Logic, Epistemology, and the Unity of Science 27

Peter Dybjer
Sten Lindström
Erik Palmgren
Göran Sundholm *Editors*

Epistemology versus Ontology

Essays on the Philosophy and Foundations of Mathematics in Honour of Per Martin-Löf

Springer

Epistemology versus Ontology

LOGIC, EPISTEMOLOGY, AND THE UNITY OF SCIENCE

VOLUME 27

Logic, Epistemology, and the Unity of Science aims to reconsider the question of the unity of science in light of recent developments in logic. At present, no single logical, semantical or methodological framework dominates the philosophy of science. However, the editors of this series believe that formal techniques like, for example, independence friendly logic, dialogical logics, multimodal logics, game theoretic semantics and linear logics, have the potential to cast new light on basic issues in the discussion of the unity of science.

This series provides a venue where philosophers and logicians can apply specific technical insights to fundamental philosophical problems. While the series is open to a wide variety of perspectives, including the study and analysis of argumentation and the critical discussion of the relationship between logic and the philosophy of science, the aim is to provide an integrated picture of the scientific enterprise in all its diversity.

For further volumes:
http://www.springer.com/series/6936

Peter Dybjer • Sten Lindström • Erik Palmgren
Göran Sundholm
Editors

Epistemology versus Ontology

Essays on the Philosophy and Foundations of Mathematics in Honour of Per Martin-Löf

Springer

Editors
Peter Dybjer
Department of Computer Science
and Engineering
Chalmers University of Technology
Göteborg
Sweden

Sten Lindström
Department of Historical, Philosophical
and Religious Studies
Umeå University
Umeå
Sweden

Erik Palmgren
Department of Mathematics
Stockholm University
Stockholm
Sweden

Göran Sundholm
Philosophical Institute
Leiden University
Netherlands

ISBN 978-94-007-4434-9 ISBN 978-94-007-4435-6 (eBook)
DOI 10.1007/978-94-007-4435-6
Springer Dordrecht Heidelberg New York London

Library of Congress Control Number: 2012942911

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Contents

Introduction

1 Background

The present anthology has its origin in an international conference that was arranged at the *Swedish Collegium for Advanced Studies* (SCAS) in Uppsala, May 5–8, 2009, “Philosophy and Foundations of Mathematics: Epistemological and Ontological Aspects”. The conference was dedicated to Per Martin-Löf on the occasion of his retirement.

The aim of the conference was to bring together philosophers, mathematicians, and logicians to penetrate both current and historically important problems in the philosophy and foundations of mathematics. Starting with the pioneering work of Dag Prawitz and Per Martin-Löf in the 1960s, Swedish logicians and philosophers have made important contributions to the foundations and philosophy of mathematics. In philosophy, one has been concerned with the opposition between constructivism and classical mathematics and the different ontological and epistemological views that are reflected in this opposition. Swedish logicians have made significant contributions to the foundations of mathematics, for example, in proof theory, proof-theoretic semantics, and constructive type theory. These contributions have had a strong impact on computer science, for example, through *Martin-Löf’s intuitionistic type theory* (*MLTT*), particularly in programming languages and proof assistants.

A suggested basis for the discussions during the conference was current *foundational frameworks* for mathematics. These frameworks give rise to – and some of them purport to solve – important epistemological and ontological problems about mathematics. The dominant, or “mainstream”, foundational framework for current mathematics is based on classical logic and set theory with the axiom of choice. Within this framework, a mathematical proof is considered rigorous if it can be formalized in Zermelo-Fraenkel set theory (ZFC), at least in principle. This framework is, however, laden with philosophical difficulties. Set-theoretic Platonism involves a transfinite hierarchy of infinite sets and is associated with serious epistemological problems. Moreover, the encoding of mathematical entities as iterative sets is unnatural and arbitrary and is not in accordance with standard mathematical practice. Two alternative foundational programmes that are actively pursued today are (1) *predicativistic constructivism* and (2) *category-theoretic foundations*:

1. Predicativistic constructivism can be based on MLTT, Aczel-Myhill's constructive set theory, or similar systems. The practice of the Bishop school of constructive mathematics fits well into this framework. Associated philosophical foundations are meaning theories in the tradition of Wittgenstein, Dummett, Prawitz, and Martin-Löf. What is the relationship between proof-theoretical semantics in the tradition of Gentzen, Prawitz, and Martin-Löf, on the one hand, and Wittgensteinian or other accounts of meaning-as-use, on the other hand? What can proof-theoretical analysis tell us about the scope and limits of constructive and (generalized) predicative mathematics? To what extent is it possible to reduce classical mathematical frameworks to constructive ones? Such reductions often reveal computational content of classical existence proofs. Is computational content enough to solve the epistemological questions?
2. Category-theoretic foundations are closely related to a structural vision of mathematics, where the unity of mathematics is obtained through a systematic use of abstraction, enabling transfer of results between seemingly unrelated fields. This has its historical roots in the works of Dedekind and Hilbert. In certain areas of mathematical practice, structuralism is strongly manifested through the extensive use of category theory. Structuralism can also be taken as an ontological view. What is the relationship between structuralism as a way of practicing mathematics and the philosophical point of view that goes under the same name? And what is the relationship between category theory and structuralism as a philosophy of mathematics?

Similarities between type theory and category theory have been observed and exploited since the early 1970s in the form of topos theory. Developments of various forms of "algebraic" set theory (Lawvere, Joyal, Moerdijk) suggest that mathematics can be fruitfully based on category theory. Philosophical critics of category theory as a foundation of mathematics include Feferman and Kreisel. Hellman and Shapiro have proposed alternative foundational frameworks for structuralist mathematics. Such systems may, for instance, be based on modal concepts and may be eliminatory with respect to higher-order mathematical objects. It is of interest to discuss structuralist views in the philosophy of mathematics in connection with developments in topos theory and category theory.

The conference was successful in bringing together a number of internationally renowned mathematicians and philosophers around common concerns. Most of the papers in this collection originate from the Uppsala Conference, but a few additional papers of relevance to the issues discussed there have been solicited especially for this volume.

2 Martin-Löf: Pioneer and Land Clearer

"Per Martin-Löf, incomparable défricheur", Jean-Yves Girard writes in the dedication in the beginning of his contribution to this volume. It is as a pioneer and "land clearer" that we know Martin-Löf – one of the principal clarifiers of the syntax and semantics of constructive mathematics.

