Topics in Modern Regularity Theory

edited by
Giuseppe Mingione

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Introduction

This volume collects the contributions of some of the leading experts in PDE who gave courses in the two intensive research periods I organized at the Centro De Giorgi of Scuola Normale Superiore at Pisa and at the University of Parma, in September 2009 and in the Spring of 2010, respectively. The speakers kindly agreed to give courses whose aims were both to review the established theory and to present the latest research developments; the notes included here summarize and extend the content of the lectures given in some of the courses offered by the schools. Specifically, the book contains three different contributions: the first one, in order of presentation, is by Ernst Kuwert & Reiner Schätzle, the second by Tristan Rivière, the third and final one by Bruno De Maria & Nicola Fusco. The first two parts are of expository character, and summarize some recent results obtained by the authors, after giving a rather general and comprehensive introduction to the subject. The third one contains some new results together with an up-to-dated presentation of the setting of problems dealt with.

I hereby take the opportunity to acknowledge the support of the European Research Council via the ERC Grant 207573 "Vectorial problems" and to thank the colleagues who were also responsible of the organization of the intensive periods, and, amongst them, especially Frank Duzaar and Juha Kinnunen.

The volume starts with the beautiful lecture notes, simply titled "The Willmore functional", by Ernst Kuwert & Reiner Schätzle. They give a very comprehensive introduction to the basic analytic aspects of the analysis of Willmore surfaces, *i.e.* the critical points of the Willmore functional, smoothly taking the reader from the basic facts to some of the most updated current research issues. After recalling the starting definitions and introducing a number of related tools, such as for instance monotonicity formulas, the authors present a careful analysis of basic aspects of the Willmore flow (the gradient flow associated to the Willmore functional) such as estimates on the maximal existence time interval and then the blow-up analysis of singularities at the time of their formation; asymptotic convergence properties are considered and studied as well. The authors then proceed in the analysis of the conformal parametrization properties of surfaces; it is here remarkable to note how Kuwert & Schätzle succeed in giving a smooth, clear and self-contained of some certainly not easy pieces of work, as for instance the study of asymptotic properties of the classical conformal parametrization of Huber made by Müller & Šverák a few years ago. Further topics treated in the notes are concerned with the removability of point singularities and applications to global existence of the Willmore flow of embedded surfaces, that the authors present in connection with the results of Robert Bryant; further connections emerge here with recent work of Rivière, partially related to the content of the subsequent chapter of this book. Finally, the authors give proofs of basic theorems in the variational analysis of Willmore functional such as those concerning compactness via the Moebius group quotients, and minimization asymptotic problems in classes of surfaces with prescribed genus.

In his "The role of conservation laws in the analysis of conformally invariant problems", Tristan Rivière gives a very comprehensive and updated presentation of regularity techniques aimed at treating conformally invariant variational problems. This is a longstanding and traditional topic in the modern Calculus of Variations. Rivière reviews a number of basic relevant techniques – as for instance compensated compactness and integrability by compensation – and then proceeds to explain the use of conservation laws in the regularity analysis of certain systems with critical growth. Finally, he explains his proof of the famous Hildebrandt's conjecture. This states the Hölder continuity of energy critical points of conformally invariant quadratic growth functionals. Rivière's proof is based on the new observation that the regularity of this problem can be treated by analyzing certain systems with antisymmetric potential; in turn Rivière's analysis identifies the central role of antisymmetry of potentials in allowing for deriving conservation laws when treating the regularity of systems with critical growth right hand side. This new and groundbreaking approach is robust enough to allow for many other applications to conformally invariant problems and in Geometric Analysis. In the last part of his notes, yet a new and surprising approach is described: the discovery of the robustness of the traditional ODE method of the variation of constants in the setting of Schrödinger systems with anti-symmetric potentials. Indeed, a new formulation of this approach is proposed and shown to be an effective tool when dealing with the regularity of more general critical growth right hand side systems.

