Chyanbin Hwu

Anisotropic Elastic Plates

Springer

Anisotropic Elastic Plates

Chyanbin Hwu

Anisotropic Elastic Plates

Springer

Chyanbin Hwu
Department of Aeronautics and Astronautics
National Cheng Kung University
No. 1 University Road
Tainan 701
Taiwan R.O.C
chwu@mail.ncku.edu.tw

ISBN 978-1-4419-5914-0 e-ISBN 978-1-4419-5915-7
DOI 10.1007/978-1-4419-5915-7
Springer New York Dordrecht Heidelberg London

Library of Congress Control Number: 2010922035

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

Due to the nature of anisotropy, composite materials are usually modeled as anisotropic elastic solids. Therefore, the researchers and engineers interested in composite materials are usually advised to get acquainted with *anisotropic elasticity*. However, only three books related to anisotropic elasticity have been published in the literature. Two of them were written in Russian by Professor S.G. Lekhnitskii and were originally published in 1947 and 1950. Their English translations were published later in 1963 and 1968. These two books are the classical books of anisotropic elasticity and have great contributions for the follow-up research. In the present book, I arrange one chapter named *Lekhnitskii formalism* introducing the classical method presented in these two books. The third book about anisotropic elasticity was written by Professor T.C.T. Ting and was published in 1996. A great contribution of Ting's book is the presentation of another systematic approach – *Stroh formalism*. Due to its importance, the Stroh formalism together with its related discussions introduced in Ting's book is summarized in Chapter 3 of this book. Owing to the publication of Lekhnitskii's and Ting's books, during the last half century numerous new advances have been achieved. Therefore, I think it is a proper time to update this topic by publishing a new book entitled *Anisotropic Elastic Plates*.

As structural elements, anisotropic elastic plates find wide applications in modern technology. The *plates* here are considered to be subjected to not only in-plane loads but also transverse loads. In other words, both plane problem and plate bending problem as well as stretching–bending coupling problem are all treated in this book. In addition to the introduction of the theory of anisotropic elasticity, several important subjects have also been discussed in this book such as interfaces, cracks, holes, inclusions, contact problems, piezoelectric materials, thermal stresses, and boundary element analysis.

Most of the materials presented in this book can be found in the journal papers written by me and my co-workers, and some others are edited from the books and journal papers written by the other researchers. Even some notations have been unified in Lekhnitskii's and Ting's books, the notations used in the new advancements including my own works are still quite varied. Without a unified notation system, it is difficult for a beginner to study the subject. Therefore, in this book all the materials collected from the published results have been rewritten using a unified notation system and some useful Appendices are provided for the symbols, sign convention, formalisms, and problem solutions.

Elasticity and *mechanics of composite materials* are two important fundamental courses for senior undergraduate students and beginning graduate students in aerospace, civil, naval and mechanical engineering, applied mechanics, and engineering science. Several textbooks have been written for the studies of these two courses. I believe this book is helpful for engineers and scientists who want to have an advanced knowledge of the theory of elasticity and mechanics of composite materials. This book is appropriate to be a university textbook for the courses such as *anisotropic elasticity*, *advanced elasticity*, and *advanced mechanics of composite materials*. It is also a good reference book for the standard courses such as *elasticity*, *mechanics of composite materials*, *fracture mechanics*, *plates and shells*, and *boundary element method* and for the advanced courses such as *micromechanics*, *contact mechanics*, *smart materials and structures*, and *thermal elasticity*.

Special features:

1. **This book connects *anisotropic elasticity* and *mechanics of composite materials*.**
   This book provides a systematic complex variable approach covering both plane problem and plate bending problem as well as stretching–bending coupling problem. The advancement of the stretching–bending coupling problem started nearly 15 years ago and hence has never been introduced in any book related to anisotropic elasticity or mechanics of composite materials. Most of the books related to anisotropic elasticity discuss only plane problem, whereas the books related to mechanics of composite materials discuss mainly the plate bending problem. Thus, we need a systematic approach to connect these two related topics.
2. **This book connects *anisotropic elasticity* and *fracture mechanics*.**
   Most of the crack problems are discussed in the books entitled *Fracture Mechanics*. Not too many books related to elasticity have special chapters named *Cracks* or *Holes* or *Inclusions* or *Wedges & Interface Corners*. I believe the arrangement of these chapters is helpful for the readers to understand the connection between elasticity and fracture mechanics.
3. **This book connects *theoretical treatment* and *numerical analysis*.**
   Most of the books related to elasticity introduce mainly the theoretical treatment of elastic deformable solids and leave the numerical analysis to special books such as finite element method or boundary element method. To let the readers see more clearly about the connection between theoretical treatment and numerical analysis, we arrange a chapter named *boundary element analysis* in this book. The boundary elements introduced in this chapter involve both two-dimensional problems and stretching–bending coupling problems.
4. **Several special topics are discussed through one systematic approach**
   In addition to cracks, holes, inclusions, wedges & interface corners, the topics such as *contact problems*, *thermoelastic problems*, *piezoelectric materials*, and

holes/cracks/ inclusions in laminates, which are important in the engineering practice, are discussed separately in the specific chapters. Through these chapters the readers can understand how to apply the method introduced in this book to treat these special and interesting problems.