Martin-Löf's work began already in the 1960s with an important contribution to the foundations of algorithmic randomness, while studying with Andrei Kolmogorov, forerunner in foundations of both probability and constructivism. With his Ph.D. thesis, "Notes on Constructive Mathematics" 1968, Martin-Löf's long exploration of the foundations of constructive mathematics started. We refer to Göran Sundholm's article for a comprehensive account of the development of this journey and of Martin-Löf's philosophical outlook. We also refer to Dag Prawitz' article that discusses Martin-Löf's philosophy in a more critical vein.

Martin-Löf's work ranges over several fields: not only constructive mathematics, mathematical logic, and philosophy but also statistics and, not least, the foundations of computer science. His influential paper "Constructive Mathematics and Computer Programming" from 1979 explains why "constructive mathematics and computer programming are the same" and why "intuitionistic type theory is a programming language". Shortly after that paper was written, computer scientists in Göteborg and at Cornell University started putting Martin-Löf's ideas into practice. Among other things, several "proof editors" or "proof assistants" based on intuitionistic type theory were developed. These are computer systems, which can be used both for formalizing constructive mathematics and for developing programs satisfying given specifications.

Intuitionistic type theory as presented in "Constructive Mathematics and Computer Programming" had an extensional equality, but did not satisfy the normalization and decidability properties of his original proposals. Martin-Löf considered this unsatisfactory, and in 1986 the theory was again revised. By changing the rules for equality, he made it "intensional" and recovered the normalization and decidability properties. Moreover, he separated an underlying "theory of types", a lambda calculus with dependent types providing a "logical framework", from a "theory of sets". The decidability of the judgements was exploited by many of the proof assistants for the theory, which were soon to be implemented. Intuitionistic type theory had now found its final form, although it has later been extended and modified in many ways by computer scientists to make it more practical. Examples of such languages based on intuitionistic type theory are ALF, Agda, and Epigram, and their impredicative cousins, such as Coq. These are all functional languages with dependent types, which incorporate a number of useful programming language features including general methods for inductive definitions and pattern matching, and modules systems.

During the 1980s, Martin-Löf also dedicated much of his time to the theory of choice sequences (infinite streams in the terminology of computer science). He developed his own approach to domain theory in 1983–1984 based on Dana Scott's neighbourhood systems and information systems. This domain theory models an extension of intuitionistic type theory with partial computations and infinite streams. He also investigated nonstandard analysis and developed in "Mathematics of Infinity" a nonstandard type theory based on the identification of propositions and types. The idea of "formal neighbourhoods" in Martin-Löf's approach to domain theory was also a first step towards the formal topology, which he began developing together with Giovanni Sambin.

3 Contributions to This Volume

We have divided this book into two parts. In the first part, we have collected papers on a more philosophical and nontechnical nature.

3.1 Part I: Philosophy of Logic and Mathematics

Many of the papers in the first part of the book concern various versions of constructivism. Mark van Atten investigates the roots of Brouwerian intuitionism in Kant's philosophy of mathematics. According to Kant, a mathematical entity exists only if it is in principle constructible in human intuition, which by its very nature is finite. On Kant's view, $\sqrt{2}$ exists as a geometrical magnitude, but not as a number. Brouwer, on the other hand, identified the irrational number $\sqrt{2}$ with a potentially infinite sequence of rational numbers. Van Atten discusses the systematic reasons why in Kant's philosophy this identification is impossible.

A special restrictive form of constructivism going back to Leopold Kronecker (1823–1891) is *finitism*, according to which (1) the natural numbers are taken as primitive, (2) all other mathematical objects ought to be constructed by finitary means from the natural numbers, and (3) all statements about numbers ought to be decidable algorithmically in finitely many steps. In his paper, W. W. Tait discusses Skolem's Primitive Recursive Arithmetic (PRA), which can be viewed as a formal realization of Kronecker's finitist programme for arithmetic. PRA does not contain bound variables, induction is over quantifier free expressions only, and definition of functions by primitive recursion is freely allowed. Tait discusses the historical roots of PRA, its relationship to the requirement of Hilbert's programme that metamathematical proofs of consistency be finitary, and its relation to Kant's philosophy of mathematics.

Gödel's two incompleteness theorems obviously had a profound influence on Hilbert's finitist consistency programme for classical mathematics. Wilfried Sieg's contribution aims at a nuanced and deepened understanding of how Gödel's results affected a transformation in proof theory between 1930 and 1934. The starting point is Gödel's announcement of a restricted form of his first incompleteness theorem in

Königsberg on 7 September 1930 and the endpoint is the first consistency proof for full arithmetic that Gentzen completed in December 1934. Sieg argues that Hilbert played a significant role in the development between these points. In his last publication "Beweis des tertium non datur" from 1931, Hilbert responds to Gödel's second incompleteness theorem – without mentioning Gödel (!) – and presented novel directions and concrete problems that needed to be addressed. Gentzen did resolve the problems in surprising new ways, but according to Sieg fully in the spirit of Hilbert's view that true contentual thinking consists in operations on proofs. The main point is that there is genuine continuity between Hilbert's "old" proof theory and the "new" proof theory initiated by Gentzen.

Dag Prawitz discusses the question whether – from an intuitionist point of view – logic is in essence epistemological or ontological. To be more specific: Are the concepts of truth and proof epistemological or ontological? Prawitz is especially concerned with examining Martin-Löf's views on this matter. Mathematical intuitionists are usually taken to view both proof and truth as epistemic concepts. That proof is an epistemic concept seems to be fairly uncontroversial, and since intuitionists define the truth of a proposition as the existence of a proof of it, it seems to follow that truth is also an epistemic notion. In his paper, Prawitz argues that Martin-Löf has changed his mind on this issue. Originally he held to the standard intuitionist view that both proof and truth are epistemic concepts. But this does not seem to be Martin-Löf's present opinion. Martin-Löf makes a distinction between two senses of proof, one ontological and one epistemological. Proofs in the ontological sense he calls *proof-objects*, and proofs in the epistemological sense he calls *demonstrations*. What intuitionists refer to as "proofs" in their explanations of meaning and truth for propositions should properly be understood as proof-objects, not demonstrations. But then it seems to follow that both truth and proof in intuitionism are ontological concepts. Prawitz gives a critical examination of what he takes to be Martin-Löf's reasons for adopting this view.

