hence $W(D,\sigma)$ is a $W(k)$-module.

Let $(V,h)$ be a hermitian form over $(D,\sigma)$, as above. Let $n = \dim_D(V)$, and let $H$ be the matrix of $h$ with respect to some $D$-basis of $V$. Let us denote by $\mathrm{Nrd} : M_n(D) \to k$ the reduced norm. The *discriminant* of $h$ is by definition $\mathrm{disc}(h) = (-1)^{n(n-1)/2}\mathrm{Nrd}(H) \in k^*/k^{*2}$.

Let $(D,\sigma)$ be a division algebra with involution over $k$. Let us denote by $J$ the sub $W(k)$-module of $W(D,\sigma)$ consisting of the hermitian forms $(V,h)$ with $\dim_D(V)$ even. Suppose that $\mathrm{cd}_2(\Gamma_k) \le d$. The following is proved in [3, Sect. 2]:

**Theorem 2.1.1.**

*(a) We have* $I^d J = 0$.
*(b) If* $\sigma$ *is of the second kind, then* $I^{d-1}J = 0$.
*(c) If* $\sigma$ *is of the first kind and of the symplectic type, then* $I^{d-2}J = 0$.

Part (a) was proved by Chabloz in [7].

The following three results follow from theorems of Parimala, Sridharan and Suresh [9], and of Berhuy [6], and are proved in [3, Sect. 2]:

**Theorem 2.1.2.** *Suppose that* $D$ *is a quaternion algebra, and that* $\sigma$ *is of the first kind and of the orthogonal type. Suppose that* $\mathrm{cd}_2(\Gamma_k) \le d$. *Let* $h \in J$*, and let* $\phi \in I^{d-1}$. *Then* $\phi \otimes h$ *is hyperbolic if and only if* $e_{d-1}(\phi) \cup (\mathrm{disc}(h)) = 0$.

**Corollary 2.1.3.** *Suppose that* $D$ *is a quaternion algebra, and that* $\sigma$ *is of the first kind and of the orthogonal type. Suppose that* $\mathrm{cd}_2(\Gamma_k) \le d$. *Let* $h$ *and* $h'$ *be two hermitian forms over* $(D,\sigma)$ *with* $\dim(h) = \dim(h')$*, and let* $\phi \in I^{d-1}$. *Then* $\phi \otimes h \simeq \phi \otimes h'$ *if and only if* $e_{d-1}(\phi) \cup (\mathrm{disc}(h)) = e_{d-1}(\phi) \cup (\mathrm{disc}(h'))$.

Let us denote by $J_2$ the sub-$W(k)$-module of $J$ consisting of the classes of the hermitian forms $h$ such that $\mathrm{disc}(h) = 1$. Then we have (cf. [3, Sect. 2]):

**Corollary 2.1.4.** *Suppose that* $D$ *is a quaternion algebra, and that* $\sigma$ *is of the first kind and of the orthogonal type. Suppose that* $\mathrm{cd}_2(\Gamma_k) \le d$. *Then* $I^{d-1}J_2 = 0$.

Let $h$ be a hermitian form over a quaternion algebra $D$ endowed with an involution $\sigma$ of the first kind and of orthogonal type. Suppose that $h \in J_2$. Then one can define the *Clifford invariant* $\mathscr{C}(h) \in Br_2(k)/(D)$, cf. [5, Sect. 2]. We have the following:

**Theorem 2.1.5.** *Suppose that* $D$ *is a quaternion algebra, and that* $\sigma$ *is of the first kind and of the orthogonal type. Suppose that* $\mathrm{cd}_2(\Gamma_k) \le d$. *Let* $h \in J_2$*, and let* $\phi \in I^{d-2}$. *Then* $\phi \otimes h$ *is hyperbolic if and only if* $e_{d-2}(\phi) \cup \mathscr{C}(h) = 0$.

*Proof.* By Berhuy [6, Th. 13], it suffices to show that $e_{d-2}(\phi)\cup\mathscr{C}(h)=0$ if and only if $e_{n,D}(\phi\otimes h)=0$ for all $n\geq 0$ (see [6, 2.2] for the definition of the invariant $e_{n,D}$). As $\mathrm{cd}_2(\Gamma_k)\leq d$, we have $e_{n,D}(\phi\otimes h)=0$ for $n>d$, so it suffices to check that $e_{d-2}(\phi)\cup\mathscr{C}(h)=0$ is equivalent with $e_{n,D}(\phi\otimes h)=0$ for all $n=0,\dots,d$. Let $k(D)$ be the function field of the quadric associated to $D$. Then $D\otimes k(D)\simeq M_2(k(D))$, and $h_{k(D)}$ corresponds via Morita equivalence to a quadratic form $q_h$ over $k(D)$. Note that $e_2(q_h)=\mathscr{C}(h)$. Similarly, the hermitian form $(\phi\otimes h)_{k(D)}$ corresponds to a quadratic form $q_{\phi h}$ over $k(D)$, and we have $q_{\phi h}\simeq\phi\otimes q_h$.

For all $n=0,\dots,d$, we have by construction that $e_{n,D}(\phi\otimes h)=0$ if and only if $e_n(q_{\phi h})=0$ (cf. [6, 2.2]). But $q_{\phi h}\simeq\phi\otimes q_h$, hence

$$e_n(q_{\phi h})=e_n(\phi\otimes q_h)=e_{n-2}(\phi)\cup e_2(q_h)=e_{n-2}(\phi)\cup\mathscr{C}(h).$$

If $n<d$, then $e_{n-2}(\phi)=0$ as $\phi\in I^{d-2}$. We have $e_d(q_{\phi h})=e_{d-2}(\phi)\cup(\mathscr{C}(h))$. Hence $e_n(q_{\phi h})=0$ for all $n\geq 0$ if and only if $e_{d-2}(\phi)\cup\mathscr{C}(h)=0$. This concludes the proof. □

Let us denote by $J_2$ the sub $W(k)$-module of $J_2$ consisting of the classes of the hermitian forms $h$ such that $\mathscr{C}(h)=0$.

**Corollary 2.1.6.** *Suppose that $D$ is a quaternion algebra, and that $\sigma$ is of the first kind and of the orthogonal type. Suppose that* $\mathrm{cd}_2(\Gamma_k)\leq d$. *Then* $I^{d-2}J_3=0$.