5. **Collection of problem solutions**
   In contrast to isotropic elastic materials that have only two elastic constants, anisotropic elastic materials may have as many as 21 elastic constants. Therefore, how to express the solutions for the problems of anisotropic elasticity in a simple and systematic way is really a big problem. Thus, even for one simple conventional problem, there may be several different kinds of mathematical expressions appeared in the literature. This also causes trouble for engineers to utilize the existing solutions, if they have difficulty in understanding the derivation details. *In this book, more than 100 problem solutions are collected in Appendix D*. To avoid the confusion caused by the symbols, *Appendix A* is provided in this book that describes the symbols, sign convention, and units. Moreover, to help the readers see clearly the unified expression used in this book, the summary of Stroh formalism is provided in *Appendix C*. Each problem collected in Appendix D is described with aid of a simple figure, and its solution is expressed in terms of the same symbol system. I believe through this collection most of the engineers and scientists can take advantage of these solutions freely and easily even they do not have enough time to understand their derivation details.

I wish to express my gratitude to my Ph.D. thesis adviser, Professor T.C.T. Ting. I am very fortunate to get into the field of anisotropic elasticity through his guidance. Several new advances of anisotropic elasticity have been achieved due to the publication of his book. Hope that the present book can also help the researchers go further. I also want to express my gratitude to my mentors, Professor C.S. Yeh of National Taiwan University and Professor W.H. Chen of National Tsing-Hua University for their guidance during my studies for B.S. and M.S. degrees. In particular, since part of this book was written during my sabbatical leave, I am grateful to Professors K. Kishimoto (Tokyo Institute of Technology), M. Omiya (Keio University), N. Miyazaki (Kyoto University), T. Ikeda (Kyoto University), Y.W. Mai (Sydney University), T. Aoki (Tokyo University), and T. Yokozeki (Tokyo University), who have helped me during my staying in their departments. Special thanks also to my assistant H.E. Shen, my former student Y.C. Liang, and my present students C.Z. Tan, T.L. Kuo, Y.C. Chen, and H.Y. Huang who helped me draw part of figures presented in this book. I would also like to thank my friends C.C. Ma, K.C. Wu, T.T. Wu of National Taiwan University, C.K. Chao of National Taiwan University of Science and Technology, and T. Chen of National Cheng Kung University for their helpful discussions during my research on anisotropic elasticity. I acknowledge the National Science Council of Taiwan for the support of my research in the area of anisotropic elasticity.

Finally, I would like to dedicate this book to my wife, Wenling, and my daughters, Frannie and Vevey, with thanks for their constant support and encouragement in everything.

Tainan, Taiwan
September, 2009

Chyanbin Hwu

Contents

Chapter 1
Linear Anisotropic Elastic Materials

The mechanical properties of materials are described by constitutive laws. There are a wide variety of materials existing in the world. We are not surprised that there are a great many constitutive laws describing an almost infinite variety of materials. What is surprising is that a simple idealized stress–strain relationship gives a good description of the mechanical properties of many elastic materials around us. In this chapter, we present the relation between stresses and strains in a linear anisotropic elastic material. By this relation, we need 21 elastic constants to describe a linear anisotropic elastic material if the materials do not possess any symmetry properties. In engineering application, this number is somewhat higher than expected. Consideration of the material symmetry will then reduce the number of elastic constants. To provide these constants obvious physical interpretation, engineering constants such as the Young's moduli, Poisson's ratios and shear moduli as well as some other behavior constants will also be introduced in this chapter. If the problems considered can be treated as a two-dimensional problem, the elastic constants needed for the analysis of the mechanical behavior of anisotropic materials can be further reduced.

1.1 Theory of Elasticity for Anisotropic Bodies

To study the behavior of an elastic continuous medium, the theory of elasticity is a generally accepted model. In this section we will briefly describe the general concept of elasticity for anisotropic bodies, such as state of stress, deformation, constitutive laws, and boundary conditions.

1.1.1 State of Stress

In the study of elasticity, the body is generally considered to be a deformable continuous medium. Usually the objectives of analysis are the determination of stresses and strains induced by the external loads. The state of stress at a given point of a continuous body, which is either in equilibrium or in motion as a result of external

C. Hwu, *Anisotropic Elastic Plates*, DOI 10.1007/978-1-4419-5915-7_1,

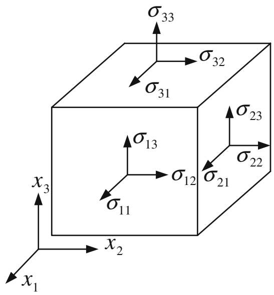

Fig. 1.1 Stress components

forces, is known to be represented by the stress components on three mutually perpendicular planes passing through the point. If the infinitesimal volume element enclosing the point is taken in the shape of a rectangular parallelepiped, with faces parallel to the coordinate planes, the stress components σ_{ij} are shown in Fig. 1.1. The first subscript of σ indicates the direction of normal to the plane, while the second indicates the direction of the stress component. The components σ_{11}, σ_{22}, and σ_{33} which are normal to the element surfaces are called *normal stresses*. The remaining components that are parallel to the surfaces are called *shear stresses*. The stress components are symmetric if the body moment is neglected. Knowing the stress components on the planes normal to the coordinate axes, we can always determine the stress vector $\mathbf{t}$ on any surface with unit outer normal vector $\mathbf{n}$, which passes through the given point. This stress vector is determined by *Cauchy's formula* as

$$t_i = \sigma_{ij} n_j, \tag{1.1}$$

in which repeated indices imply summation through 1–3. This rule of summation convention will be applied in the whole text unless stated otherwise. The derivation of symmetry property and Cauchy's formula can be found in any standard text of elasticity such as Sokolnikoff (1956). The stress components in the other coordinate systems are determined like a Cartesian tensor of rank 2, i.e.,

$$\sigma^*_{pq} = \Omega_{pi}\Omega_{qj}\sigma_{ij}, \tag{1.2}$$

where Ω_{pi} are the direction cosines between rotated (starred) and original (unstarred) axes.