Two papers in this collection concern Wittgenstein's philosophy of mathematics. Sören Stenlund discusses Wittgenstein's philosophy of mathematics in the "middle period" (roughly 1929–1936). Stenlund is concentrating on the change in Wittgenstein's thinking that takes place mainly in the beginning of the 1930s. By examining certain crucial features in this change, he tries to show that Wittgenstein received decisive impulses and ideas from new developments in mathematics and natural science at the time. Hertz, Hilbert, and Einstein are important sources of inspiration. Stenlund argues that these ideas affected not only Wittgenstein's thinking about mathematics but also his thinking about language and the nature of philosophy in general.

Juliet Floyd discusses a series of remarks that Wittgenstein wrote on July 30, 1947. It begins with the remark "Turing's 'machines': these machines are humans who calculate. And one might express what he says also in the form of games". Immediately after his remark about "Turing's machines", Wittgenstein formulates what he calls a "variant" of Cantor's diagonal proof. Wittgenstein's argument has the form of a language game that involves a rule, which is circular and cannot be executed. Floyd presents and assesses Wittgenstein's variant, claiming that it is a distinctive form of proof, and an elaboration rather than a rejection of Turing or Cantor.

In Sambin's article, we find a discussion on how "real" and "ideal" are treated differently in relation to constructive and classical mathematics. He shows that the communication between the two is much stronger in the constructive stance and exemplifies with constructive topology and his and Maietti's so-called Minimalist Foundations.

Jan Smith's paper is a contribution to evolutionary epistemology. The question discussed here is whether our ability to reason logically and develop mathematics can be explained in evolutionary terms. Smith also poses the question, "Given that evolution explains why there is mathematics, can it single out any of the views on the foundations of mathematics as the correct one?" His answer is negative, "Although Formalism, Platonism and Intuitionism have very different explanations of mathematics, it seems to me to be possible for a devotee of any of them to argue for an evolutionary origin".

3.2 Part II: Foundations

We now turn to the contributions to the second part of the book.

The first constructive versions of Zermelo-Fraenkel Set Theory arose in the work of Harvey Friedman and John Myhill. Myhill introduced a theory called Constructive Set Theory intended to be able to formalize Bishop-style constructive mathematics. It was based on intuitionistic logic and avoided impredicative construction principles such as power sets and full separation. A convincing argument for it being truly constructive and predicative (in the generalized sense) was given by Peter Aczel, who constructed a model of the theory inside Martin-Löf's Intuitionistic Type Theory (MLTT). Based on this model, he developed the axioms further to what is now commonly called Constructive ZF (CZF), or Aczel-Myhill Set Theory, and taken to be one of the standard systems for formalizing constructive mathematics. Aczel's model, viewing sets as (infinitary) trees and their equality governed by bisimilarity, has proved very fruitful and flexible. It gives a standard method for comparing type theories and set theories. Rathjen's contribution "Constructive Zermelo-Fraenkel Set Theory, Power Sets and the Calculus of Constructions" deals with the extension of CZF with power sets and its relation to a type theory akin to Coquand and Huet's Calculus of Constructions, as well as an extension of Kripke-Platek set theory.

Väänänen argues in the contribution "Second Order Logic, Set Theory and Foundations of Mathematics" that the difference between classical second-order logic and set theory is, contrary to widespread belief, illusory when it comes to questions of categoricity.

The view that Category Theory gives a good realization of structuralist philosophy of mathematics has been well defended by several authors such as Colin McLarty and Steve Awodey. A first case demonstrating this was Lawvere's

Elementary Theory of the Category of Sets (ETCS) from 1964. In the contribution by Palmgren, a constructive version of ETCS is investigated. It is related to Bishop's informal set theory that was presented by Bishop in 1967. Bishop was essentially taking a type-theoretic view of sets, namely that a set is a type together with a defined equivalence relation, that is, in modern parlance a *setoid*. The setoids of a type theory behaves largely as sets from a category-theoretic point of view, depending of course on axioms on the underlying type theory. A perhaps surprising feature of MLTT is that the *axiom of choice is a theorem* of the theory. It is a triviality when understood that it refers to choices over types and that the existential quantifier is interpreted by the Σ-type construction. However, to make this version of choice valid for setoids, one needs a uniform method of introducing minimal equivalence relations on types. This is what the identity types of MLTT do. These constructions do more than was probably intended: the identity types of standard (intensional) MLTT introduce a natural groupoid structure on types, as was discovered by Hofmann and Streicher in 1994. A groupoid is the category where all arrows are invertible. It can be considered as a common generalization of a group and an equivalence relation and figures prominently in Homotopy Theory, as groupoids of paths in space. Around 2005–2006 Steve Awodey and Vladimir Voevodsky discovered, independently, a deeper and surprising relation between homotopy theory and MLTT. Awodey gives in his contribution "Type Theory and Homotopy Theory" an overview of this active field of research.

There are at least two major classes of models for constructive systems. One may be called the realizability class, which includes the Brouwer-Heyting-Kolmogorov interpretation, and which often give computational interpretation to the system. Another may loosely be called the forcing class of models, which include Kripke- and Beth-semantics and more generally sheaf models. Models of this class are often devoid of direct computational sense, but are instead capable of expressing epistemic states. However, Coquand and Jaber show in their contribution that certain forcing models of type theory can indeed be given natural computational interpretations.

Girard started his programme Geometry of Interaction to give interpretations of dynamical aspects of logic, such as Gentzen's cut-elimination, in terms of operator algebras. In his contribution "Normativity in logic", he considers the application of this method to an example of computational complexity.

The meaning explanations of MLTT are discussed in Dybjer's contribution to this volume and are considered from a program testing point of view. A type-theoretic judgement is interpreted as a conjecture about program correctness that can be tested by computing the output of the program for all inputs that are possible to generate. Furthermore, testing of higher-order functions becomes an interactive process akin to game semantics.

Logical semantics of natural languages was pioneered by Frege and further developed in detail by Montague using an extension of classical simple type theory, Montague's intensional logic. Aarne Ranta started, as a student of Martin-Löf, to investigate what MLTT with its rich structure of contexts and dependent types could say more about natural languages. Notable successes include the treatment of anaphora and discourses. Ranta has since then developed a type theory, the Grammatical Framework (GF), adapted to describe languages and to make machine translations between multiple languages. In his contribution, he gives an introduction to machine translation and GF.