*Proof.* This is an immediate consequence of 2.1.5. □

# 3 Galois Cohomology of Unitary Groups

Let $k$ be a field of characteristic $\neq 2$. Let $A$ be a finite dimensional $k$-algebra, and let $\sigma: A\to A$ be a $k$-linear involution. Let $U_A$ be the linear algebraic group over $k$ defined by

$$U_A(E)=\{x\in A_E\mid x\sigma(x)=1\}$$

for any commutative $k$-algebra $E$. The group $U_A$ is called the *unitary group* of $(A,\sigma)$. Let us denote by $U'_A$ the connected component of the identity. Let $\phi$ be a quadratic form of dimension $n$, set $A_\phi=M_n(k)\otimes_k A$, and let $\sigma_\phi: A_\phi\to A_\phi$ be the involution given by the tensor product of the involution on $M_n(k)$ induced by $\phi$ with the involution $\sigma$.

Recall that $k_s$ is a separable closure of $k$, that $\Gamma_k=\mathrm{Gal}(k_s/k)$, and that for any linear algebraic group $U$, we use the standard notation $H^1(k,U)=H^1(\Gamma_k,U(k_s))$ (see [11, 12] for basic facts concerning nonabelian Galois cohomology). With the notation as above, we have natural maps $H^1(k,U'_A)\to H^1(k,U_A)$ and $H^1(k,U_A)\to H^1(k,U_\phi)$.

Let $R_A$ be the radical of the algebra $A$, and set $\overline{A}=A/R_A$. We have

$$\overline{A}\simeq A_1\times\cdots\times A_s\times(A_{s+1}\times A'_{s+1})\times\cdots\times(A_m\times A'_m),$$

where $A_i$ is a simple algebra for all $i = 1,\dots,m$, with $\sigma(A_i) = A_i$ for $i = 1,\dots,s$ and $\sigma(A_i) = A_i'$ for $i = s+1,\dots,m$. Let $a_i$ be the index of $A_i$.

**Theorem 3.1.1.** *Suppose that* $\mathrm{cd}_2(\Gamma_k) \le d$.

*(i) Let $\phi \in I^d$. Then the map $H^1(k,U_A) \to H^1(k,U_\phi)$ is trivial.*
*(ii) Let $\phi \in I^{d-1}$. Suppose that if $\sigma_i$ is orthogonal and $i = 1,\dots,s$, then $a_i = 1$. Then the composition $H^1(k,U_A') \to H^1(k,U_A) \to H^1(k,U_\phi)$ is trivial.*

*Proof.* The projection $A \to \overline{A}$ induces a bijection of pointed sets $H^1(k,U_A) \to H^1(k,U_{\overline{A}})$. Let $F_i$ be the maximal subfield of the center of $A_i$ such that $\sigma_i$ is $F_i$-linear if $i = 1,\dots,s$, and let $U_i$ be the norm-one group of $(A_i,\sigma_i)$. For $i = s+1,\dots,m$, let $F_i$ be the center of $A_i$, and let $U_i$ be the norm-one group of $((A_i \times A_i),\sigma_i)$. Then $U_i$ is a linear algebraic group defined over $F_i$ for all $i = 1,\dots,m$. We have a bijection of pointed sets

$$H^1(k,U_A) \to \prod_{i=1,\dots,m} H^1(F_i,U_i).$$

If $i = s+1,\dots,m$, then $U_i$ is a general linear group, hence $H^1(F_i,U_i) = 0$. Hence we have a bijection of pointed sets $H^1(k,U_A) \to \prod_{i=1,\dots,s} H^1(F_i,U_i)$. For all $i = 1,\dots,s$, the simple algebra $A_i$ is a matrix algebra over a division algebra with involution $D_i$. The group $U_i$ is the unitary group of a hermitian form $h_i$ over $D_i$, and it is well-known that $H^1(F_i,U_i)$ is in bijection with the isomorphism classes of the hermitian forms over $D_i$ of the same dimension as $h_i$.

Let $R_\phi$ be the radical of $A_\phi$. The map $f : H^1(k,U_A) \to H^1(k,U_\phi)$ induces $\overline{f} : H^1(k,U_{\overline{A}}) \to H^1(k,U_{\overline{\phi}})$, and

$$f_i : H^1(F_i,U_i) \to H^1(F_i,U_{M_n(A_i)})$$

for all $i = 1,\dots,s$. The image of the isomorphism class of the hermitian form $h_i'$ is the hermitian form $\phi \otimes h_i'$.

If $\phi \in I^d$, then by 2.1.1 (a) $\phi \otimes h_i \simeq \phi \otimes h_i'$ for every hermitian form $h_i'$ with $\dim(h_i') = \dim(h_i)$. This implies that $f_i$ is trivial for all $i = 1,\dots,s$, hence $\overline{f}$ is trivial. Therefore $f$ is trivial, which proves (i).

Let us prove (ii). Let us denote by $f'$ the composition $H^1(k,U_A') \to H^1(k,U_A) \to H^1(k,U_\phi)$. Let $U_i'$ be the connected component of the identity in $U_i$. If $\sigma_i$ is unitary or symplectic, then $U_i = U_i'$. If $\sigma_i$ is orthogonal, then by hypothesis $a_i = 1$, so $H^1(F_i,U_i)$ is in bijection with the isomorphism classes of the hermitian (actually, quadratic) forms over $D_i = F_i$ of the same dimension and same discriminant as $h_i$.

Let us denote by $f_i'$ the composition $H^1(F_i,U_i') \to H^1(F_i,U_i) \to H^1(k,U_{M_r(A_i)})$ for all $i = 1,\dots,s$. Suppose that $\phi \in I^{d-1}$. Then 2.1.1 (b) and (c) imply that $f_i'$ has trivial image if $\sigma_i$ is unitary or symplectic. If $\sigma_i$ is orthogonal, then we have $d_i = 1$, therefore by 1.2.3 the map $f_i'$ is trivial. This implies that $f'$ is trivial, and this completes the proof. □

## 4 Systems of Quadratic and Hermitian Forms

Let $V$ be a finite dimensional $k$-vector space, and let $S = \{q_1, \dots, q_r\}$ be a *system of quadratic forms*, $q_i : V \times V \to k$. We say that two systems $S$ and $S' = \{q'_1, \dots, q'_r\}$ are *isomorphic* if there exists a $k$-linear isomorphism $f : V \to V$ such that $q'_i(fx, fy) = q_i(x,y)$ for all $x, y \in V$ and for all $i = 1, \dots, r$.