There exists a particular set of coordinate axes with respect to which all the shear stress components are zero. These coordinate axes are called the *principal axes*, and the corresponding stress components are called the *principal stresses*. By this definition, the principal axes and principal stresses can be determined by solving the following eigenvalue problem:

$$(\sigma_{ij} - \sigma\delta_{ij})n_i = 0, \tag{1.3}$$

in which δ_{ij} is defined as $\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ when $i \neq j$, and is called the *Kronecker delta*.

The stress components in a continuous body which is in equilibrium must satisfy the *equilibrium equations*, which in Cartesian coordinates are

$$\sigma_{ij,j} + f_i = 0, \tag{1.4}$$

where f_i designate the body forces referred to a unit volume in directions x_1, x_2, and x_3, and a comma stands for differentiation. Equations of motion of a continuous medium differ from equilibrium equations only by having inertia terms $\rho \ddot{u}_i$ placed at the right hand side of (1.4) instead of zero, where ρ is the material density and u_i is the displacement in the x_i-direction and the double dot denotes twice differentiation with respect to time.

Note that all the relations written above for the stresses are designated to every point of a continuous solid, which is nothing to do with the material properties, structure types and sizes. Therefore, it should be valid for the studies of isotropic elasticity or anisotropic elasticity, micromechanics or macromechanics, etc.

1.1.2 Deformation

Forces applied to solids cause deformation. When the relative position of points in a continuous body is altered, the body is *strained*. The change in the relative position of points is a *deformation*. All material bodies are to some extent deformable. If there exists an ideal body which is nondeformable such that the distance between every pair of its points remains invariant throughout the history of the body, we call it a *rigid body*. The motion of a rigid body is usually described by translation and rotation. A deformable solid will experience an additional change in shape, i.e., deformation.

One of the major objectives of elasticity theory is the determination of the deformation of the solid from some reference configuration. There are two modes of description of the deformation of a continuous medium, the *Lagrangian* and *Eulerian*. The Lagrangian description employs the coordinates of a typical particle in the initial state as the independent variables, while in the case of Eulerian coordinates the independent variables are the coordinates of a material point in the deformed state. When the deformation is infinitesimal, these two viewpoints coalesce and no distinction between them will be made. Let the variable (a_1, a_2, a_3) refer to any particle in the original configuration of the body, and let (x_1, x_2, x_3) be the coordinates of that particle when the body is deformed. The deformation of the body is known if x_1, x_2, x_3 are known functions of a_1, a_2, a_3. The displacement vector $\mathbf{u}$ is defined by its components

$$u_i = x_i - a_i. \tag{1.5}$$

Since the displacements defined in (1.5) may include the rigid body motion and deformation in which the former induces no stress. Thus, the displacements themselves are not directly related to the stress. To relate deformation with stress, we must consider the stretching and distortion of the body, which is related to the change in distance between any two points of the body. For this purpose, the Lagrangian and Eulerian strain tensors are defined as

$$
\begin{aligned}
L_{ij} &= \frac{1}{2}\left(\frac{\partial u_i}{\partial a_j} + \frac{\partial u_j}{\partial a_i} + \frac{\partial u_k}{\partial a_i}\frac{\partial u_k}{\partial a_j}\right), \\
E_{ij} &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right).
\end{aligned} \tag{1.6}
$$

If the components of displacement are such that their first derivatives are so small that the squares and products of the partial derivatives of u_i can be neglected, both of the Lagrangian and Eulerian strain tensors reduce to *Cauchy's infinitesimal strain tensor*

$$
\varepsilon_{ij} = \frac{1}{2}(u_{i,\,j} + u_{j,\,i}). \tag{1.7}
$$

Like the stresses, the components of the strains that reflect the stretching or shortening of the body, i.e., ε_{11}, ε_{22} and ε_{33} are called *normal strains*. The remaining components related to the distortion of the body are called *shear strains*. Sometimes $\gamma_{ij} = 2\varepsilon_{ij},\ i \neq j$ are used to represent *engineering shear strains*.

Same as the stress components, according to the transformation law of the tensor of rank two the strains in the other coordinate systems can be calculated by

$$
\varepsilon^{*}_{pq} = \Omega_{pi}\Omega_{qj}\varepsilon_{ij}. \tag{1.8}
$$

The principal strains and the principal axes of strains can also be determined by solving the following eigenvalue problem:

$$
(\varepsilon_{ij} - \varepsilon\delta_{ij})n_i = 0. \tag{1.9}
$$

With reference to the definition given in (1.7), it is clear that a sufficiently well-behaved displacement field u_i will generate an equally well-behaved strain field by differentiation. The converse, however, is not necessarily true. It is not always possible to find a continuous, single-valued displacement field for any set of six well-behaved scalar functions ε_{ij} by integration of (1.7). For this reason we need to have the compatibility conditions for the strain fields to insure the existence of a single-valued, continuous displacement field for simply connected continuous body. The equations of *compatibility* obtained by St. Venant for infinitesimal strains are

$$
\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0. \tag{1.10}
$$

The system of equations (1.10) consists of $3^4 = 81$ equations, but some of these are identically satisfied, and some are repetitions. Considering the symmetry of the strains, there are only six strain components for three-dimensional problems. Since the six strains are defined in terms of three displacement functions, then only three independent compatibility equations within (1.10) are essential. In the case of two-dimensional problems, only three strain components and two displacement functions are necessary. Thus, only one independent compatibility equation should be satisfied for two-dimensional problems, which can be written as

$$2\varepsilon_{12,12} = \varepsilon_{11,22} + \varepsilon_{22,11}. \tag{1.11}$$

Note that all the relations written above for the strains are true for any continuous body, both elastic and inelastic, and are nothing to do with the material properties, structure types and sizes. Derivation of these formulas can be found in textbooks of the theory of elasticity such as (Sokolnikoff, 1956).