The Lorenzen-Prawitz inversion principle for natural deduction gives a way of deriving the elimination rules from given introduction rules. This is an important principle used as well in Martin-Löf type theory. In Setzer's contribution to this volume, he considers an inversion principle that goes in the opposite direction, generating introduction rules from elimination rules. For the so-called coinductively defined types – a typical example is a potentially infinite stream of data – this is shown to be the natural approach. Meaning explanations are given for this new class of types.

Acknowledgments

We thank the Swedish Collegium for Advanced Study (SCAS) in Uppsala and the Centre for Interdisciplinary Mathematics at Uppsala University for invaluable organizational support in connection with the Uppsala Conference held in May 2009 dedicated to Per Martin-Löf. We are especially grateful to the director of SCAS, Professor Björn Wittrock, for supporting the project throughout and to the people at SCAS for all their efforts in connection with the conference. We are grateful to Professor Dag Prawitz (Stockholm University), Professor Sören Stenlund (Uppsala University), and Professor Viggo Stoltenberg-Hansen (Uppsala University) for scientific advice and support. We also wish to thank Anna Eriksson-Trenter, Anton Hedin, Tore Hållander, Kristina Lindgren, and Olov Wilander for all their help with organizational matters. We gratefully acknowledge the financial support from the Swedish Research Council, The Royal Swedish Academy of Sciences, The Royal Swedish Academy of Letters, History and Antiquities, and the Departments of Mathematics and the Departments of Philosophy in Stockholm and Uppsala.

Göteborg	Peter Dybjer
Umeå	Sten Lindström
Stockholm	Erik Palmgren
Leiden	Göran Sundholm

On the Philosophical Work of Per Martin-Löf

Göran Sundholm

Per Martin-Löf began his work on logic and foundational issues in 1966 with a definition of the notion of a random sequence that has become classic. It was continued with the doctoral dissertation *Notes on Constructive Mathematics* that was written in 1968. Martin-Löf spent the academic year 1968–1969 in the United States, first at Chicago where, in December 1968, W. A. Howard gave him a copy of the handwritten manuscript *The formulae-as-types notion of construction* (that was subsequently published (1980) in the Curry Festschrift). In it an isomorphism à la Curry is established between axiomatic Hilbert-style systems for predicate logic and arithmetic and matching calculi in combinatory logic. The correspondence struck Martin-Löf as being of profound significance and he was determined to understand it fully. His first contribution to this end, as pointed out by Howard in a note added to the published paper, was to carry over the idea from Howard's Hilbert-style calculi to the framework of Gentzen's Natural Deduction that was known to him from Dag Prawitz's (1965) dissertation. In Martin-Löf's formulation, proof-theoretic reductions of natural-deduction derivations correspond to conversions of terms in an enriched lambda calculus. The now customary "Curry-Howard" isomorphism between natural-deduction derivations and the terms of a matching lambda-calculus (rather than the combinatory logic that was used by Howard) was then written up in a paper on *Infinite terms and a system of natural deduction* that was circulated in March 1969. Armed with the insight that natural-deduction derivations and lambda-calculus terms are essentially equivalent, Martin-Löf began a search for the optimal way of proving normalization results for systems of natural deduction. His investigation of iterated generalized inductive definitions, which was completed by March 1970, carried over the computability method of W. W. Tait (1967) to proofs of normalization for natural-deduction derivations, and the paper was presented at the *Second Scandinavian Logic Symposium* at Oslo in June 1970. Kreisel (1975, p. 177, footnote 6) reports that when travelling – with Prawitz and Martin-Löf – by train to Oslo, he mentioned J.-Y. Girard's very recent, but still unpublished, work on how to extend Gödel's *Dialectica* interpretation to second-order arithmetic ("Analysis") and gave Girard's manuscript to Martin-Löf, who already at the Oslo meeting convinced himself that Girard's insight would carry over into a normalization theorem also for second-order logic, giving a procedure of normalization, rather than completeness of the cut-free sequent-calculus rules for second-order logic (as in Prawitz's (1967) elegant demonstration of Takeuti's conjecture). After the Oslo meeting, Girard, Martin-Löf, and Prawitz all gave normalization proofs for second-order logic that were published in its *Proceedings* (Fenstad (1971)), and Girard (1971, p. 64) confirms that "les remarques de M. Martin-Löf sur la possibilité d'une démonstration de normalisation de l'Analyse ... on été déterminantes pour la suite de ce travail". Martin-Löf quickly extended the computability approach also to the intuitionistic simple theory of types (1970c). A characterization of the provable well-orderings of the theory of species also belongs to this period (1970d) of great creativity. Finally, in the autumn of 1970, the first system of Intuitionistic Type Theory was designed and presented in a seminar lecture at Stockholm as a deliberate attempt to extend the computability approach that had so amply demonstrated its worth at the Oslo conference to even stronger systems. Its main building blocks were Gentzen's natural-deduction style of formalization, with proof-theoretic reduction steps from Prawitz (1965), Gödel's *Dialectica* system T (1958), the Curry-Howard isomorphism, and Tait's (1967) computability method for normalization proofs. The written presentation, in a preprint dated February, revised October (1971a), was submitted for publication to *Acta Mathematica*. Martin-Löf's invited lecture at the Bucharest fourth LMPS conference in 1971b was also devoted to his novel intuitionistic type theory. However, both submissions were withdrawn from publication pursuant to Girard's discovery that the extreme impredicativity of the system allowed for the derivation of a version of Burali-Forti's paradox. Martin-Löf's revised version of his Intuitionistic Type Theory from 1972 was finally published in the proceedings of the conference at Venice that was held in 1995 to commemorate *Twenty-five Years of Constructive Type Theory*, and the proceedings of the 1973 Bristol Logic Colloquium contain the first published presentation of a – predicative – version of the type theory.