Let $K$ be a quadratic extension of $k$, and let $W$ be a finite dimensional $K$-vector space. Let us denote by $x \mapsto \bar{x}$ the involution of $K$ given by the unique nontrivial $k$-automorphism of $K$. We can then consider *systems of hermitian forms* $\Sigma = \{h_1, \dots, h_r\}$. We say that two systems $\Sigma$ and $\Sigma' = \{h_1, \dots, h_r\}$ are *isomorphic* if there exists a $K$-linear isomorphism $f : W \to W$ such that $h'_i(fx, fy) = h_i(x,y)$ for all $x, y \in W$ and for all $i = 1, \dots, r$.

Let $\phi$ be a quadratic form over $k$. Then the tensor product $\phi \otimes S$ of $\phi$ with a system of quadratic forms is a system of quadratic forms, and the tensor product $\phi \otimes S$ of $\phi$ with a system of hermitian forms is a system of hermitian forms.

**Theorem 4.1.1.** *Suppose that* $\mathrm{cd}_2(\Gamma_k) \leq d$.

*(i) Let $S$ and $S'$ be two systems of quadratic forms, and suppose that $S$ and $S'$ are isomorphic over $k_s$. Let $\phi \in I^d$. Then $\phi \otimes S \simeq \phi \otimes S'$.*
*(ii) Let $\Sigma$ and $\Sigma'$ be two systems of hermitian forms, and suppose that $\Sigma$ and $\Sigma'$ are isomorphic over $k_s$. Let $\phi \in I^{d-1}$. Then $\phi \otimes \Sigma \simeq \phi \otimes \Sigma'$.*

*Proof.* Write $A(S)$ for the set of all $(e, f) \in \mathrm{End}(V) \times \mathrm{End}(V)$ such that $q_i(ex, y) = q_i(x, fy)$ for all $i = 1, \dots, r$ and write $A(\Sigma)$ for the set of $(e, f) \in \mathrm{End}(W) \times \mathrm{End}(W)$ such that $h_i(ex, y) = h_i(x, fy)$ for all $i = 1, \dots, r$. These are the algebras associated to the systems $S$ and $\Sigma$, cf. [1]. They are endowed with involutions defined by $(e, f) \mapsto (f, e)$, and the automorphism groups of the systems can be identified with the unitary groups of these algebras, see [1] for details.

The isomorphism classes of the systems of quadratic forms that become isomorphic to $S$ over $k_s$ are in bijection with $H^1(k, U_{A(S)})$. Hence (i) follows from 3.1.1(i).

The isomorphism classes of the systems of hermitian forms that become isomorphic to $\Sigma$ over $k_s$ are in bijection with $H^1(k, U_{A(\Sigma)})$. As the forms are hermitian with respect to the nontrivial involution $x \mapsto \bar{x}$, the decomposition of the algebra $A(\Sigma)$ as in Sect. 3 has no orthogonal components. Therefore, the hypothesis of 3.1.1(ii) are satisfied, and this implies part (ii) of the theorem. □

## 5 *G*-Quadratic Forms

Let $G$ be a finite group, and let us denote by $k[G]$ the associated group ring. A *G-quadratic form* is a pair $(M, q)$, where $M$ is a $k[G]$-module that is a finite dimensional $k$-vector space, and $q : M \times M \to k$ is a nondegenerate symmetric bilinear form such that

$$q(gx, gy) = q(x, y) \quad \text{for all } x, y \in M \text{ and all } g \in G.$$

We say that two $G$-quadratic forms $(M,q)$ and $(M',q')$ are *isomorphic* if there exists an isomorphism of $k[G]$-modules $f : M \to M'$ such that $q(f(x), f(y)) = q'(x,y)$ for all $x, y \in M$. If this is the case, we write $(M,q) \simeq_G (M',q')$, or $q \simeq_G q'$.

If $\phi$ is a quadratic form over $k$, and $q$ a $G$-quadratic form, then the tensor product $\phi \otimes q$ is a $G$-quadratic form.

For any $G$-quadratic form $(M,q)$, let

$$A = A(M,q) = \left\{ (e,f) \in \mathrm{End}_{k[G]}(M) \times \mathrm{End}_{k[G]}(M) \,\middle|\, \begin{array}{l} q(ex,y) = q(x,fy) \text{ for all} \\ x,y \in M \end{array} \right\}$$

be the associated algebra (cf. [1]). Then the unitary group $U_A$ can be identified with the group of automorphisms of $(M,q)$, and the set of isomorphism classes of $G$-quadratic forms $(M,q')$ is in bijection with the set $H^1(k,U_A)$.

**Theorem 5.1.1.** *Suppose* $\mathrm{cd}_2(\Gamma_k) \le d$. *Let* $(M,q)$ *and* $(M,q')$ *be two G-quadratic forms.*

*(i) Let* $\phi \in I^d$. *Then* $\phi \otimes q \simeq \phi \otimes q'$.
*(ii) Suppose moreover that G has no quotient of order 2, and let* $\phi \in I^{d-1}$. *Then* $\phi \otimes q \simeq \phi \otimes q'$.

*Proof.* Part (i) follows immediately from 3.1.1(i). In order to prove part (ii), note that as $G$ has no quotient of order 2, the group $G$ has no nontrivial orthogonal characters, hence $U_A = U'_A$. Therefore, 3.1.1(ii) implies part (ii) of the theorem. □

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# On Saltman's $p$-Adic Curves Papers

Eric Brussel

*For Parimala on her 60th birthday*

**Summary** We present a synthesis of Saltman's work (Adv. Math. 43(3):250–283, 1982; J. Alg. 314(2):817–843, 2007) on the division algebras of prime-to-$p$ degree over the function field $K$ of a $p$-adic curve. Suppose $\Delta$ is a $K$-division algebra. We prove that (a) $\Delta$'s degree divides the square of its period; (b) if $\Delta$ has prime degree (different from $p$), then it is cyclic; (c) $\Delta$ has prime index different from $p$ if and only if $\Delta$'s period is prime, and its ramification locus on a certain model for $K$ has no "hot points".



These notes are based on Saltman's seminal work on the Brauer group $\mathrm{Br}(K)$ of the function field $K$ of a $p$-adic curve $X_{\mathbb{Q}_p}$. In [S1, S2] Saltman showed the index of an element in the prime-to-$p$ part of $\mathrm{Br}(K)$ divides the square of its period, and that all division algebras of prime degree $q \neq p$ are cyclic. He also gave a geometric criterion for a class of prime period $q \neq p$ to have index $q$. We reprove these theorems, modulo some of [S2, Sect. 1], expanding some proofs, consolidating some results, and providing some additional background along the way.