1.1.3 Constitutive Laws

In continuous media the state of stress is completely determined by the stress tensor σ_{ij}, and the state of deformation by the strain tensor ε_{ij}. If a material deforms as it is loaded and will return to its original dimensions during unloading, it is called an *elastic material*. In other words, an elastic material has a one-to-one analytical relation between the stresses and strains. If the materials obey a linear relationship between stresses and strains, which is usually called the *generalized Hooke's law*, they are linear elastic materials. When a linear elastic material is maintained at a fixed temperature and the stresses vanish when the strains are all zero, i.e., the initial unstrained state of the solid is unstressed, the generalized Hooke's law can be written as

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}, \tag{1.12}$$

where C_{ijkl} are *elastic constants* which characterize the elastic behavior of the solid. Since C_{ijkl} is a fourth rank tensor, there are 81 elastic constants. Consideration of the symmetry properties of stresses and strains as well as the elastic and symmetric characteristics of the materials will reduce the number of elastic constants, which will be discussed detailedly in the following sections.

1.1.4 Boundary Conditions

From the previous sections, we know that the basic equations for the anisotropic elasticity consist of equilibrium equations for the static loading conditions (1.4), strain–displacement relations for the small deformations (1.7) as well as the stress–strain laws for the linear anisotropic elastic solids (1.12). These three equation sets constitute 15 partial differential equations with 15 unknown functions, u_i, ε_{ij}, σ_{ij}, $i, j = 1, 2, 3$, in terms of three coordinate variables x_i, $i = 1, 2, 3$. All these basic equations and unknown functions are designated to every point of the elastic body. A general solution for these 15 unknown functions satisfying 15 basic equations has been derived by complex variable formulation (Ting, 1996). Since all the equations stated are designated to points of the elastic body without considering the structure type and size, their associated general solutions can be applied to the studies of micromechanics or macromechanics, etc. If the structures constructed by the anisotropic elastic body are clearly defined and their associated loading and boundary conditions are well described, the undetermined functions of the general solutions and hence the 15 unknown functions will then be uniquely determined through the satisfaction of the boundary conditions. In other words, the state of stress and deformation will be determined by taking into account the boundary conditions. Depending on what is given at the boundary, there are several distinct problems. Generally, they are separated into the following three types: the first fundamental problem, the second fundamental problem, and the third fundamental problem.

First fundamental problem is also called *traction-prescribed problem*. It is a problem that surface tractions and body forces are given on the elastic bodies. Surface tractions are force distributions which are applied to the surface of the solid, whereas body forces act on the internal matter of the solid. Examples of body forces are the action of gravity and magnetic attraction or repulsion. From (1.4) we see that the consideration of body forces will not induce too much difficulty if the solutions of the homogeneous parts have been found. Hence, in the whole text, the body forces will be neglected unless stated otherwise. The traction boundary conditions can usually be written as

$$\mathbf{t} = \hat{\mathbf{t}}, \text{ along the body surface,} \tag{1.13}$$

where $\mathbf{t}$ is the stress vector defined in (1.1), and the overhat denotes its prescribed value induced by the external forces.

Second fundamental problem is also called *displacement-prescribed problem*. It is a problem that displacements are prescribed on the body surface. The boundary conditions in this case are

$$\mathbf{u} = \hat{\mathbf{u}}, \text{ along the body surface.} \tag{1.14}$$

Third fundamental problem is also called *mixed boundary value problem*. In this case external forces are given on one part of the surface and displacements on another. This type of problem also includes problems where tangential forces and normal displacements are given on the surface, or normal forces and tangential displacements are given, etc. Mathematically, they can be written as

$$\begin{aligned} \mathbf{t} = \hat{\mathbf{t}}, &\text{ along part of the surface,} \\ \text{and } \mathbf{u} = \hat{\mathbf{u}}, &\text{ along the remaining part of the surface;} \end{aligned} \tag{1.15a}$$

or,

$$t_1 = \hat{t}_1,\ u_2 = \hat{u}_2,\ u_3 = \hat{u}_3,\ \text{or the other combinations.} \tag{1.15b}$$

1.2 Three-Dimensional Constitutive Relations

1.2.1 Generalized Hooke's Law

As stated in Section 1.1.3, if the materials obey a linear relationship between stresses and strains, the constitutive law may be expressed by using the generalized Hooke's law, i.e.,

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}. \tag{1.16}$$

Since σ_{ij} and ε_{ij} are tensors of order two, C_{ijkl} is a tensor of order four according to the quotient law. Consequently, the elastic constants transform according to the rule

$$C^*_{pqrs} = \Omega_{pi}\,\Omega_{qj}\,\Omega_{rk}\,\Omega_{sl}\,C_{ijkl}. \tag{1.17}$$

The elastic tensor C_{ijkl} may vary from point to point of the medium. If the C_{ijkl} are independent of the position of the point, the medium is called *elastically homogeneous*. In this book, most of our attention is confined to those media in which the C_{ijkl} do not vary throughout the region under consideration.