The task that faced a mathematical logician from Frege's days until 1930 was a challenging one: to graft the Fregean notion of a formal system onto the Aristotelian conception of demonstrative science. That is, one should design a sizeable formal system with clearly delimited axioms and rules of inference. The system should be adequate for the needs of analysis after the then novel fashion of Weierstraß and Dedekind. In particular, it should admit of classical logic (as well as impredicative quantification). The axioms and (primitive) rules of inference should be rendered immediately evident from the meaning explanations for the (primitive signs of the) formal language of the formal system in question. Frege, Whitehead-Russell, Lesniewski, and the early logical works of Curry, Church, and Quine, are good examples here; as is well known, the early logicist attempts were not successful, owing to the use of axioms, such as Reducibility, that were not rendered evident by the relevant meaning-explanations. Carnap's contribution to the famous conference on the philosophy and foundations of mathematics at Königsberg in November 1930 can be seen as the last stand of Logicism. Thereafter, it seemed clear that Logicism was no longer a live option. This meant that the foundations of mathematics were faced with the two horns of a dilemma: either we retain classical logic in a mathematical object-language, but give up hope for meaning explanations, or we insist on retaining a contentual language with meaning explanations, but have to jettison classical logic. Hilbert in his programme chose the first option, whereas the second one was preferred by Brouwer and elaborated by Heyting in his articles on the formalization and interpretation of intuitionist logic.

Martin-Löf's first logical writings belong to the metamathematical paradigm. The formal languages and systems dealt with are principally objects of metamathematical study, and in that spirit normalization *theorems* are established for the early versions of type theory and other systems. However, in 1974, stimulated by reading Wittgenstein, as well as by listening to Michael Dummett's lecture *The Philosophical Basis of Intuitionistic Logic* at the Bristol Logic Colloquium 1973, Martin-Löf turned to the theory of meaning, and thereby brought type theory within the contentual approach in logic. Peter Hancock's words in the lectures that marked the meaning-theoretical turn (1975b, p. 13) speak with quiet confidence and could be used unchanged also today:

> Especially in its later stages, Martin-Löf's work has been developed with the principal aim that it should admit a detailed and coherent semantics. We are not concerned here, except by implication, to subject other languages to a destructive criticism. We are ignorant of a comparable project that has been carried through for a different language. So we do not have to say at this stage why our account is to be preferred to another. You will just have to make up your mind about that if and when there *is* another account.

He has devoted himself ever since to the realization of the second horn of the above dilemma in the foundations of mathematics with a project that is comparable to that of Frege's *Grundgesetze*, but without classical logic and impredicativity: design a full-scale formal language, with explicit meaning-explanations, that is adequate for the needs of mathematical analysis in the style of Errett Bishop's (1967) revolutionary constructive presentation. In the first two papers after his meaning-theoretical turn, Martin-Löf made an experimental attempt to view the natural-deduction *elimination* rules as basic, or meaning giving. A trace of this approach can be found in the sole reference (1991, p. 280) to Martin-Löf in Michael Dummett's William James lectures (that were given at Harvard in 1976, shortly after Martin-Löf's course at All Souls in Michaelmas Term 1975, and was later published as Dummett (1975)). The paper (1975a), written jointly with Peter Hancock, on primitive recursive arithmetic, has attracted a measure of attention also as a contribution to the proper interpretation of mathematics in Wittgenstein's *Tractatus*, whereas the lecture notes (1975b) gave meaning explanations for the language of type theory. The experiment with the elimination rules as basic was soon abandoned, though, and the lecture notes left uncompleted, after it became clear that the approach could not be carried through. In Martin-Löf's (1979) lecture at LMPS 6 in Hannover, the pattern of meaning explanations based on the introduction-rule constructors that yield canonical proof-objects is introduced. These explanations have essentially remained constant ever since, particularly in the Padova lecture notes from 1980 by Giovanni Sambin that were published by Bibliopolis. (The only notable change from the expositions in 1979 and 1980 lies in Martin-Löf's current use of *intensional* identity rather than the previous extensional one.) In this mature form, Martin-Löf's constructive type theory, from the point of view of the foundations of mathematics, constitutes an impressive, mathematically precise rendering of the BHK meaning-explanations that were first given by Arend Heyting in 1930.

Around this time, that is, the late 1970s, Martin-Löf also began a study of the philosophy of Edmund Husserl and from then on the phenomenological perspective has been a rich source of inspiration. In conversation Martin-Löf has indicated that he regards his syntactic-semantic method of logical exploration as a version of phenomenology. To this time belongs also a second period of experimentation, in which Martin-Löf attempted to avoid the use of type-theoretical abstractions, and instead explored the alternative of working within predicate logic, eschewing the use of proof-objects and the form of judgement a: proof(A), working instead with the predicate logic form A true as principal form of judgement. This work culminated in a set of influential lectures that were given in 1983 at Siena, on which basis a compact course was published, and where considerable systematic effort was spent on relating the work to issues in the philosophy of logic. With the creation of the higher type structure, Martin-Löf returned to a type-theoretical formulation of his ideas. The full language of type theory, now using both sets and types, has been in place since 1986, when it was first presented in a lecture at Edinburgh. A lecture series given at Florence in 1987 gave meaning explanations also for the higher type theory. It was further extended in 1992 with a calculus of explicit substitutions in order to make the treatment fully formal. In 1993, Martin-Löf gave a semester-long series of lectures on the *Philosophical Aspects of Type Theory* at Leiden, in which he presented the meaning explanations in full detail and dealt with a number of topics from the philosophy of logic and language.

Martin-Löf's mature philosophical outlook is characterized by three main tenets that make it unique among contemporary philosophical positions within the foundations of mathematics. First, and most significantly, constructive type theory is an *interpreted formal language*. The importance of this cannot be stressed firmly enough. Today, as a rule, the metamathematical "expressions" employed in mathematical logic are mere objects of study, but do not express. On the contrary, they are objects that may serve as referents of real expressions. In constructive type theory, on the other hand, the expressions used are real expressions that carry meaning. In a nutshell, the language is endowed with meaning by turning the proof-theoretic reductions into steps of meaning explanation. Just like the formulae of Frege's ideography, or of the language of *Principia Mathematica*, the type-theoretic formulae are actually intended to say something. They do not essentially serve as objects of metamathematical study. This, of course, in no way precludes Martin-Löf's contentual system from being studied metamathematically.

Secondly, Martin-Löf has restored the notion of *judgement* to pride of place in logic. In the metamathematical tradition, the same well-formed formulae serve in different roles: wff's are built up from wff's that have been generated earlier, for instance, when φ and χ are wff's, then so is $(\varphi \supset \chi)$. Furthermore, a wff φ may be a derived theorem, that is, the end-formula of a closed derivation (with no open assumptions). From Martin-Löf's contentual point of view, a proposition A is a set of (canonical) proof-objects. Such propositions serve as building blocks for more complex propositions. The contentual role of a derivational end-formula, however, is not propositional; here we do not have just an occurrence of the proposition A, but an *assertion* that

proposition A is true.