De Maria & Fusco present a contribution titled “Regularity properties of equilibrium configurations of epitaxially strained elastic films” which is devoted to the mathematical study of the morphological instabilities of interfaces generated by the competition between elastic energy and surface tension, the so-called stress driven rearrangement instabilities (SDRI) of surfaces and interfaces in solids. Such a topic has been the focus of a rapidly growing interest of the applied and computational communities, also in view of its several important technological applications. Besides this, their paper can be also seen as a fine contribution to the regularity theory of the so-called free-discontinuity problems. Morphological instabilities occur, for instance, in the hetero-epitaxial growth of thin films for systems with a lattice mismatch between film and substrate. When the film is grown on a flat substrate, its profile remains flat until a critical value of the thickness is reached, after which the free surface develops corrugations, material clusters, and, possibly, cusp singularities. This is commonly referred to as the Asaro-Grinfeld-Tiller (AGT) instability, after the name of the scientists who started such theoretical investigations. Several numerical and theoretical studies have been carried out to study quantitative and qualitative properties of equilibrium configurations of strained epitaxial films. Although very insightful, most of these works lack rigorous mathematical content. Eventually, the foundations of a rigorous mathematical treatment have been given in works by Grinfeld (Soviet Physics Doklady, 1986), Bonnetier & Chambolle (SIAM J. Appl. Math., 2002), and more recently in work by Fonseca, Fusco, Leoni and Morini (ARMA 2007), who developed a complete regularity theory for a variant of the Bonnetier-Chambolle functional, modeling the case of an infinitely thick elastic substrate. Following the path set by Fonseca, Fusco, Leoni and Morini, De Maria & Fusco extend all these results to the functional originally considered by Bonnetier & Chambolle, which deals with the case of a rigid substrate. The main technical achievement is the rigorous validation of zero contact angle condition. The proof of this fact turns out to be considerably more difficult than in the case considered by Fonseca, Fusco, Leoni and Morini, since the presence of a Dirichlet condition at the interface between film and substrate poses non-trivial additional difficulties. The regularity results established by De Maria & Fusco presented here have been in fact used in several subsequent papers.

Parma, October 2011

Giuseppe Mingione

The Willmore functional

Ernst Kuwert and Reiner Schätzle

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1 Introduction to Geometry

1.1 Introduction

For an immersed closed surface $f : \Sigma \to \mathbb{R}^n$ the Willmore functional is defined by

$$\mathcal{W}(f) = \frac{1}{4} \int_\Sigma |\vec{\mathbf{H}}|^2 \, d\mu_g.$$

Here $g = f^* g_{\text{euc}}$ denotes the pull-back metric of the Euclidean metric under f, that is in local coordinates

$$g_{ij} := \langle \partial_i f, \partial_j f \rangle.$$

Moreover, $g = \det(g_{ij}), (g^{ij}) = (g_{ij})^{-1}$ and for the induced area measure

$$\mu_f = \mu_g = \sqrt{g}\, \mathcal{L}^2.$$

The second fundamental form of f is the normal projection of the second derivatives of f

$$A_{ij} := (\partial_{ij} f)^\perp.$$

We define the mean curvature vector and the tracefree second fundamental form by

$$\vec{\mathbf{H}} = g^{ij} A_{ij} \quad \text{and} \quad A^0_{ij} = A_{ij} - \frac{1}{2} \vec{\mathbf{H}} g_{ij}.$$

The Gauß curvature can be written by the Gauß equations, see [dC, Section 6, Proposition 3.1], as

$$K = \langle A(e_1, e_1), A(e_2, e_2)\rangle - \langle A(e_1, e_2), A(e_1, e_2)\rangle, \tag{1.1.1}$$

and combining with the inequality of geometric and arithmetic mean

$$|K| \le |A|^2/2. \tag{1.1.2}$$

In any orthonormal basis e_1, e_2 of the tangent space, we see

$$\begin{aligned} \vec{\mathbf{H}} &= A(e_1, e_1) + A(e_2, e_2), \\ |A|^2 &= |A(e_1, e_1)|^2 + |A(e_1, e_2)|^2 + |A(e_2, e_1)|^2 + |A(e_2, e_2)|^2. \end{aligned}$$

We calculate

$$\begin{aligned} |\vec{\mathbf{H}}|^2 &= |A(e_1, e_1)|^2 + 2\langle A(e_1, e_1), A(e_2, e_2)\rangle + |A(e_2, e_2)|^2 \\ &= |A|^2 + 2\big(\langle A(e_1, e_1), A(e_2, e_2)\rangle - \langle A(e_1, e_2), A(e_1, e_2)\rangle\big), \end{aligned} \tag{1.1.3}$$

hence by (1.1.1)