Since C_{ijkl} is a fourth rank tensor, there are $3^4 = 81$ elastic constants. Inasmuch as the stress components are symmetric, which is contingent upon the vanishing of the body moment, an interchange of the indices i and j in (1.16) does not alter these formulas, so that

$$C_{ijkl} = C_{jikl}. \tag{1.18}$$

Equation (1.18) reduces the number of independent elastic constants to $3 \times 3 \times 6 = 54$. Moreover, through the symmetry of the strain tensor, which can be observed either from the Lagrangian description or from the Eulerian description, a further reduction of the elastic constants may be made. That is, $\varepsilon_{ij} = \varepsilon_{ji}$, and therefore $\sigma_{ij} = C_{ijkl}\varepsilon_{kl} = C_{ijkl}\varepsilon_{lk} = C_{ijlk}\varepsilon_{kl}$, which may lead to $\varepsilon_{kl}(C_{ijkl} - C_{ijlk}) = 0$. Since this equality must hold for arbitrary values of ε_{kl}, it looks like we may conclude that

$$C_{ijkl} = C_{ijlk}. \tag{1.19}$$

However, ε_{kl} are not actually arbitrary, they are restricted by the symmetry conditions. For example, expansion of $\varepsilon_{kl}(C_{ijkl} - C_{ijlk}) = 0$ may lead to $\varepsilon_{12}(C_{ij12} - C_{ij21}) + \varepsilon_{21}(C_{ij21} - C_{ij12}) + \cdots = 0$. Given that $\varepsilon_{12} = \varepsilon_{21}$, etc., this equation will automatically be satisfied and no conclusion such as (1.19) can be made. Hence, the above proof generally seen in the textbook is not correct. The correct proof can be found in Sokolnikoff (1956), in which the elastic tensor is separated into two parts, i.e., $C_{ijkl} = C'_{ijkl} + C''_{ijkl}$ where $C'_{ijkl} = (C_{ijkl} + C_{ijlk})/2$ and $C''_{ijkl} = (C_{ijkl} - C_{ijlk})/2$. From the definition, it can easily be seen that C'_{ijkl} is symmetric and C''_{ijkl} is skew symmetric with respect to k and l. Thus, (1.16) may lead to $\sigma_{ij} = C'_{ijkl}\varepsilon_{kl}$ since the sum of $C''_{ijkl}\varepsilon_{kl}$ will be identical to zero due to the skew symmetry of C''_{ijkl} and symmetry of ε_{kl}. Hence, if we redefine the elastic constants by using C'_{ijkl}, the linear relationship still preserves and moreover the newly defined elastic constants will be symmetry with respect to k and l. Therefore, the symmetry property shown in (1.19) will be used in the theory of anisotropic elasticity. This symmetry relation now leads a further reduction of the number of the elastic constants to $6 \times 6 = 36$.

Additional restrictions are possible if we consider elastic materials only. The strain energy density of a material can be calculated by $W = \int_{\varepsilon_{ij}^0}^{\varepsilon_{ij}^f} \sigma_{ij} d\varepsilon_{ij}$. If a material is elastic, its strain energy should be independent of the loading and unloading path. Mathematically speaking, the integrand should be a total differential, i.e., $\sigma_{ij} d\varepsilon_{ij} = dW$. By using the stress–strain relation given in (1.16), we get $C_{ijkl}\varepsilon_{kl} d\varepsilon_{ij} = dW$. The elastic tensor is therefore related to the strain energy density through the differentiation, i.e., $C_{ijkl} = \partial^2 W/\partial\varepsilon_{kl}\partial\varepsilon_{ij}$. Since the differentiations with respect to ε_{ij} and ε_{kl} are interchangeable, we may get an additional symmetry conditions for the elastic tensor, i.e.,

$$C_{ijkl} = C_{klij}. \tag{1.20}$$

The symmetries (1.18), (1.19), and (1.20) result in $(6^2 - 6)/2 + 6 = 21$ independent elastic constants for the most general case of anisotropy.

Due to the symmetry properties, the number of independent elastic constants has been drastically decreased from 81 to 21. To avoid dealing with double sums, a *contracted notation* has been introduced as

$$\begin{aligned} &\sigma_{11} = \sigma_1,\ \sigma_{22} = \sigma_2,\ \sigma_{33} = \sigma_3,\ \sigma_{23} = \sigma_4,\ \sigma_{31} = \sigma_5,\ \sigma_{12} = \sigma_6, \\ &\varepsilon_{11} = \varepsilon_1,\ \varepsilon_{22} = \varepsilon_2,\ \varepsilon_{33} = \varepsilon_3,\ 2\varepsilon_{23} = \varepsilon_4,\ 2\varepsilon_{31} = \varepsilon_5,\ 2\varepsilon_{12} = \varepsilon_6. \end{aligned} \tag{1.21}$$

The generalized Hooke's law (1.16) and the symmetry conditions of the elastic tensor C_{ijkl} shown in (1.18), (1.19), and (1.20) can therefore be written as

$$\sigma_p = C_{pq}\varepsilon_q,\ C_{pq} = C_{qp},\ p, q = 1, 2, \ldots, 6, \tag{1.22a}$$

or, in matrix notation,

$$\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon},\ \mathbf{C} = \mathbf{C}^T. \tag{1.22b}$$

Note that σ_p, C_{pq}, ε_q are not tensor quantities and therefore cannot be transformed as tensors. C_{pq} is sometimes called *stiffness matrix*. The transformation between C_{ijkl} and C_{pq} is accomplished by the replacement of the subscript according to the following rules for ij (or kl) $\leftrightarrow$ p (or q):

$$11 \leftrightarrow 1,\ 22 \leftrightarrow 2,\ 33 \leftrightarrow 3,\ 23(\text{or } 32) \leftrightarrow 4,\ 31(\text{or } 13) \leftrightarrow 5,\ 12(\text{or } 21) \leftrightarrow 6. \tag{1.23}$$