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Saltman based his analysis on four observations: First, that while $K$ is the function field of the curve $X_{\mathbb{Q}_p}$ over $\mathbb{Q}_p$, it is also the function field of a regular model $X$ of that curve, over $\mathbb{Z}_p$; second, that associated to every $\alpha \in \mathrm{H}^2(K)$ is a divisor $D_\alpha \subset X$, which may be assumed to have normal crossings; third, that purity holds for all such $X$; and fourth, that the unramified Brauer group $\mathrm{H}^2_{\mathrm{nr}}(K)$ is trivial. The goal in splitting, then, is to construct a field extension $L/K$ over which $\alpha$ is unramified:

$$\partial_w(\alpha_L) = 0 \quad \text{for all } w \in V(L).$$

In particular, we avoid the construction of an explicit model for $X_L$, over which $\alpha$ would have zero ramification divisor. Constructing $L$ turns out to be tricky, mainly due to the prospect of divisors in the ramification locus of $\alpha_L$, on any (hypothetical) model of $X_L$ that are centered on closed points of $X$. The general strategy is to use local equations for $D_\alpha$ to construct an element $f \in K$, set $L = K(f^{1/n})$, compute the residue of $\alpha_L$ at all $w$ using a local structure theorem (this uses purity), and adjust $f$ accordingly, if necessary. The methods are extremely valuation-theoretic. If we can manage $\alpha_L = 0$ and $\mathrm{ind}(\alpha) = n$ is prime, we have cyclicity, by a theorem of Albert.

It is possible to see why the index should divide the square of the period; in light of the triviality of $\mathrm{H}^2_{\mathrm{nr}}(K)$, it is traceable to the dimension of $X$ (see also the excellent summary of [S1] in the review [C-T2], which includes additional citations and context). The existence of a radical splitting field in prime degree is not apparent, even locally, and the proof is quite technical.

**Examples.** The following two examples illustrate the approach. Write

$$K^* \longrightarrow \mathrm{H}^1(K, \mu_n) \quad \text{via} \quad a \longmapsto (a)$$

for the Kummer homomorphism, suppressing the $n$ because it is always implicit. We write $a \cdot b$ for the cup product of cohomology classes $a$ and $b$, but we write

$$(a,b) \overset{\mathrm{df}}{=} a \cdot (b) \in \mathrm{H}^p(K, \mu_n^{\otimes(p-1)})$$

when $a \in \mathrm{H}^{p-1}(K, \mu_n^{\otimes(p-2)})$ and $b \in K^*$, especially when $a \in \mathrm{H}^1(K, \mathbb{Z}/n)$.

Suppose $\alpha = (\theta, \pi)$ for $\theta \in \mathrm{H}^1(K, \mathbb{Z}/n)$ and $\pi \in K^*$, and let $L = K(\pi^{1/n})$. Since $(\pi)_L = 0$, we compute $\alpha_L = (\theta, \pi)_L = 0$, so $L$ splits $\alpha$. More difficult: Suppose $\alpha = (\theta_0, \pi_0) + (\theta_1, \pi_1)$, where $\operatorname{div} \pi_0$ and $\operatorname{div} \pi_1$ are prime divisors on $X$, and $\theta_i \in \mathrm{H}^1_{\mathrm{nr}}(K, \mathbb{Z}/n)$. Let $L = K((\pi_0\pi_1)^{1/n})$. Since $(\pi_0\pi_1)_L = 0 = (\pi_0)_L + (\pi_1)_L$, $\alpha_L = (-\theta_0 + \theta_1, \pi_1)_L$. If $w \in V(L)$, then

$$\partial_w(\alpha_L) = w(\pi_1)(-\theta_0 + \theta_1)_{\kappa(w)}.$$

If $w$'s center on $X$ has codimension one, then it is easy to show $w(\pi_1) = 0 \pmod n$. However, if the center is a closed point, in general $w(\pi_1) \neq 0$, and since we cannot expect $(-\theta_0 + \theta_1)_{\kappa(w)}$ to be zero, the strategy fails in general. We try to salvage it as follows. Let $w_0$ be a discrete valuation on $L$ extending the valuation $v_0$ on $K$ defined by $\pi_0$, and let $\alpha(w_0) = \alpha_{K_{v_0}(\theta_0)}$ be the value of $\alpha$ at $w_0$. Then $\alpha(w_0)$ is defined over $\kappa(v_0)$, and we can prove

$$\partial_w(\alpha_L) = w(\pi_1)\,\partial_{v_0,\bar{v}_1}(\alpha(w_0))$$

where $\bar{v}_1$ is the valuation induced by the image of $\pi_1$ in $\kappa(v_0)$. If we could zero out $\alpha(w_0)$, we could solve the problem for all $w$. This strategy turns out to be viable. If $\alpha$ has prime degree, then $\alpha(w_0)$ is cyclic, and the parameter part of $\alpha(w_0)$ varies with $f = \pi_0\pi_1$. Finding an "anti-parameter" $u \in K$, we zero out $\alpha(w_0)$ by replacing $f$ with $uf$. But now, $\operatorname{div} uf$ may no longer match $D_\alpha$ and if $(\operatorname{div} u) \cap D_\alpha$ is not empty, there is a problem.

In general, there are other, more serious problems. $D_\alpha$ may not be principal in the first place, forcing us to contend with the extra "rogue" components in any $\operatorname{div} f$. If $D_\alpha$ has more than one nodal point, there is a globalization problem. In the end we produce $f$ by a type of successive approximation, and at each stage we must check that the earlier stages have not come undone.

## 1 Discrete Valuations

Let $X$ be an integral noetherian scheme, and let $K$ be a field containing the function field $F(X)$ of $X$. If $v$ is a discrete valuation on $K$, let $\mathrm{O}_v \subset K$ denote the corresponding discrete valuation ring, $\mathfrak{p}_v \subset \mathrm{O}_v$ the prime ideal, and $\kappa(v)$ the residue field. $\operatorname{Spec} K$ is an open subset of $\operatorname{Spec} \mathrm{O}_v$, so $\operatorname{Spec} K \to X$ induces a rational map $\operatorname{Spec} \mathrm{O}_v \dashrightarrow X$. We say $v$ has a *center on* $X$ if this is a morphism, and then denote by $x_v$ the image of the closed point $\operatorname{Spec} \kappa(v)$. Thus, $v$ has a center on $X$ if there is a point $x_v \in X$ such that $\mathrm{O}_{X,x_v} \subset \mathrm{O}_v$ and $\mathfrak{m}_{x_v} = \mathfrak{p}_v \cap \mathrm{O}_{X,x_v}$ (see [Liu, Def. 8.3.17]). We then say $v$ is an *$X$-valuation of $K$*. Denote the set of all normalized $X$-valuations of $K$ by