$$|\vec{\mathbf{H}}|^2 = |A|^2 + 2K. \tag{1.1.4}$$

Likewise

$$\begin{aligned} \frac{1}{2}|A^0|^2 &= \left|\frac{A(e_1, e_1) - A(e_2, e_2)}{2}\right|^2 + |A(e_1, e_2)|^2 \\ &= \frac{1}{4}|A(e_1, e_1)|^2 + \frac{1}{4}|A(e_2, e_2)|^2 + \frac{1}{2}|A(e_1, e_2)|^2 \\ &\quad - \frac{1}{2}\Big(\langle A(e_1, e_1), A(e_2, e_2)\rangle - \langle A(e_1, e_2), A(e_1, e_2)\rangle\Big) \\ &= \frac{1}{4}|A|^2 - \frac{1}{2}\Big(\langle A(e_1, e_1), A(e_2, e_2)\rangle - \langle A(e_1, e_2), A(e_1, e_2)\rangle\Big) \end{aligned}$$

and using (1.1.3)

$$\frac{1}{2}|A^0|^2 = \frac{1}{4}|\vec{\mathbf{H}}|^2 - \langle A(e_1, e_1), A(e_2, e_2)\rangle + \langle A(e_1, e_2), A(e_1, e_2)\rangle. \tag{1.1.5}$$

Again by (1.1.1), we obtain

$$\frac{1}{4}|\vec{\mathbf{H}}|^2 - K = \frac{1}{2}|A^0|^2 \tag{1.1.6}$$

and combining with (1.1.4)

$$|A|^2 = 2|A^0|^2 + 2K. \tag{1.1.7}$$

For a closed surface Σ the integral over the Gauß curvature is given by the Gauß-Bonnet theorem as

$$\int_\Sigma K \, d\mu_g = 2\pi(\Sigma), \tag{1.1.8}$$

where $\chi(\Sigma)$ denotes the Euler characteristic of Σ. This yields with (1.1.4) and (1.1.6)

$$\mathcal{W}(f) = \frac{1}{4}\int_\Sigma |A|^2 \, d\mu_g + \pi\chi(\Sigma) = \frac{1}{2}\int_\Sigma |A^\circ|^2 \, d\mu_g + 2\pi\chi(\Sigma). \tag{1.1.9}$$

We finish this introduction establishing a lower bound in codimension one.

Proposition 1.1.1 ([Wil65]). *For any embedding $f : \Sigma \to \mathbb{R}^3$ of a closed surface Σ, we have*

$$\mathcal{W}(f) \geq 4\pi$$

and equality implies that f parametrises a round sphere.

Proof. We consider $\Sigma \subseteq \mathbb{R}^3$. Let $\nu : \Sigma \to S^2$ be the unique smooth outer normal at Σ. For any unit vector ν_0, we choose $x_0 \in \Sigma$ with

$$\langle x_0, \nu_0 \rangle := \max_{x\in\Sigma} \langle x, \nu_0 \rangle$$

and see

$$\Sigma \subseteq \{ y \in \mathbb{R}^3 \mid \langle y - x_0, \nu_0 \rangle \leq 0 \}.$$

Therefore $\{\nu_0\}^\perp$ is a supporting hyperplane of Σ at x_0 and

$$\nu(x_0) = \nu_0 \quad \text{and} \quad K(x_0) \geq 0.$$

As $\nu_0 \in S^2$ was arbitrary, we get

$$\nu(K \geq 0) = S^2. \tag{1.1.10}$$

We define the scalar second fundamental form

$$h_{ij} = \langle A_{ij}, \nu \rangle = \langle \partial_{ij} f, \nu \rangle.$$

Clearly, the Gauß curvature is the determinant

$$K = \det_g(h_{ij}).$$

As $\partial f \perp \nu$, we get

$$h_{ij} = -\langle \partial_i \nu, \partial_j f \rangle.$$

Considering $g_{ij} = \delta_{ij}$ at some point, we get that $\partial_1 f, \partial_2 f, \nu$ is an orthonormal basis of $\mathbb{R}^3$ and observing $\partial \nu \perp \nu$, as $|\nu| = 1$,

$$\partial_i \nu = -h_{i1} \partial_1 f - h_{i2} \partial_2 f.$$

We calculate the Jacobian

$$(J_g \nu)^2 = \det(\langle \partial_i \nu, \partial_j \nu \rangle) = \det(h_{ij})^2 = K^2,$$

hence

$$J_g \nu = |K|.$$

By (1.1.6), (1.1.10) and the area formula, we get

$$\mathcal{W}(\Sigma) \geq \int\limits_{[K>0]} K \, d\mu_g = \int\limits_{[K\geq 0]} J_g \nu \, d\mu_g \geq \mathcal{H}^2(\nu(K \geq 0)) = \mathcal{H}^2(S^2) = 4\pi,$$