Martin-Löf's propositions are given via *proof conditions*, that is, to each proposition A there is associated a type proof(A) that is explained in terms of how canonical proof-objects for A may be formed and when two such proof-objects are the same. Truth of propositions is explained by the "truth-maker" analysis

proposition A is true = proof(A) exists,

where the existence in question is the constructive Brouwer-Weyl notion that is explained in terms of possession of an instance. It should be noted that these are proofs of *propositions*. Previously in the history of logic, all proving (better: all *demonstration*) took place at the level of assertions, and not at the level of propositions (that are traditionally seen as unasserted contents of theorems). Such proofs of propositions were first considered in Heyting's seminal writings from the early 1930s. Martin-Löf's crucial notion of a judgement is explained in terms of an *assertion condition* that lays down what one has to know in order to have the right to make the judgement in question. Thus for instance, in order to have the right to make the judgement that the proposition A&B is true, one has to have exhibited a proof-object c that either is of, or evaluates to, the form <a, b>, where a is a proof-object for the proposition A and b is a proof-object for the proposition B. This reintroduction of judgements into logic, with the concomitant distinction between judgements and propositions, also leads to a crucial distinction between (epistemic) *demonstrations* of judgements, and *proofs*, in the sense of proof-objects, of propositions. This distinction between propositions and judgements, and the ensuing distinction between proofs of propositions and demonstrations of judgements, also brings about a corresponding distinction between (epistemic) *inference* from premise to conclusion judgements and relations of *consequence* between antecedent and consequent propositions.

Finally, the emphasis on judgement and the acquaintance with the works of Husserl has led Martin-Löf to an uncommon epistemological perspective. In contemporary analytical philosophy, epistemology takes a very "factual", almost ontological stance. Knowledge is invariably seen from a third person perspective as an ontological state that obtains in the world and makes true propositions such that *agent A knows proposition p*. Here the main concern is not what it is to know something oneself, but rather what it is for *someone else* to know something. Thus, on the linguistic level, one is concerned with the meaning of the locution "A knows p" rather than with "I know p". Martin-Löf's approach to meaning, on the other hand, is squarely *first person*. To him every assertion by means of an utterance of a declarative sentence contains an *implicit* first-person knowledge claim. Martin-Löf explains a declarative by means of an "assertion condition" that lays down *what one has to know* in order to have the right to make the assertion by means of an utterance of the declarative in question. Accordingly, the *legitimacy* of the counter-question *How do you know that?* serves as a criterion by which assertoric uses of declarative sentences can be recognized.

The effect of this can be seen, for instance, in Moore's paradox concerning an assertion by means of:

> "It is raining, but I do not believe it."

As is well known, the use of the present tense and the first person is essential here. No paradox results in the imperfect: "It was raining, but I did not believe it." Similarly, non-assertoric occurrences of the crucial sentence are not problematic: "If it is raining, but I do not believe it, then I will get wet when I go outside." Also any third-person assertion by me, of "It is raining, but X does not believe it," where X is placeholder for the name of a person, is not paradoxical, even though we may choose X = Göran Sundholm here. Only the first person poses problems, owing to the implicit first-person "I know" that is contained in every assertion and that contradicts the second clause of the Moorean assertion. Martin-Löf's insistence on this kind of first-person knowledge comes out time and again in his philosophy, for instance, in the quotation above from 1975b, but perhaps most clearly in his pertinent request to the reader in the first full contentual exposition of constructive type theory (1979, p. 166):

> For each of the rules of inference, the reader is asked to try to make the conclusion evident on the presupposition that he knows the premises. This does not mean that further verbal explanations are of no help in bringing about an understanding of the rules, only that this is not the place for such detailed explanations. But there are also certain limits to what verbal explanations can do when it comes to justifying axioms and rules of inference. In the end, everybody must understand for himself.

In a series of published philosophical lectures from 1985 to 2004, Martin-Löf has explored central notions within the philosophy of logic, such as judgement, evidence, rightness, and knowledge, as well as in the philosophy of mathematics, for instance concerning the Axiom of Choice, and spelled out consequences for his views on metaphysics and epistemology. He has, however, been a frequent invited speaker at conferences and the list of topics covered in unpublished material is long: it comprises Frege's distinction between *Sinn* and *Bedeutung*, intensionality of objects, Tarski's truth definition and the notion of a model for type theory, predicativity in mathematics, propositions versus contents, categories, and general methodology in the philosophy of logic. In recent years, since his Gödel lecture *The two layers of logic* at the annual meeting of the Association for Symbolic Logic at Montréal in 2006, Martin-Löf's philosophical work has been directed to the question, Logic, epistemological or ontological? that also gave the title for his lecture at the Uppsala meeting.

Works by Per Martin-Löf

Martin-Löf has often been reticent in publishing his papers and sometimes there is a considerable delay between the distribution of the preprint version and its ultimate publication, if any. I therefore cite his works after the year of distribution rather than of publication.

Martin-Löf, Per. 1966. The definition of random sequences. *Information and Control* 9(6): 602–619.

Martin-Löf, Per. 1968. *Notes on constructive mathematics* (doctoral dissertation). Stockholm: Almqvist & Wiksell, 1970.

Martin-Löf, Per. 1969. Infinite terms and a system of natural deduction. *Compositio Mathematica* 24 (1972): 93–103.

Martin-Löf, Per. 1970a. Hauptsatz for the intuitionistic theory of iterated inductive definitions. In *Proceedings of the 2nd Scandinavian logic symposium*, ed. J.E. Fenstad, 179–216. Amsterdam: North-Holland Publishing Company, 1971.

Martin-Löf, Per. 1970b. Hauptsatz for the intuitionistic theory of species. In *Proceedings of the 2nd Scandinavian logic symposium*, ed. J.E. Fenstad, 217–233. Amsterdam: North-Holland Publishing Company, 1971.

Martin-Löf, Per. 1970c. Hauptsatz for intuitionistic simple type theory. In *Logic, methodology and philosophy of science IV, Proceedings of the fourth international congress for logic, methodology and philosophy of science, Bucharest, 1971*, ed. P. Suppes, L. Henkin and A. Joja, 279–290. Amsterdam: North-Holland Publishing Company, 1973.