$$V_X(K).$$

If $S$ is a fixed base, or by default if $S = \operatorname{Spec} \mathbb{Z}$, write $V(K)$ instead of $V_S(K)$. Similarly, we write $V_x(K)$ for those centered on $x$, and $V(X)$ for the discrete valuations on $F(X)$ that have codimension-one center on $X$. Substitute $\kappa(x)$ and $A$ for $x$ and $X$ when $x = \operatorname{Spec} \kappa(x)$ and $X = \operatorname{Spec} A$. If $x_v$ belongs to the set $X^{(1)}$ of codimension-one points on $X$, we have a prime divisor

$$D_v \stackrel{\mathrm{df}}{=} \overline{\{x_v\}}.$$

**Remark 1.1.** If $v \in V_x(K)$, then $\kappa(x) \subset \kappa(v)$, and $v \in V_{\mathrm{O}_{X,z}}(K)$ for all $z \in \overline{\{x\}}$. If $\phi : Y \to X$ is a morphism of integral schemes, then $V_y(F(Y)) \subset V_{\phi(y)}(F(Y))$ for all $y \in Y$, and conversely, if $\phi$ is proper then $V_Y(F(Y)) = V_X(F(Y))$.

**Lemma 1.2.** *Suppose $A = (A, \mathfrak{m}, k)$ is a two-dimensional regular local domain with fraction field $K$, and $\mathfrak{m} = (\pi_0, \pi_1)$. Suppose $f \in A$, $L = K(f^{1/n})$, and $v \in V(K)$*

*extends to* $w \in V(L)$*. Then* $w|_K = e(w/v)\,v$*, and* $e(w/v) = n/\gcd(v(f),n)$*. In particular, if* $v(f) \in (\mathbb{Z}/n)^*$*, then* $v$ *is totally ramified in* $L$*.*

*Proof.* Since $\mathfrak{p}_v \cap A = \mathfrak{m}$, $k \subset \kappa(v)$, both $v$ and $w$ are normalized; so $e(w/v) = [\mathbb{Z} : w(K^*)]$. After replacing $f$ by a prime-to-$n$ power of $f$ if necessary, we may assume $v(f) = m$ divides $n$. At the completions

$$w = \frac{e(w/v)}{[L_w : K_v]} v \circ N_{L_w/K_v}$$

so $w|_K = e(w/v)\,v$. Over $K_v$, we write $f = \pi_v^m u$, where $v(u) = 0$. The polynomial $T^m - u$ is separable over $\kappa(v)$, and $K_v(u^{1/m})$ is unramified by Hensel's lemma. Then $T^{n/m} - u^{1/m}\pi_v$ is an Eisenstein polynomial over $K_v(u^{1/m})$, and since $L_w = K_v(u^{1/m}, (u^{1/m}\pi_v)^{n/m})$, we conclude $e(w/v) = n/m$. □

## 2 Residue Map

The material in this section is standard; see [C-T, GMS] or [GS] for additional background. Let $X$ be a scheme, and let $n$ be invertible on $X$, meaning that $n$ is prime to the residue characteristic of $\kappa(x)$ for all $x \in X$. We write

$$\mathrm{H}^q(X) \stackrel{\mathrm{df}}{=} \mathrm{H}^q(X_{\text{ét}}, \mu_n^{\otimes(q-1)})$$

If $f : Y \to X$ is a map of schemes, then $f^*\mathbb{Z}/n = \mathbb{Z}/n$ and we have a restriction map

$$\mathrm{res}_{X|Y} : \mathrm{H}^q(X) \to \mathrm{H}^q(Y).$$

If $\alpha \in \mathrm{H}^q(X)$, we will sometimes write $\alpha_Y$ instead of $\mathrm{res}_{X|Y}(\alpha)$, and if $X = \operatorname{Spec} A$ and/or $Y = \operatorname{Spec} B$, we write $\mathrm{H}^q(A)$ and $\alpha_B$.

Let $K$ be a field, and let $v \in V(K)$. Suppose $n$ is prime to $\operatorname{char}\kappa(v)$. By [C-T, 3.10, p. 26], for any $r \in \mathbb{Z}$ and $q \geq 1$ we have an exact sequence

$$0 \to \mathrm{H}^q(\mathrm{O}_v) \to \mathrm{H}^q(K) \xrightarrow{\partial_v} \mathrm{H}^{q-1}(\kappa(v)) \to 0 \tag{2.1}$$

such that if $\pi_v \in \mathrm{O}_v$ is a uniformizer and $\theta \in \mathrm{H}^{q-1}(\mathrm{O}_v)$, then

$$\partial_v(\theta \cdot (\pi_v)) = \theta_{\kappa(v)}. \tag{2.2}$$

The map $\partial_v$ is called the *residue map*, the images are called *residues*, and $\alpha \in \mathrm{H}^q(K)$ is said to be *unramified at* $v$ if $\partial_v(\alpha) = 0$.

**Definition 2.3.** If $\alpha \in \mathrm{H}^q(K)$ and $T = X$, $A$, etc., the *ramification locus of* $\alpha$ *with respect to* $T$ is the set

$$\mathrm{div}_T(\alpha) \stackrel{\mathrm{df}}{=} \{v \in V(T) : \partial_v(\alpha) \neq 0\}.$$

The *ramification locus* or *ramification divisor for* $\alpha$ *on* $X$ is

$$D_\alpha \overset{\text{df}}{=} \Big\{ \sum_v D_v : v \in \operatorname{div}_X(\alpha) \Big\}.$$

**Diagrams 2.4.** If $L/K$ is a finite separable extension and $w \in V(L)$, then for $v = w|_K$ we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}^q(\mathrm{O}_w) & \longrightarrow & \mathrm{H}^q(L) & \xrightarrow{\partial_w} & \mathrm{H}^{q-1}(\kappa(w)) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow{\scriptstyle e(w/v)\,\mathrm{res}_{\kappa(v)|\kappa(w)}} & & \\ 0 & \longrightarrow & \mathrm{H}^q(\mathrm{O}_v) & \longrightarrow & \mathrm{H}^q(K) & \xrightarrow{\partial_v} & \mathrm{H}^{q-1}(\kappa(v)) & \longrightarrow & 0. \end{array}$$