in particular $[K > 0] \neq \emptyset$. In case of equality, we see using (1.1.10)

$$0 = \int\limits_{[K>0]} \left(\frac{1}{4} |\vec{\mathbf{H}}|^2 - K \right) d\mu_g = \int\limits_{[K>0]} \frac{1}{2} |A^0|^2 \, d\mu_g,$$

hence $A^0 \equiv 0$ in $[K > 0]$. By a theorem of Codazzi, f parametrises in any connected component Ω of $[K > 0]$ a piece of a round sphere $\partial B_R(a)$, in particular $K \equiv 1/R^2$ is constant in Ω. Therefore Ω is closed, hence $\Omega = [K > 0] = \Sigma$ and $A^0 \equiv 0$ on Σ, as Σ is connected and $[K > 0] \neq \emptyset$. Then $f : \Sigma \to \partial B_R(a)$ is a local diffeomorphism, hence a covering map, and, as $\partial B_R(a)$ is simply connected, $f : \Sigma \xrightarrow{\approx} \partial B_R(a)$ is a diffeomorphism. □

1.2 Conformal invariance

Clearly, the Willmore functional is invariant under isometries. Oberserving for $\lambda > 0$

$$\vec{\mathbf{H}}_{\lambda f} = \lambda^{-1} \vec{\mathbf{H}} \quad \text{and} \quad \mu_{\lambda f} = \lambda^2 \mu_f,$$

we see

$$\mathcal{W}(\lambda f) = \mathcal{W}(f),$$

and the Willmore functional is also invaraint under homotheties.

More general the Willmore functional is invariant under conformal transformations. We start with the following pointwise invariance.

Proposition 1.2.1 ([Ch74]). *Let* M *be a* n*-dimensional manifold with two conformal metrics* $\bar{g} = e^{2u} g$ *and* $\Sigma \subseteq M$ *be a* m*-dimensional submanifold. Then the second fundamental forms* A *and* $\bar{A}$ *of* Σ *with respect to the ambient metrics* g *and* $\bar{g}$ *satisfy*

$$\bar{A}_{ij} - A_{ij} = -g_{ij} \operatorname{grad}_g^{\perp} u \tag{1.2.1}$$

in local charts on Σ*, where* $\operatorname{grad}_g u^k = g^{kl} \partial_l u$ *denotes the gradient of* u *with respect to* g *and* $.^{\perp}$ *denotes the component normal to* Σ *with respect to either* g *or* $\bar{g}$*. In particular*

$$\bar{A}^0_{ij} = A^0_{ij} \tag{1.2.2}$$

for the tracefree second fundamental forms and for a surface Σ *that is* $m = 2$

$$|\bar{A}^0|^2_{\bar{g}}\, \mu_{\bar{g}} = |A^0|^2_g\, \mu_g. \tag{1.2.3}$$

First we compute the difference of Christoffel symbols for conformal metrics.

Proposition 1.2.2. *Let* M *be a* n*-dimensional manifold with two conformal metrics* $\bar{g} = e^{2u} g$*. The difference of the Christoffel symbols in local coordinates is a tensor and is given by*

$$T^k_{ij} := \bar{\Gamma}^k_{ij} - \Gamma^k_{ij} = \delta^k_i \partial_j u + \delta^k_j \partial_i u - g_{ij} g^{kl} \partial_l u \tag{1.2.4}$$

and in conformal coordinate $g_{ij} = e^{2v} \delta_{ij}$

$$\begin{aligned} T^1_{11} &= \partial_1 u, \quad & T^1_{12} &= T^1_{21} = \partial_2 u, \quad & T^1_{22} &= -\partial_1 u, \\ T^2_{11} &= -\partial_2 u, \quad & T^2_{12} &= T^2_{21} = \partial_1 u, \quad & T^2_{22} &= \partial_2 u. \end{aligned} \tag{1.2.5}$$