The relations between stresses and strains written in (1.16) must be reversible, and we can write

$$\varepsilon_{ij} = S_{ijkl}\sigma_{kl}, \tag{1.24}$$

where S_{ijkl} are the *compliances* which are components of a fourth rank tensor. They also possess the full symmetry conditions like (1.18), (1.19), and (1.20), i.e.,

$$S_{ijkl} = S_{jikl},\ S_{ijkl} = S_{ijlk},\ S_{ijkl} = S_{klij}. \tag{1.25}$$

Similar to the contracted notation introduced for the elastic tensor C_{ijkl}, the compliance tensor S_{ijkl} can also be contracted according to the rules shown in (1.23) except suitable factors should be added as (Ting, 1996)

$$\begin{aligned} S_{ijkl} &= S_{pq}, \quad \text{if both } p,\ q \leq 3, \\ 2S_{ijkl} &= S_{pq}, \quad \text{if either } p \text{ or } q \leq 3, \\ 4S_{ijkl} &= S_{pq}, \quad \text{if both } p,\ q > 3. \end{aligned} \tag{1.26}$$

With (1.25) and (1.26), the stress–strain law (1.24) in contracted notation is

$$\varepsilon_p = S_{pq}\sigma_q,\quad S_{pq} = S_{qp}, \tag{1.27a}$$

or, in matrix notation,

$$\boldsymbol{\varepsilon} = \boldsymbol{S}\boldsymbol{\sigma},\ \boldsymbol{S} = \boldsymbol{S}^T. \tag{1.27b}$$

Substitution of (1.27b) into (1.22b) yields

$$\mathbf{C}\boldsymbol{S} = \mathbf{I} = \boldsymbol{S}\mathbf{C}, \tag{1.28}$$

where $\mathbf{I}$ is the 6×6 unit matrix. The relation (1.28) indicates that the stiffness matrix $\mathbf{C}$ and the compliance matrix $\boldsymbol{S}$ are the inverses of each other.

1.2.2 Material Symmetry

With the foregoing reduction from 81 to 21 independent constants, the stress–strain relations (1.22) are

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}, \tag{1.29}$$

which is the most general expression within the framework of linear elasticity. In engineering applications, σ_i and ε_i, $i = 4, 5, 6$ are usually replaced by the notation τ_{ij} and γ_{ij} to represent the *engineering shear stresses* and *strains*. Actually, the relations in (1.29) are referred to characterizing *anisotropic* materials since there are no planes of symmetry for the material properties. An alternative name for such an anisotropic material is a *triclinic* material. For most elastic solids, the number of independent elastic constants is far smaller than 21. The reduction is caused by the existence of material symmetry. If there is one plane of material symmetry such as the plane $x_3 = 0$, the stress–strain relations reduce to

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}. \tag{1.30}$$

Such a material is termed *monoclinic* or *aelotropic*, which has 13 independent elastic constants. To prove the relation (1.30), the symmetry with respect to the plane $x_3 = 0$ is expressed by the statement that the C_{ij} are invariant under the transformation

$$x_1 = x_1^*, \; x_2 = x_2^*, \; x_3 = -x_3^*. \tag{1.31}$$

The direction cosines Ω_{pi} of this transformation is

$$\mathbf{\Omega} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{1.32}$$

One may prove (1.30) by directly using the transformation given in (1.17) and letting $C^*_{ijkl} = C_{ijkl}$ and using the contraction notation defined in (1.23). If one is not familiar with the transformation of fourth rank tensors, (1.30) can be proved indirectly through the transformation of stresses and strains. Since the stresses and strains are tensors of rank 2, they can be transformed according to the following transformation:

$$\sigma^*_{pq} = \Omega_{pi}\Omega_{qj}\sigma_{ij}, \quad \varepsilon^*_{pq} = \Omega_{pi}\Omega_{qj}\varepsilon_{ij}. \tag{1.33}$$

From (1.32), (1.33) and the contracted notation defined in (1.23), it is seen that

$$\begin{aligned} &\sigma_i^* = \sigma_i, \quad \varepsilon_i^* = \varepsilon_i, \quad i = 1,\ 2,\ 3,\ 6, \\ &\sigma_4^* = -\sigma_4, \quad \varepsilon_4^* = -\varepsilon_4, \quad \sigma_5^* = -\sigma_5, \quad \varepsilon_5^* = -\varepsilon_5. \end{aligned} \tag{1.34}$$

Consider the transformed coordinate $(x_1^*,\ x_2^*,\ x_3^*)$, the first equation of (1.29) can be written as

$$\sigma_1^* = C_{11}^*\varepsilon_1^* + C_{12}^*\varepsilon_2^* + C_{13}^*\varepsilon_3^* + C_{14}^*\varepsilon_4^* + C_{15}^*\varepsilon_5^* + C_{16}^*\varepsilon_6^*. \tag{1.35}$$

Since the elastic constants are invariant under the transformation with respect to $x_3 = 0$, and the stresses and strains in the transformed coordinate are obtained in (1.34), (1.35) can now be rewritten as

$$\sigma_1 = C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + C_{13}\varepsilon_3 - C_{14}\varepsilon_4 - C_{15}\varepsilon_5 + C_{16}\varepsilon_6. \tag{1.36}$$

Comparison of this equation with the expression of σ_1 given by (1.29) shows that

$$C_{14} = C_{15} = 0. \tag{1.37}$$

Similarly, by considering $\sigma_2^*, \ldots, \sigma_6^*$, we can get the results shown in (1.30) for a material with the symmetry property with respect to the plane $x_3 = 0$.