Let $K_v$ denote the completion of $K$ with respect to $v$, and let $\widehat{\mathrm{O}}_v$ be the valuation ring of $v$ on $K_v$. Then the inflation map $\mathrm{H}^q(\kappa(v)) \to \mathrm{H}^q(\widehat{\mathrm{O}}_v)$ is an isomorphism, and we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}^q(\kappa(v)) & \longrightarrow & \mathrm{H}^q(K_v) & \xrightarrow{\partial_v} & \mathrm{H}^{q-1}(\kappa(v)) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \| & & \\ 0 & \longrightarrow & \mathrm{H}^q(\mathrm{O}_v) & \longrightarrow & \mathrm{H}^q(K) & \xrightarrow{\partial_v} & \mathrm{H}^{q-1}(\kappa(v)) & \longrightarrow & 0 \end{array}$$

If $\pi_v \in K_v$ is a uniformizer, then any $\alpha \in \mathrm{H}^q(K_v)$ may be expressed as

$$\alpha = \alpha^\circ + (\theta_v, \pi_v)$$

where $\alpha^\circ \in \mathrm{H}^q(\kappa(v))$ and $\theta_v = \partial_v(\alpha) \in \mathrm{H}^{q-1}(\kappa(v))$ are defined over $\widehat{\mathrm{O}}_v$ via inflation. The element $\alpha^\circ$, of course, depends on the choice of $\pi_v$. If $\alpha \in \mathrm{H}^p(K_v)$ and $\beta \in \mathrm{H}^q(K_v)$, then by (2.2) we have the formula

$$\partial_v(\alpha \cdot \beta) = \alpha \cdot \partial_v(\beta) + (-1)^q \partial_v(\alpha) \cdot \beta + \partial_v(\alpha) \cdot \partial_v(\beta) \cdot (-1) \tag{2.5}$$

interpreted over $K_v$.

**Value of a class 2.6.** Suppose $\alpha \in \mathrm{H}^q(K)$, $L/K$ is a finite separable extension, $w \in V(L)$, and $\theta_w = \partial_w(\alpha_L)$. Define the *value of* $\alpha$ *at* $w$ to be

$$\alpha(w) = \begin{cases} \alpha_{L_w} & \text{if } \theta_w = 0 \\ \alpha_{L_w(\theta_w)} & \text{if } q = 2 \end{cases}$$

where $L_w(\theta_w)/L_w$ is the cyclic extension defined by the inflation of the character $\theta_w$ from $\kappa(w)$ to $L_w$. Note in any case $\alpha(w)$ is defined over $\kappa(w)(\theta_w)$.

# 3 Surfaces

We state some well-known facts and definitions. See [Liu, Lbm, Si] for additional background and proofs.

**Definition 3.1.** Let $S$ be a Dedekind scheme.

1. A *projective curve* over a field $k$ is a subscheme of $\mathbb{P}^m_k$ for some $m \geq 0$, whose irreducible components are one-dimensional.
2. A *projective flat S-curve* $X$ is a 2-dimensional, integral, projective, flat $S$-scheme.
3. An *arithmetic surface* $X \to S$ is a regular projective flat $S$-curve.

**Remarks 3.2.** If $X \to S$ is an arithmetic surface, then for each closed point $z \in X^{(2)}$, $A = \mathrm{O}_{X,z}$ is a two-dimensional regular noetherian local domain, which is factorial by Auslander-Buchbaum's theorem ([Mat, 20.3]). If $S$ is excellent, then $X$ is excellent, and $A$ is excellent. Since $X \to S$ is proper, $V(K) = V_X(K)$ by the valuative criterion for properness.

If $X$ is a normal projective flat $S$-curve, and $S$ is a one-dimensional Dedekind scheme, then the generic fiber $X_K$ is a nonsingular integral curve over $K$, and each closed fiber $X_s$, $s \in S$, is a projective curve over $\kappa(s)$ ([Liu, 8.3.3]). If $X \to S$ is arithmetic, then each $X_s \to \kappa(s)$ is a local complete intersection ([Liu, 8.3.6]), hence $X_s$ has no embedded points. $X_s$ is generally not reduced. Every effective irreducible divisor $D \subset X$ is either a component of some $X_s$ ($D$ is *vertical*), or the closure of a closed point $x \in X_K$ of the generic fiber ($D$ is *horizontal*).

**Intersections on an arithmetic surface 3.3.** See [Liu,Lbm]. Let $X$ be an arithmetic surface. Since $X$ is normal, every prime (Weil) divisor defines a discrete valuation $v \in V(X)$, via the local ring of its generic point. If $f \in K$, then

$$\operatorname{div} f = \sum_{V(X)} v(f) D_v,$$

the sum being finite since $X$ is noetherian. If $D$ and $E$ are two effective divisors with no common irreducible components, the intersection multiplicity $(D \cdot E)_z$ at a closed point $z$ is defined to be

$$(D \cdot E)_z \overset{\text{df}}{=} \operatorname{length}_{\mathrm{O}_{X,z}} \mathrm{O}_{X,z}/(I_{D,z} + I_{E,z})$$

where $I_D$ and $I_E$ are the ideal sheaves, and the subscript $z$ means “localized at $z$”. If $D$ is prime and $g \in K$ is a local equation for $E$ at $z$, then $(D \cdot E)_z = v_{\bar{z}}(g)$, where $v_{\bar{z}}$ is the valuation on $D$ induced by $z$.

**Definition 3.4 (Normal crossings).** An effective divisor $D$ on a regular noetherian scheme $X$ has *normal crossings at* $z \in X$ if $X$ is regular at $z$, and for some system of parameters $f_1, \dots, f_n$ for $\mathrm{O}_{X,z}$, there is an integer $0 \leq m \leq n$ and integers $r_1, \dots,$ $r_m \geq 1$, such that $I_{D,z}$ is generated by $f_1^{r_1} \cdots f_m^{r_m}$. We say $D$ has *normal crossings on*

$X$ if it has normal crossings at every point. If $D$ has normal crossings, we will also say that the set $\{v_i\}$ of discrete valuations centered on the generic points of $D$ has normal crossings.

This definition, from [Liu, 9.1.6], is different from the relative version in [SGA1, XIII, 2.1.0]. We obtain that version by replacing "normal crossings" by "strictly normal crossings" and "smooth over" by "regular". A normal crossings divisor need not be reduced, but its irreducible components are nonsingular, and meet each other transversally, meaning their local equations form part of a system of parameters at each intersection point.

**Existence of a Model 3.5.** *Let $S$ be an excellent one-dimensional affine Dedekind scheme, with function field $F$, and let $K$ be a field extension of $F$ of transcendence degree one. Then there exists a normal connected projective curve $X_F$ with function field $K$, and a normal projective flat $S$-curve $X$ with generic fiber $X_F$. If $F$ is perfect, $X_F$ is smooth over $F$.*

In fact, there is a category equivalence between normal connected projective curves over $F$ and function fields of transcendence degree one over $F$, where the morphisms between curves are dominant morphisms ([Liu, 7.3.13]). For curves, normal and regular are the same, and if $F$ is perfect, regular and smooth are the same ([Liu, 4.3.33]), so $X_F$ is smooth if $F$ is perfect. The rest is stated in [Liu, 10.1.6].