Proof. We calculate

$$\begin{aligned} 2\bar{\Gamma}^k_{ij} &= \bar{g}^{kl}(\partial_i \bar{g}_{jl} + \partial_j \bar{g}_{li} - \partial_l \bar{g}_{ij}) \\ &= g^{kl}(\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij}) \\ &\quad + e^{-2u} g^{kl}(g_{jl}\partial_i e^{2u} + g_{li}\partial_j e^{2u} - g_{ij}\partial_l e^{2u}) \\ &= 2\Gamma^k_{ij} + 2g^{kl}(g_{jl}\partial_i u + g_{li}\partial_j u - g_{ij}\partial_l u) \\ &= 2\Gamma^k_{ij} + 2(\delta^k_j \partial_i u + \delta^k_i \partial_j u - g^{kl} g_{ij} \partial_l u), \end{aligned}$$

which is (1.2.4), and (1.2.5) follows easily by direct evaluation. □

Proof of Proposition 1.2.1. Let Γ_{ij}^k and $\bar{\Gamma}_{ij}^k$ denote the Christoffel symbols of g and $\bar{g}$ and put $T_{ij}^k := \bar{\Gamma}_{ij}^k - \Gamma_{ij}^k$ which is a tensor. We calculate the covariant derivatives of a vectorfield $X = X^k \partial_k$ on M as

$$\nabla_i^{\bar{g}} X^k = \nabla_i^g X^k + T_{il}^k X^l.$$

We may consider $\Sigma = B_1^m(0) \times \{0\} \subseteq B_1^n(0) = M$ and $g_{ij}(0) = \delta_{ij}$. Then $\partial_1, \ldots, \partial_m$ are tangential to Σ and $\partial_{m+1}, \ldots, \partial_n$ form a basis of $T_0^\perp \Sigma$. Choosing $X = \partial_j, i, j = 1, \ldots, m$, we get for the normal projections of the covariant derivatives which are the second fundamental form

$$\bar{A}_{ij} - A_{ij} = \nabla_i^{\bar{g},\perp} \partial_j - \nabla_i^{g,\perp} \partial_j = T_{ij}^k \partial_k^\perp = \sum_{k=m+1}^n T_{ij}^k \partial_k. \qquad (1.2.6)$$

Using (1.2.4) and observing that $i, j = 1, \ldots, m, k = m+1, \ldots, n$ yields $g_{ik}(0) = g_{jk}(0) = 0$, as $g_{rs}(0) = \delta_{rs}$, hence

$$T_{ij}^k = -g_{ij} g^{kl} \partial_l u \quad \text{in } 0.$$

Plugging into (1.2.6), we obtain

$$\bar{A}_{ij} - A_{ij} = -\sum_{k=m+1}^n g_{ij} g^{kl} \partial_l u \partial_k = -g_{ij} g^{kl} \partial_l u \partial_k^\perp = -g_{ij} \operatorname{grad}_g^\perp u \quad \text{in } 0.$$

This equation is tensorial, and we obtain (1.2.1) on Σ. As the difference is a multiple of either metric g or $\bar{g}$, the tracefree parts coincide which is (1.2.2). Finally

$$\begin{aligned} |\bar{A}^0|_{\bar{g}}^2 \sqrt{\bar{g}} &= \bar{g}^{ik} \bar{g}^{jl} \bar{g}(\bar{A}_{ij}^0, \bar{A}_{kl}^0) \sqrt{\bar{g}} \\ &= e^{-4u} g^{ik} g^{jl} e^{2u} g(A_{ij}^0, A_{kl}^0) \sqrt{\det(e^{2u} g_{ij})} = |A^0|_g^2 \sqrt{g} \end{aligned}$$

which yields (1.2.3). □

We consider an immersion $f : \Sigma \to \Omega$ of a closed surface into an open set $\Omega \subseteq \mathbb{R}^n$. Let $\Phi : \Omega \xrightarrow{\approx} \Omega' \subseteq \mathbb{R}^n$ be a conformal diffeomorphism with pull-back metric $\bar{g} = \Phi^* g_{\text{euc}} = e^{2u} g_{\text{euc}}$. We calculate with (1.1.9) and (1.2.3)

$$\begin{aligned} \mathcal{W}(\Phi \circ f) &= \frac{1}{2} \int_\Sigma |A_{\Phi \circ f}^0|^2 \, d\mu_{\Phi \circ f} + 2\pi \chi(\Sigma) \\ &= \frac{1}{2} \int_\Sigma |A_f^0|_{\bar{g}}^2 \, d\mu_{\bar{g}} + 2\pi \chi(\Sigma) \\ &= \frac{1}{2} \int_\Sigma |A_f^0|^2 \, d\mu_f + 2\pi \chi(\Sigma) = \mathcal{W}(f), \end{aligned}$$

and the Willmore functional is invariant under conformal diffeomorphisms.