If a material has two orthogonal planes of material symmetry, it can be proved that the symmetry will exist relative to a third mutually orthogonal plane (Ting, 1996). Such materials are said to be *orthotropic* (or *rhombic*). By a way similar to that shown for the monoclinic materials, the elastic constants for the orthotropic materials in coordinates aligned with principal material directions can be proved to be

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}, \tag{1.38}$$

which has nine independent elastic constants.

If at every point of a material there is one plane in which the mechanical properties are equal in all directions, then the material is termed *transversely isotropic*. If, for example, the $x_3 = 0$ plane is the special plane of isotropy, the elastic constants of this kind of materials can be proved to be

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & (C_{11} - C_{12})/2 \end{bmatrix}, \tag{1.39}$$

which have only five independent constants.

The greatest reduction in the number of elastic constants is obtained when the material is symmetric with respect to any plane and any axis, or say, the elastic properties are identical in all directions. Such materials are called *isotropic* materials. Although there are an infinite number of symmetry planes for isotropic materials, to determine the structure of the elastic constants C_{ij} no more than three symmetry planes are required (Ting, 1996). For example (Sokolnikoff, 1956), we may determine the elastic constants by considering the invariance about the new coordinate axes obtained by rotating the x_1, x_2, x_3-system through a right angle about the x_1-axis, then another new coordinate by a rotation of axes through a right angle about the x_3-axis, finally by a rotation of 45° about the x_3-axis. It can be proved that for any isotropic material, exactly two independent elastic constants characterize the material and the structure of the elastic constants C_{ij} is

$$\begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & (C_{11}-C_{12})/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (C_{11}-C_{12})/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (C_{11}-C_{12})/2 \end{bmatrix}. \tag{1.40}$$

By letting

$$C_{12} = \lambda \text{ and } (C_{11} - C_{12})/2 = \mu, \tag{1.41}$$

where λ and μ are the *Lame constants*, the generalized Hooke's law for an isotropic material can then be written in the following form:

$$\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij}, \ i, j = 1, 2, 3. \tag{1.42}$$

Restrictions on elastic constants are due to the positive definiteness of the strain energy which implies that the stiffness (or the compliances) matrices must be positive definite. The necessary and sufficient conditions for C_{ij} (or S_{ij}) to be positive definite are that the eigenvalues of C_{ij} (or S_{ij}) are positive, or alternatively all the leading principal minors of the stiffness (or compliance) matrix are positive. The obtained restrictions on elastic constants can then be used to examine experimental data to see if they are physically consistent within the framework of the mathematical elasticity model.

1.2.3 Engineering Constants

Engineering constants (also known as *technical constants*) are generalized Young's moduli, Poisson's ratios, and shear moduli as well as some other behavior constants. These constants are measured in simple tests such as uniaxial tension or pure shear tests. Thus, these constants with their obvious physical interpretation have more direct meaning than the components of the relatively abstract compliance and stiffness matrices discussed previously. Most simple tests are performed with a known load or stress. The resulting displacement or strain is then measured. Thus, the components of the compliance matrix are determined more directly than those of the stiffness matrix. For a general anisotropic material, the compliance matrix components in terms of the engineering constants are

$$\boldsymbol{S} = \begin{bmatrix} \frac{1}{E_1} & \frac{-\nu_{21}}{E_2} & \frac{-\nu_{31}}{E_3} & \frac{\eta_{1,23}}{G_{23}} & \frac{\eta_{1,31}}{G_{31}} & \frac{\eta_{1,12}}{G_{12}} \\ \frac{-\nu_{12}}{E_1} & \frac{1}{E_2} & \frac{-\nu_{32}}{E_3} & \frac{\eta_{2,23}}{G_{23}} & \frac{\eta_{2,31}}{G_{31}} & \frac{\eta_{2,12}}{G_{12}} \\ \frac{-\nu_{13}}{E_1} & \frac{-\nu_{23}}{E_2} & \frac{1}{E_3} & \frac{\eta_{3,23}}{G_{23}} & \frac{\eta_{3,31}}{G_{31}} & \frac{\eta_{3,12}}{G_{12}} \\ \frac{\eta_{23,1}}{E_1} & \frac{\eta_{23,2}}{E_2} & \frac{\eta_{23,3}}{E_3} & \frac{1}{G_{23}} & \frac{\mu_{23,31}}{G_{31}} & \frac{\mu_{23,12}}{G_{12}} \\ \frac{\eta_{31,1}}{E_1} & \frac{\eta_{31,2}}{E_2} & \frac{\eta_{31,3}}{E_3} & \frac{\mu_{31,23}}{G_{23}} & \frac{1}{G_{31}} & \frac{\mu_{31,12}}{G_{12}} \\ \frac{\eta_{12,1}}{E_1} & \frac{\eta_{12,2}}{E_2} & \frac{\eta_{12,3}}{E_3} & \frac{\mu_{12,23}}{G_{23}} & \frac{\mu_{12,31}}{G_{31}} & \frac{1}{G_{12}} \end{bmatrix}, \tag{1.43}$$