**Strong Desingularization 3.6.** *Let $X$ be a two-dimensional, excellent, reduced, noetherian scheme. Then $X$ admits a desingularization in the strong sense, i.e., a proper birational morphism $f : X' \to X$ with $X'$ regular, and $f$ an isomorphism above every regular point of $X$. In particular, this holds for $X$ a projective flat $S$-curve over an excellent Dedekind scheme. If $S$ is affine, $X' \to S$ is then an arithmetic surface.*

This is a theorem of Lipman ([Liu, 8.3.44]). To prove the last statement, note that since $f$ is birational, $X'$ is integral, and $X \to S$ is flat, the induced map $X' \to S$ is flat, since $X'$ dominates $S$ ([Liu, 4.3.9]). Since $S$ is affine, $f$ is projective by [Liu, 8.3.50], and since the composition of projective morphisms is projective, $X' \to S$ is an arithmetic surface.

**Embedded Resolution 3.7.** *Let $X \to S$ be an arithmetic surface over an excellent Dedekind scheme, and let $D$ be a divisor on $X$. Then there exists a projective birational morphism $f : X' \to X$ with $X'$ an arithmetic surface, such that $f^*D$ has normal crossings.*

This is [Liu, 9.2.26]. We will need the following lemma from the proof.

**Blowup Lemma 3.8.** *Suppose $X \to S$ and $D$ are as in the statement of Embedded Resolution (3.7), $f : X' \to X$ blows up a closed point $z \in X$, and $D$ has normal crossings at $z$. Then $X' \to S$ is an arithmetic surface, and $f^*D$ has normal crossings at every point of the exceptional fiber $E$. Moreover, $E$ meets the irreducible components of the strict transform $\widetilde{D}$ in at most two points, and these points are rational over $\kappa(z)$.*

One of Saltman's basic observations is that the ramification locus of a Brauer class on an arithmetic surface can be conditioned to have normal crossings.

**Existence of Surfaces 3.9.** *Let $S$ be an excellent one-dimensional affine Dedekind scheme, and let $s \in S$ be a closed point. Suppose $F = F(S)$, and $K$ is a field of transcendence degree one over $F$. Then for any $\alpha \in \mathrm{H}^2(K)$, there exists an arithmetic surface $X \to S$ with function field $K = F(X)$ and fiber $X_s$, such that the union $D_\alpha \cup X_s$ has normal crossings on $X$.*

*Proof.* By Existence of a Model 3.5 there exists a normal projective flat $S$-curve $X'' \to S$ with function field $K$. Therefore by Strong Desingularization 3.6, there exists an arithmetic surface $X' \to S$ with function field $K$. Suppose $\alpha \in \mathrm{H}^2(K)$. Then by Embedded Resolution (3.7), there exists an arithmetic surface $X \to S$ and a projective birational morphism $f : X \to X'$ such that $f^*(D'_\alpha \cup X'_s)$ has normal crossings, where $D'_\alpha$ is the divisor of $\alpha$ on $X'$. Since the irreducible components of $f^*(D'_\alpha \cup X'_s)$ equal those of $D_\alpha \cup X_s$, this proves the result. □

## 4 Unramified Brauer Group and Arithmetic Surfaces

By definition, if $K$ is a field, $S = \operatorname{Spec} R$ is a base, and $V(K)$ is the set of discrete $S$-valuations on $K$,

$$\mathrm{H}^2_{\mathrm{nr}}(K/R) \overset{\mathrm{df}}{=} \bigcap_{V(K)} \mathrm{H}^2(\mathrm{O}_v).$$

We write $\mathrm{H}^2_{\mathrm{nr}}(K)$ if the sum is over all discrete valuations on $K$.

**Purity for curves 4.1.** Let $C$ be a smooth integral projective curve over a field $k$, with function field $F$. Then

$$\mathrm{H}^2(C) = \mathrm{H}^2_{\mathrm{nr}}(F/k).$$

This is proved in the affine case for the Brauer group in [Ho, Th.], and extended using the fact that Br is a Zariski sheaf, as in [S1, Th. 1.4].

Let $X$ be a two-dimensional regular noetherian scheme. Then by [GB, II, 2.7], $\mathrm{Br}(X) = \mathrm{H}^2(X, \mathrm{G}_m)$. Let

$$\bar{\mathrm{H}}^2(X) \overset{\mathrm{df}}{=} {}_n\mathrm{H}^2(X, \mathrm{G}_m)$$

denote the $n$-torsion subgroup, where $n$ is as in Sect. 2. For any regular quasi-compact integral scheme $X$ with function field $K$, the natural map $\mathrm{H}^2(X, \mathrm{G}_m) \to \mathrm{H}^2(K, \mathrm{G}_m)$ is injective [M, III.2.22]. Thus we may view $\bar{\mathrm{H}}^2(X)$ as the image of $\mathrm{H}^2(X)$ in $\mathrm{H}^2(K)$. Since $\mathrm{H}^2(X) \to \mathrm{H}^2(K)$ factors through $\mathrm{H}^2(\mathrm{O}_v)$ for any $v \in V_X(K)$, we have

$$\bar{\mathrm{H}}^2(X) \subset \bigcap_{V_X(K)} \mathrm{H}^2(\mathrm{O}_v).$$

Using Auslander–Goldman's theorem [AG, Prop. 7.4] and the fact that Br is a Zariski sheaf, it is not hard to prove the following ([GB, II, Prop. 2.3], or [S1, Th. 1.4]).

**Purity for Surfaces 4.2.** *If $X$ is a regular integral noetherian surface, then*

$$\bar{\mathrm{H}}^2(X) = \bigcap_{V(X)} \mathrm{H}^2(\mathrm{O}_v).$$

*If $X$ is projective over $S = \operatorname{Spec} R$, then $\bar{\mathrm{H}}^2(X) = \mathrm{H}^2_{\mathrm{nr}}(K/R)$.*

We will use purity for surfaces in two ways, first to show that when $S = \operatorname{Spec} \mathbb{Z}_p$, to split $\alpha \in \mathrm{H}^2(K)$ it is sufficient to construct a finite separable extension $L/K$ over which $\partial_w(\alpha) = 0$ for all $w \in V(L)$ (Theorem 4.5); and second, to prove a local structure theorem for $\mathrm{H}^2(K)$, which provides the computational foothold we need in order to construct such an $L$ locally.