Now we want to extend the definition of the Willmore functional from Euclidean target to general n-dimensional manifold M with metric g, see [Wei78]. We consider an immersion $f : \Sigma \to M$ of a closed surface. The Gauß equations (1.1.1) extend in M, see [dC, Section 6, Proposition 3.1], using the Riemann curvature tensors R_Σ and R_M of Σ and M to

$$\begin{aligned}&\langle A(e_1, e_1), A(e_2, e_2)\rangle - \langle A(e_1, e_2), A(e_1, e_2)\rangle \\ &\quad = R_\Sigma(e_1, e_2, e_1, e_2) - R_M(e_1, e_2, e_1, e_2) = K_\Sigma - K_M^\Sigma,\end{aligned}$$

where K_Σ is Gauß curvature of Σ and K_M^Σ is the sectional curvature of M with respect to the tangent space of Σ. Recalling (1.1.5), which holds true in general M, we obtain

$$\frac{1}{4}|\vec{\mathbf{H}}|^2 + K_M^\Sigma = \frac{1}{2}|A^0|^2 + K_\Sigma.$$

The integral over Σ with respect to the area measure μ_g of the first term on the right hand side is a conformal invariant by Proposition 1.2.1, and the integral over the second term is a topological invariant. This yields the following definition and proposition.

Proposition 1.2.3. *For a immersion $f : \Sigma \to M$ of a closed surface Σ into a n-dimensional manifolds M with metric g, we define the Willmore functional*

$$\mathcal{W}(f) = \mathcal{W}(f, g) := \int_\Sigma \left(\frac{1}{4}|\vec{\mathbf{H}}|^2 + K_M^\Sigma\right) \mathrm{d}\mu_g. \tag{1.2.7}$$

The Willmore functional is invariant under conformal changes of the metric, that is

$$\mathcal{W}(f, \bar{g}) = \mathcal{W}(f)$$

for any conformal metric $\bar{g} = e^{2u} g$.

For the special case of a sphere $M = S^n$ with canonical metric, we have $K_{S^n}^\Sigma \equiv 1$ and get for $f : \Sigma \to S^n$ that

$$\mathcal{W}(f) = \frac{1}{4}\int_\Sigma |\vec{\mathbf{H}}|^2 \,\mathrm{d}\mu_g + \mathrm{Area}(f). \tag{1.2.8}$$

1.3 The Euler Lagrange equation

Critical points of the Willmore functional are called Willmore immersions or Willmore surfaces. Here we derive the Euler Lagrange equation for the Willmore functional.

We consider a smooth one-parameter family of immersions $f_t : \Sigma \to \mathbb{R}^n$ with $\partial_t f_{t,|t=0} = V$ normal along f. We get in local coordinates

$$g_{t,ij} = \langle \partial_i f_t, \partial_j f_t \rangle$$

and

$$\partial_t g_{ij} = \langle \partial_i f, \partial_j V \rangle + \langle \partial_j f, \partial_i V \rangle = -2\langle A_{ij}, V \rangle, \tag{1.3.1}$$

as $\partial f \perp V$ and $A = \partial^2 f^\perp$. Then as $g^{ij} g_{jk} = \delta^i_k$

$$\partial_t g^{ij} = -g^{ik} \partial_t g_{kl} g^{lj} = 2 g^{ik} g^{jl} \langle A_{kl}, V \rangle \tag{1.3.2}$$

and

$$\partial_t g = \partial_t \det(g^{ij}_{t=0} g_{jk}) g = \mathrm{tr}(g^{ij} \partial_t g_{jk}) g = -2 g^{ij} \langle A_{ij}, V \rangle g = -2\langle \vec{\mathbf{H}}, V \rangle g,$$

hence

$$\partial_t \mu_g = \partial_t \sqrt{g} \mathcal{L}^2 = -\langle \vec{\mathbf{H}}, V \rangle \sqrt{g} \mathcal{L}^2 = -\langle \vec{\mathbf{H}}, V \rangle \mu_g. \tag{1.3.3}$$

Next we write ∇_i for the covariant derivative, $\nabla_i^\perp$ for its normal projection and $\partial_t^\perp$ for the normal projection of the time derivative. We recall the Weingarten equations

$$\begin{aligned} A_{ij} &= (\partial_{ij} f)^\perp = \partial_{ij} f - \langle \partial_{ij} f, \partial_k f \rangle g^{kl} \partial_l f \\ &= \partial_{ij} f - \Gamma^k_{ij} \partial_k f = \nabla_i \nabla_j f. \end{aligned}$$