Martin-Löf, Per. 1970d. A construction of the provable wellorderings of the theory of species. In *Logic, meaning and computation*, ed. A. Anthony Anderson and M. Zeleny, 343–352. Dodrecht: Kluwer, 1997.

Martin-Löf, Per. 1971a. An intuitionistic theory of types, unpublished preprint.

Martin-Löf, Per. 1971b. On the strength of intuitionistic reasoning, preprint of a contribution to the Symposium on Perspectives in the Philosophy of Mathematics, *LMPS* IV, Bucharest, September 1971.

Martin-Löf, Per. 1972. An intuitionistic theory of types. In *Twenty-five years of constructive type theory*, ed. G. Sambin and Jan J. Smith, 127–172. Oxford: Clarendon Press, 1998.

Martin-Löf, Per. 1973. An intuitionistic theory of types: Predicative part. In *Logic colloquium'73*, ed. H.E. Rose and J. Shepherdson, 73–118. Amsterdam: North-Holland Publishing Company, 1975.

Martin-Löf, Per. 1975a. Syntax and semantics of the language of primitive recursive functions (written jointly with Peter Hancock), unpublished preprint.

Martin-Löf, Per. 1975b. Syntax and semantics of mathematical language (Notes by Peter Hancock on a lecture series given at All Souls College, Oxford, Michaelmas term, 1975), unpublished preprint.

Martin-Löf, Per. 1979. Constructive mathematics and computer programming. In *Logic, methodology and philosophy of science VI, Proceedings of the 1979 international congress at Hannover, Germany*, ed. L.J. Cohen, J. Los, H. Pfeiffer and K.-P. Podewski, 153–175. Amsterdam: North-Holland Publishing Company, 1982.

Martin-Löf, Per. 1980. Intuitionistic type theory: Notes by Giovanni Sambin of a series of lectures given in Padua, June 1980. Napoli: Bibliopolis 1984.

Martin-Löf, Per. 1983. On the meanings of the logical constants and the justifications of the logical laws. *Nordic Journal of Philosophical Logic* 1(1): 11–60, 1996 ("Siena lectures").

Martin-Löf, Per. 1985. Truth of a proposition, evidence of a judgement, validity of a proof. *Synthese* 73: 407–420.

Martin-Löf, Per. 1988. Mathematics of infinity. In *Colog 88*, ed. P. Martin-Lof and G. Mints, 146–197. Berlin: Springer, 1990.

Martin-Löf, Per. 1991. A path from logic to metaphysics. In *Atti del Congresso Nuovi Problemi della Logica e della Filosofia della Scienza, Viareggio, 8–13 gennaio, 1990*, ed. G. Corsi and G. Sambin, 141–149. Bologna: CLUEB, 1991.

Martin-Löf, Per. 1992. Analytic and synthetic judgements in type theory. In *Kant and contemporary epistemology*, ed. P. Parrini, 87–99. Dordrecht: Kluwer, 1994.

Martin-Löf, Per. 1994. Verificationism then and now. In *The foundational debate: Complexity and constructivity in mathematics and physics*, ed. W. DePauli-Schimanovich et al, 187–196. Dordrecht: Kluwer, 1995.

Martin-Löf, Per. 1995. Truth and knowability: On the principles C and K of Michael Dummett. In *Truth in mathematics*, ed. H.G. Dales and G. Oliveri, 105–114. Oxford: Clarendon Press, 1998.

Martin-Löf, Per. 2004. 100 years of Zermelo's Axiom of Choice: What was the problem with it? In *Logicism, intuitionism, and formalism – What has become of them*?, ed. S. Lindström, E. Palmgren, K. Segerberg and V. Stoltenberg-Hansen, 209–219. Dordrecht: Springer, 2009.

Martin-Löf, Per. 2008. The Hilbert-Brouwer controversy resolved? In *One hundred years of intuitionism (1907–2007). The Cerisy conference*. Publications of the Henri Poincaré Archives. Science around 1900, ed. M. van Atten, P. Boldini, M. Bourdeau and G Heinzmann, 243–256. Basel: Birkhäuser, 2008.

Works by Others

Bishop, Errett. 1967. *Foundations of constructive analysis*. New York: McGraw-Hill.

Dummett, Michael. 1975. The philosophical basis of intuitionistic logic. In *Logic colloquium'73*, ed. H.E. Rose and J. Shepherdson, 5–40. Amsterdam: North-Holland Publishing Company.

Dummett, Michael. 1991. *The logical basis of metaphysics*. London: Duckworth.

Fenstad, J.E. (ed.). 1971. *Proceedings of the second Scandinavian logic symposium*. Amsterdam: North-Holland Publishing Company.

Girard, J.-Y. 1971. Une extension de l'interprétation de Gödel à l'analyse, et son application à l'élimination des coupures dans l'analyse et la théorie des types. In *Proceedings of the 2nd Scandinavian logic symposium*, ed. J.E. Fenstad, 63–92. Amsterdam: North-Holland Publishing Company, 1971.

Gödel, Kurt. 1958. Über eine bisher noch nocht benützte Erweiterung des finiten Standpunktes. *Dilalectica* 12: 280–287.

Howard, W.A. 1980. The formulae-as-types notion of construction. In *To H. B. Curry: Essays on combinatory logic, lambda calculus and formalism*, ed. J.P. Seldin and J.R. Hindley, 479–490. London: Academic.

Kreisel, Georg. 1975. Observations on a recent generalization of completeness theorems due to Schütte. In *ISILC Proof Theory Symposion, Dedicated to Kurt Schütte on the occasion of his 65th birthday. Proceedings of the International Summer Institute and Logic Colloquium, Kiel 1974. Springer Lecture Notes in Mathematics*, vol. 500, ed. J. Diller and G.H. Müller, 164–181. Berlin: Springer.

Prawitz, Dag. 1965. *Natural deduction* (doctoral dissertation). Stockholm: Almqvist & Wiksell. (Second edition at Dover Publishing Company, 2006).

Prawitz, Dag. 1967. Completeness and Hauptsatz for second order logic. *Theoria* 33: 246–258.

Prawitz, Dag. 1971. Ideas and results in proof theory. In *Proceedings of the 2nd Scandinavian logic symposium*, ed. J.E. Fenstad, 235–307. Amsterdam: North-Holland Publishing Company, 1971.

Tait, W. W. 1967. Intensional interpretations of functionals of finite type. *Journal of Symbolic Logic* 32: 198–212.