**Lemma 4.3.** *Let $S$ be a henselian local scheme, let $Y \to S$ be a proper $S$-scheme, and let $Y_0 \to S_0$ be the closed fiber. Then for all $q \geq 0$,*

$$\mathrm{H}^q(Y) = \mathrm{H}^q(Y_0).$$

This is a corollary to the proper base change theorem ([GB, III, 3] or [M, VI.2.7]).

**Lemma 4.4.** *Let $Y_0$ be a normal crossings divisor on an arithmetic surface $Y$, and let $\{D_i\}$ denote $Y_0$'s irreducible components. Then the natural map $\mathrm{H}^2(Y_0) \to \bigoplus_i \mathrm{H}^2(D_i)$ is injective.*

This is [S1, Lemma 3.2], and it is also in [GB, III]. The following fundamental result is [S2, Th. 0.9].

**Theorem 4.5.** *If $L$ is the function field of a $p$-adic curve, and $n$ is prime-to-$p$, then $\mathrm{H}^2_{\mathrm{nr}}(L) = 0$. Thus if $\alpha_L \in \mathrm{H}^2(L)$, and $\partial_w(\alpha_L) = 0$ for all $w \in V(L)$, then $\alpha_L = 0$.*

*Proof.* Let $S = \operatorname{Spec} \mathbb{Z}_p$. By Existence of Surfaces (3.9) there is an arithmetic surface $Y \to S$ with function field $L$, such that the closed fiber $Y_0$ has normal crossings. Every discrete valuation on $L$ has a center on $S$, so $\mathrm{H}^2_{\mathrm{nr}}(L/\mathbb{Z}_p) = \mathrm{H}^2_{\mathrm{nr}}(L)$. We then have $\mathrm{H}^2_{\mathrm{nr}}(L) = \bar{\mathrm{H}}^2(Y)$ by Purity for Surfaces (4.2), and $\mathrm{H}^2(Y) = \mathrm{H}^2(Y_0)$ by Lemma 4.3. Therefore since $\mathrm{H}^2(Y) \to \bar{\mathrm{H}}^2(Y)$ is surjective, it suffices to show $\mathrm{H}^2(Y_0) = 0$. The second statement then follows immediately.

Let $k = \mathbb{F}_p$. $Y_0$ is projective over $k$ by [Liu, 8.3.6(a)], and since $Y_0$ has normal crossings, the irreducible components $C$ of $Y_0$ are regular and projective over $k$, and $\mathrm{H}^2(Y_0) \subset \bigoplus \mathrm{H}^2(C)$ by Lemma 4.4. Let $F = k(C)$. Since $k$ is finite, $V_k(F) = V(F)$ is the set of all normalized discrete valuations on $F$, and $C$ is smooth over $k$ by [Liu, 8.3.33]. Therefore, by Purity for Curves (4.1),

$$\mathrm{H}^2(C) = \bigcap_{V(F)} \mathrm{H}^2(\mathrm{O}_v)$$

On the other hand, by Class Field Theory we have an exact sequence

$$0 \to \mathrm{H}^2(F) \to \bigoplus_{V(F)} \mathrm{H}^1(\kappa(v)) \to \mathrm{H}^1(k) \to 0$$

(see [M, III.2.22(g)] and [GB, III, Remark 2.5b]). We conclude $\mathrm{H}^2(C) = 0$ by (2.1), hence $\mathrm{H}^2(Y_0) = 0$, as desired. □

## 5 Modified Picard Group

We briefly summarize [S2, Sect. 1], which shows how, on an arithmetic surface over a complete discrete valuation ring, we can represent a divisor class whose restriction to the closed fiber is an $n$-th power by a divisor that is itself an $n$-th power, while avoiding a fixed finite set of closed points. The results of this section will not be applied until the very end.

Let $X$ be a projective scheme with no embedded points over an affine scheme. This hypothesis applies to an arithmetic surface over an affine Dedekind scheme, or one of its closed fibers. Let $\underline{z} = \{z_i\}_I \subset X$ be a finite set of closed points, and let $\iota : \underline{z} \to X$ be the closed immersion so that $\iota_* \mathrm{O}^*_{\underline{z}} = \bigoplus_I \iota_{i*} \kappa(z_i)^*$. Let ${}_{\underline{z}}\mathrm{O}^*_X$ denote the sheaf of units with value 1 at each $z_i$, defined by the exact sequence

$$1 \longrightarrow {}_{\underline{z}}\mathrm{O}^*_X \longrightarrow \mathrm{O}^*_X \longrightarrow \iota_* \mathrm{O}^*_{\underline{z}} \longrightarrow 1.$$

Let $K_X$ denote the (quasi-coherent) sheaf of total fractions of $X$. We have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & {}_{\underline{z}}\mathrm{O}^*_X & \longrightarrow & K^*_X & \longrightarrow & K^*_X/{}_{\underline{z}}\mathrm{O}^*_X & \longrightarrow & 1 \\ & & \downarrow & & \| & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{O}^*_X & \longrightarrow & K^*_X & \longrightarrow & K^*_X/\mathrm{O}^*_X & \longrightarrow & 1 \end{array} \tag{5.1}$$

Set

$$\mathrm{Div}\, X = \mathrm{H}^0(X, K^*_X/\mathrm{O}^*_X) \qquad \mathrm{Pic}\, X = \mathrm{H}^1(X, \mathrm{O}^*_X)$$
$${}_{\underline{z}}\mathrm{Div}\, X = \mathrm{H}^0(X, K^*_X/{}_{\underline{z}}\mathrm{O}^*_X) \qquad {}_{\underline{z}}\mathrm{Pic}\, X = \mathrm{H}^1(X, {}_{\underline{z}}\mathrm{O}^*_X)$$

${}_{\underline{z}}\mathrm{Div}\, X = \mathrm{H}^0(X, K^*_X/{}_{\underline{z}}\mathrm{O}^*_X)$ is the set of $\{(U_j, f_j)\}$ such that not only is $f_j/f_k$ a unit on $U_j \cap U_k$, but $(f_j/f_k)(z) = 1$ for each $z \in \underline{z} \cap (U_j \cap U_k)$. By definition, we have an exact sequence

$$1 \longrightarrow \iota_* \mathrm{O}^*_{\underline{z}} \longrightarrow K^*_X/{}_{\underline{z}}\mathrm{O}^*_X \longrightarrow K^*_X/\mathrm{O}^*_X \longrightarrow 1. \tag{5.2}$$