We calculate

$$\begin{aligned} \partial_t^\perp A_{ij} &= (\partial_{ij} V)^\perp - \langle \partial_{ij} f, \partial_k f \rangle g^{kl} \partial_t^\perp \partial_l f \\ &= (\partial_{ij} V - \Gamma^k_{ij} \partial_k V)^\perp = (\nabla_i \nabla_j V)^\perp \\ &= \nabla_i^\perp \nabla_j^\perp V + \nabla_i^\perp \Big(\langle \partial_j V, \partial_k f \rangle g^{kl} \partial_l f \Big) \\ &= \nabla_i^\perp \nabla_j^\perp V - \langle A_{jk}, V \rangle g^{kl} \nabla_i^\perp \partial_l f \\ &= \nabla_i^\perp \nabla_j^\perp V - \langle A_{jk}, V \rangle g^{kl} A_{il} \end{aligned} \tag{1.3.4}$$

and

$$\begin{aligned}
\partial_t^\perp \vec{\mathbf{H}} &= \partial_t^\perp (g^{ij} A_{ij}) = g^{ij} \Big(\nabla_i^\perp \nabla_j^\perp V - \langle A_{jk}, V \rangle g^{kl} A_{il} \Big) \\
&\quad + 2 g^{ik} g^{jl} \langle A_{kl}, V \rangle A_{ij} \\
&= \Delta^\perp V + g^{ik} g^{jl} \langle A_{ij}, V \rangle A_{kl} \\
&= \Delta^\perp V + g^{ik} g^{jl} \Big\langle A_{ij}^0 + \frac{1}{2} \vec{\mathbf{H}} g_{ij}, V \Big\rangle \Big(A_{kl}^0 + \frac{1}{2} \vec{\mathbf{H}} g_{kl} \Big) \\
&= \Delta^\perp V + g^{ik} g^{jl} \langle A_{ij}^0, V \rangle A_{kl}^0 \\
&\quad + \frac{1}{2} g^{ik} g^{jl} g_{ij} A_{kl}^0 \langle \vec{\mathbf{H}}, V \rangle + \frac{1}{2} g^{ik} g^{jl} g_{kl} \langle A_{ij}^0, V \rangle \vec{\mathbf{H}} \\
&\quad + \frac{1}{4} g^{ik} g^{jl} g_{ij} g_{kl} \langle \vec{\mathbf{H}}, V \rangle \vec{\mathbf{H}} \\
&= \Delta^\perp V + g^{ik} g^{jl} \langle A_{ij}^0, V \rangle A_{kl}^0 + \frac{1}{2} \langle \vec{\mathbf{H}}, V \rangle \vec{\mathbf{H}},
\end{aligned}$$

where $\Delta^\perp$ denotes the Laplacian in the normal bundle. Combining we get

$$\begin{aligned}
\partial_t \Big(|\vec{\mathbf{H}}|^2 \mu_g \Big) &= 2 \langle \partial_t^\perp \vec{\mathbf{H}}, \vec{\mathbf{H}} \rangle \mu_g + |\vec{\mathbf{H}}|^2 \partial_t \mu_g \\
&= \Big(2 \langle \Delta^\perp V, \vec{\mathbf{H}} \rangle + 2 g^{ik} g^{jl} \langle A_{ij}^0, \vec{\mathbf{H}} \rangle \langle A_{kl}^0, V \rangle \\
&\qquad + \langle \vec{\mathbf{H}}, V \rangle |\vec{\mathbf{H}}|^2 - |\vec{\mathbf{H}}|^2 \langle \vec{\mathbf{H}}, V \rangle \Big) \mu_g \\
&= 2 \Big(\langle \Delta^\perp V, \vec{\mathbf{H}} \rangle + g^{ik} g^{jl} \langle A_{ij}^0, \vec{\mathbf{H}} \rangle \langle A_{kl}^0, V \rangle \Big) \mu_g .
\end{aligned} \tag{1.3.5}$$