Notes on the Contributors

Steve Awodey is professor of philosophy at Carnegie Mellon University. His research is focused on Logic and Category Theory, as well as the History and Philosophy of Logic. In addition to numerous research articles, he is the author of a textbook, *Category Theory* (Oxford Logic Guides 52). His most recent work is centered on the recently discovered connection between constructive logic and homotopy theory.

Thierry Coquand is professor of computing science at the University of Gothenburg. His main research areas are constructive mathematics and intuitionistic type theory.

Peter Dybjer is professor of computing science at Chalmers University of Technology. His research is in the junction of computer science and logic with a focus on intuitionistic type theory, program verification, and programming language semantics.

Juliet Floyd is professor of philosophy at Boston University, working primarily on the interplay between logic, mathematics, and philosophy in late nineteenth and early-twentieth-century philosophy, as well as on Kant, aesthetics, and eighteenth-century philosophy. She has written articles on Kant, Frege, Russell, Wittgenstein, Quine, Rawls, and Gödel and edited (with Sanford Shieh) *Future Pasts: The Analytic Tradition in Twentieth Century Philosophy* (Oxford University Press, 2001; Oxford Scholarship Online, 2004). Her current research examines Wittgenstein's reactions to Turing in the mid-1930s.

Jean-Yves Girard is a logician working in proof theory. His contributions include a proof of strong normalization in a system of second-order logic called system F; the invention of linear logic; the geometry of interaction; and ludics. He is senior researcher (directeur de recherches) at the Centre National de la Recherche Scientifique (CNRS) affiliated with the Institute of Mathematics at Luminy.

Guilhem Jaber is a Ph.D. student at the Ecole des Mines de Nantes, working on forcing and realizability, and their application to proofs of program correctness.

Sten Lindström is professor of theoretical philosophy at Umeå University. His main research areas are philosophical logic, philosophy of language, and philosophy of mathematics.

Erik Palmgren is professor of mathematical logic at Stockholm University. His research revolves around constructive mathematics and its foundations, type theory, and categorical logic.

Dag Prawitz is professor emeritus of theoretical philosophy at Stockholm University. His main research has been in proof theory and theory of meaning.

Aarne Ranta is professor of computer science at the University of Gothenburg. He was a Ph.D. student of Per Martin-Löf in Stockholm in 1987–1990. Ranta's work is focused on the analysis of natural languages in type theory and on computer programs that apply the analysis in machine translation and human–computer interaction. Most of this work is carried out in GF, Grammatical Framework, which is a grammar formalism based on type theory.

Michael Rathjen is professor of pure mathematics at the University of Leeds, England. His main research area is mathematical logic, especially proof theory, infinitary logics, type theory, non-classical set theories, and theories based on intuitionistic logic.

Giovanni Sambin is professor of mathematical logic at the University of Padua. He wrote up the lecture notes that until this day is the only textbook by Per Martin-Löf on his constructive type theory. He put type theory into action, giving life to formal topology (i.e., topology as developed over type theory), a field which he has always worked on since then. He has always cultivated an active interest in the foundations of mathematics.

Anton Setzer is a reader at the Department of Computer Science at Swansea University. He has a Diploma (German M.Sc.) and Ph.D. in mathematics from the University of Munich (Germany), and a docentship in Mathematical Logic from Uppsala University (Sweden). He held research assistantships in Munich, Uppsala (Sweden), Gothenburg (Sweden), Leeds (England), and had longer visits at Hiroshima and Kobe (Japan), the Mittag-Leffler Institute (Stockholm), and the Newton Institute (Cambridge). He is specialized in proof theory, especially of extensions of Martin-Löf Type Theory, and on theoretical basis of the interactive theorem prover and dependently typed programming language Agda.

Jan Smith obtained his Ph.D. in mathematics from the University of Gothenburg in1978, supervised by Per Martin-Löf. His research has since then mainly been on type theory as a logical framework for programming and he is now professor in computing science at Chalmers University of Technology. Over the years, Jan Smith has remained in close contact with Per Martin-Löf on research questions related to type theory.

Wilfried Sieg is the Patrick Suppes Professor of Philosophy and Mathematical Logic at Carnegie Mellon University. He works in proof theory, philosophy and history of modern mathematics, computation theory and, relatedly, in the foundations of cognitive science. Most relevant to his paper in this volume is his editorial work concerning Kurt Gödel, David Hilbert, and Paul Bernays. As to Hilbert, he is editing (with William Ewald) Hilbert's unpublished notes for lectures on arithmetic and logic from the 1890s to the early 1930s.

Sören Stenlund is professor emeritus of philosophy at Uppsala University. He is the author of *Language and Philosophical Problems* (Routledge 1990) and of the article *Different senses of finitude: an inquiry into Hilbert's finitism* (Synthese). He has published several other books and articles on various themes in the philosophies of language, logic, and mathematics. Problems concerning the nature and history of philosophy are other themes dealt with in Stenlund's publications, some of which are available only in Swedish.

Göran Sundholm studied at Lund, Uppsala, and Oxford and has held the Chair of Logic at Leiden University since 1987, serving as Visiting Professor also at the universities of Utrecht, Siena, Stockholm, Nancy, and Paris. He has written extensively on the philosophy of mathematical constructivism, applying the perspective of Martin-Löf's Type Theory, as well as on the history of modern logic, and is a frequent speaker on such topics at international conferences.

William Tait is a professor emeritus of philosophy at the University of Chicago. He has worked in logic, principally in proof theory, and in the foundations of mathematics and its history.

Jouko Väänänen is a professor of mathematics at the Department of Mathematics and Statistics of the University of Helsinki. He is also affiliated to the Institute of Logic, Language and Computation (ILLC) of the University of Amsterdam. He specializes in mathematical logic, set theory, model theory, and foundations of mathematics. Recently he has focused on the logic of concepts of dependence and independence, and on second-order logic. His "Dependence Logic" appeared in 2007 and "Models and Games" in 2011, both published by Cambridge University Press. He is currently writing a monograph on second-order logic.

Mark van Atten is senior researcher (directeur de recherches) at the Centre National de la Recherche Scientifique (CNRS), affiliated with the Institut d'Histoire et de Philosophie des Sciences et des Techniques (IHPST), Paris. His main interests being phenomenology and the foundations of mathematics, he has published on Brouwer, Gödel, Husserl, and Leibniz.

Part I
Philosophy of Logic and Mathematics