where $E_1,\ E_2,\ E_3$ are the *Young's moduli* in $x_1,\ x_2$, and x_3 directions, respectively; ν_{ij} is the *Poisson's ratio* for transverse strain in the x_j-direction when stressed in the x_i-direction, that is, $\nu_{ij} = -\varepsilon_j/\varepsilon_i$ for $\sigma_i = \sigma$ and all other stresses are zero; $G_{23},\ G_{31},\ G_{12}$ are the *shear moduli* in the x_2x_3, x_3x_1, and x_1x_2 planes, respectively; $\eta_{i,ij}$ is the *coefficient of mutual influence of the first kind* which characterizes stretching in the x_i-direction caused by shear in the x_ix_j-plane, that is, $\eta_{i,ij} = \varepsilon_i/\gamma_{ij}$ for $\tau_{ij} = \tau$ and all other stresses are zero; $\eta_{ij,i}$ is the *coefficient of mutual influence of the second kind* which characterizes shearing in the x_ix_j-plane caused by a normal stress in the x_i-direction, that is, $\eta_{ij,i} = \gamma_{ij}/\varepsilon_i$ for $\sigma_i = \sigma$ and all other stresses are zero; $\mu_{ij,kl}$ is the *Chentsov coefficient* which characterizes the shearing strain in the x_ix_j-plane due to shearing stress in the x_kx_l-plane, that is, $\mu_{ij,kl} = \gamma_{ij}/\gamma_{kl}$ for $\tau_{kl} = \tau$ and all other stresses are zero.

Due to the symmetry of the compliance matrix, the Poisson's ratios, the coefficients of mutual influences, and the Chentsov coefficients are subject to the following reciprocal relations:

$$\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j},\ \frac{\eta_{i,jk}}{G_{jk}} = \frac{\eta_{jk,i}}{E_i},\ \frac{\mu_{ij,kl}}{G_{kl}} = \frac{\mu_{kl,ij}}{G_{ij}}. \tag{1.44}$$

Full matrix shown in (1.43) for the general anisotropic materials also indicates that application of a normal stress leads not only to extension in the direction of the stress and contraction perpendicular to it, but to shearing deformation. Conversely, shearing stress causes extension and contraction in addition to the distortion of shearing deformation. For example, the out-of-plane shearing strains of an anisotropic material due to in-plane shearing stress and normal stresses are

$$\gamma_{13} = \frac{\eta_{1,31}\sigma_1 + \eta_{2,31}\sigma_2 + \mu_{12,31}\tau_{12}}{G_{31}},\ \gamma_{23} = \frac{\eta_{1,23}\sigma_1 + \eta_{2,23}\sigma_2 + \mu_{12,23}\tau_{12}}{G_{23}}, \tag{1.45}$$

wherein both the Chentsov coefficients and the coefficients of mutual influence of the first kind are required. Note that neither of these shear strains arise in an orthotropic material unless it is stressed in directions other than the principal material directions. In such cases, the Chentsov coefficients and the coefficients of mutual influence would be obtained from the transformed compliances.

From (1.28) we know by inversion of (1.43) the stiffness matrix components C_{ij} in terms of the engineering constants can be obtained. However, since the compliance matrix shown in (1.43) is a 6×6 full symmetric matrix, it is not easy to get the analytical expression of its inverse matrix. Following we just list the stiffness matrix components C_{ij} for the orthotropic materials.

Orthotropic Materials

$$\mathbf{C} = \begin{bmatrix} \dfrac{1-\nu_{23}\nu_{32}}{E_2E_3\Delta} & \dfrac{\nu_{12}+\nu_{32}\nu_{13}}{E_1E_3\Delta} & \dfrac{\nu_{13}+\nu_{12}\nu_{23}}{E_1E_2\Delta} & 0 & 0 & 0 \\ & \dfrac{1-\nu_{13}\nu_{31}}{E_1E_3\Delta} & \dfrac{\nu_{23}+\nu_{21}\nu_{13}}{E_1E_2\Delta} & 0 & 0 & 0 \\ & & \dfrac{1-\nu_{12}\nu_{21}}{E_1E_2\Delta} & 0 & 0 & 0 \\ & \text{symm.} & & G_{23} & 0 & 0 \\ & & & & G_{31} & 0 \\ & & & & & G_{12} \end{bmatrix}, \tag{1.46a}$$

where

$$\Delta = \frac{1-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}-\nu_{31}\nu_{13}-2\nu_{21}\nu_{32}\nu_{13}}{E_1E_2E_3}, \tag{1.46b}$$

and the symmetry conditions give us

$$\begin{aligned} &\frac{\nu_{21}+\nu_{31}\nu_{23}}{E_2E_3\Delta} = \frac{\nu_{12}+\nu_{32}\nu_{13}}{E_1E_3\Delta}, \quad \frac{\nu_{31}+\nu_{21}\nu_{32}}{E_2E_3\Delta} = \frac{\nu_{13}+\nu_{12}\nu_{23}}{E_1E_2\Delta}, \\ &\frac{\nu_{32}+\nu_{12}\nu_{31}}{E_1E_3\Delta} = \frac{\nu_{23}+\nu_{21}\nu_{13}}{E_1E_2\Delta}. \end{aligned} \tag{1.47}$$

1.3 Two-Dimensional Constitutive Relations

1.3.1 Isotropic Materials

Two-dimensional problems usually considered in isotropic elasticity fall into two physical distinct types. One of these arises in the study of deformation of large cylindrical bodies acted by the external forces so distributed that the component of deformation in the direction of the axis of the cylinder vanishes and the remaining components do not vary along the length of the cylinder. This is the class of problems in *plane deformation* or *plane strain*. Take the cross section of the cylinder be a plane parallel to x_1x_2-plane, the state of plane deformation may be characterized by