Integrating yields

$$\begin{aligned}
\frac{d}{dt} \mathcal{W}(f_t) &= \frac{1}{2} \int_\Sigma \Big(\langle \Delta^\perp V, \vec{\mathbf{H}} \rangle + g^{ik} g^{jl} \langle A_{ij}^0, \vec{\mathbf{H}} \rangle \langle A_{kl}^0, V \rangle \Big) \, \mathrm{d}\mu_g \\
&= \frac{1}{2} \int_\Sigma \langle \Delta^\perp \vec{\mathbf{H}} + g^{ik} g^{jl} \langle A_{ij}^0, \vec{\mathbf{H}} \rangle A_{kl}^0, V \rangle \, \mathrm{d}\mu_g .
\end{aligned} \tag{1.3.6}$$

We abbreviate for a normal field ϕ

$$Q(A^0)\phi := g^{ik} g^{jl} \langle A_{ij}^0, \phi \rangle A_{kl}^0 \tag{1.3.7}$$

and

$$\delta \mathcal{W}(f) := \frac{1}{2} \Big(\Delta^\perp \vec{\mathbf{H}} + Q(A^0) \vec{\mathbf{H}} \Big). \tag{1.3.8}$$

Now we consider a general smooth one-parameter family of immersions $f_t : \Sigma \to \mathbb{R}^n$ with $\partial_t f_t = V_t \in \mathbb{R}^n$. We decompose

$$V_t = N_t + df.\xi_t$$

in N_t normal along f and $df.\xi_t$ tangential along f and $\xi_t \in T\Sigma$. We solve the ordinary differential equation

$$\phi_0 = id_\Sigma \quad \text{and} \quad \partial_t \phi_t = -\xi_t(\phi_t).$$

Clearly, $\phi_t : \Sigma \xrightarrow{\approx} \Sigma$ is a one-parameter family of diffeomorphisms. We put $\tilde{f}_t = f_t \circ \phi_t$ and see

$$\partial_t \tilde{f}_t = (\partial_t f_t) \circ \phi_t + df_t.\partial\phi_t = V_t(\phi_t) - df_t.\xi_t(\phi_t) = N_t(\phi_t)$$

which is normal along f. By parameter invariance of the Willmore functional and (1.3.6), we get

$$\frac{d}{dt}\mathcal{W}(f_t) = \frac{d}{dt}\mathcal{W}(\tilde{f}_t) = \int_\Sigma \langle \delta\mathcal{W}(f), N\rangle \, \mathrm{d}\mu_g = \int_\Sigma \langle \delta\mathcal{W}(f), \partial_t f\rangle \, \mathrm{d}\mu_g,$$

since $\delta\mathcal{W}(f)$ is normal along f. We have proved the following proposition.

Proposition 1.3.1. *For a smooth one-parameter family of immersions $f_t : \Sigma \to \mathbb{R}^n$ the first variation of the Willmore functional is given by*

$$\frac{d}{dt}\mathcal{W}(f_t) = \int_\Sigma \langle \delta\mathcal{W}(f), \partial_t f\rangle \, \mathrm{d}\mu_g. \tag{1.3.9}$$

f is called a Willmore immersion, if this vanishes, hence if

$$\delta\mathcal{W}(f) = \frac{1}{2}\Big(\Delta^\perp \vec{\mathbf{H}} + Q(A^0)\vec{\mathbf{H}}\Big) = 0 \quad \text{on } \Sigma. \tag{1.3.10}$$

In case of an immersion $f : \Sigma \to S^n$ into a sphere, we obtain the following proposition.

Proposition 1.3.2. *For an immersion $f : \Sigma \to S^n \subseteq \mathbb{R}^{n+1}$ into the sphere, we see for the second fundamental forms $A_{f,\mathbb{R}^{n+1}}$ respectively A_{f,S^n} as immersions into $\mathbb{R}^{n+1}$ respectively S^n*

$$\begin{gathered} A_{f,\mathbb{R}^{n+1}} = A_{f,S^n} - fg, \quad \vec{\mathbf{H}}_{f,\mathbb{R}^{n+1}} = \vec{\mathbf{H}}_{f,S^n} - 2f, \quad A^0_{f,\mathbb{R}^{n+1}} = A^0_{f,S^n}, \\ \Delta_g^{\mathbb{R}^{n+1},\perp}\vec{\mathbf{H}}_{f,\mathbb{R}^{n+1}} + Q(A^0_{f,\mathbb{R}^{n+1}})\vec{\mathbf{H}}_{f,\mathbb{R}^{n+1}} = \Delta_g^{S^n,\perp}\vec{\mathbf{H}}_{f,S^n} + Q(A^0_{f,S^n})\vec{\mathbf{H}}_{f,S^n}. \end{gathered} \tag{1.3.11}$$