Solid Mechanics and its Applications

Edward B. Magrab

Vibrations of Elastic Systems

With Applications to MEMS and NEMS

Springer

Vibrations of Elastic Systems

SOLID MECHANICS AND ITS APPLICATIONS
Volume 184

Series Editor: G.M.L. GLADWELL
Department of Civil Engineering
University of Waterloo
Waterloo, Ontario, Canada N2L 3GI

Aims and Scope of the Series

The fundamental questions arising in mechanics are: Why?, How?, and How much? The aim of this series is to provide lucid accounts written by authoritative researchers giving vision and insight in answering these questions on the subject of mechanics as it relates to solids.

The scope of the series covers the entire spectrum of solid mechanics. Thus it includes the foundation of mechanics; variational formulations; computational mechanics; statics, kinematics and dynamics of rigid and elastic bodies: vibrations of solids and structures; dynamical systems and chaos; the theories of elasticity, plasticity and viscoelasticity; composite materials; rods, beams, shells and membranes; structural control and stability; soils, rocks and geomechanics; fracture; tribology; experimental mechanics; biomechanics and machine design.

The median level of presentation is the first year graduate student. Some texts are monographs defining the current state of the field; others are accessible to final year undergraduates; but essentially the emphasis is on readability and clarity.

For further volumes:
http://www.springer.com/series/6557

Edward B. Magrab

Vibrations of Elastic Systems

With Applications to MEMS and NEMS

Springer

Prof. Edward B. Magrab
Department of Mechanical Engineering
University of Maryland
College Park, MD 20742
USA
ebmagrab@umd.edu

ISSN 0925-0042
ISBN 978-94-007-2671-0 e-ISBN 978-94-007-2672-7
DOI 10.1007/978-94-007-2672-7
Springer Dordrecht Heidelberg London New York

Library of Congress Control Number: 2011941768

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

For
June Coleman Magrab
Still my muse after all these years

Preface

Vibrations occur all around us: in the human body, in mechanical systems and sensors, in buildings and structures, and in vehicles used in the air, on the ground, and in the water. In some cases, these vibrations are undesirable and attempts are made to avoid them or to minimize them; in other cases, vibrations are controlled and put to beneficial uses. Until recently, many of the application areas of vibrations have been largely concerned with objects having one or more of its dimensions being tens of centimeters and larger, a size that we shall denote as the macro scale. During the last decade or so, there has been a large increase in the development of electromechanical devices and systems at the micrometer and nanometer scale. These developments have lead to new families of devices and sensors that require consideration of factors that are not often important at the macro scale: viscous air damping, squeeze film damping, viscous fluid damping, electrostatic and van der Waals attractive forces, and the size and location of proof masses. Thus, with the introduction of these sub millimeter systems, the range of applications and factors has been increased resulting in a renewed interest in the field of the vibrations of elastic systems.

The main goal of the book is to take the large body of material that has been traditionally applied to modeling and analyzing vibrating elastic systems at the macro scale and apply it to vibrating systems at the micrometer and nanometer scale. The models of the vibrating elastic systems that will be discussed include single and two degree-of-freedom systems, Euler-Bernoulli and Timoshenko beams, thin rectangular and annular plates, and cylindrical shells. A secondary goal is to present the material in such a manner that one is able to select the least complex model that can be used to capture the essential features of the system being investigated. The essential features of the system could include such effects as in-plane forces, elastic foundations, an appropriate form of damping, in-span attachments and attachments to the boundaries, and such complicating factors as electrostatic attraction, piezoelectric elements, and elastic coupling to another system. To assist in the model selection, a very large amount of numerical results has been generated so that one is also able to determine how changes to boundary conditions, system parameters, and complicating factors affect the system's natural frequencies and mode shapes and how these systems react to externally applied displacements and forces.

The material presented is reasonably self-contained and employs only a few solution methods to obtain the results. For continuous systems, the governing equations and boundary conditions are derived from the determination of the contributions to the total energy of the system and the application of the extended Hamilton's principle. Two solution methods are used to determine the natural frequencies and mode shapes for very general boundary conditions, in-span attachments, and complicating factors such as in-place forces and elastic foundations. When possible, the Laplace transform is used to obtain the characteristic equation in terms of standard functions. For these systems, numerous special cases of the very general solutions are obtained and tabulated. Many of these analytically obtained results are new. For virtually all other cases, the Rayleigh-Ritz method is used. Irrespective of the solution method, almost all solutions that are derived in this book have been numerically evaluated by the author and presented in tables and annotated graphs. This has resulted in a fair amount of new material.

The book is organized into seven chapters, six of which describe different vibratory models for micromechanical systems and nano-scale systems and their ranges of applicability. In Chapter 2, single and two degree-of-freedom system models are used to obtain a basic understanding of squeeze film damping, viscous fluid loading, electrostatic and van der Waals attractive forces, piezoelectric and electromagnetic energy harvesters, enhanced piezoelectric energy harvesters, and atomic force microscopy. In Chapters 3 and 4, the Euler-Bernoulli beam is introduced. This model is used to determine: the effects of an in-span proof mass and a proof mass mounted at the free boundary of a cantilever beam; the applicability of elastically coupled beams as a model for double-wall carbon nanotubes; its use as a biosensor; the frequency characteristics of tapered beams and the response of harmonically base-driven cantilever beams used in atomic force microscopy; the effects of electrostatic fields, with and without fringe correction, on the natural frequency; the power generated from a cantilever beam with a piezoelectric layer; and to compare the amplitude frequency response of beams for various types of damping at the macro scale and at the sub millimeter scale. Also determined in Chapter 3 is when a single degree-of-freedom system can be used to estimate the lowest natural frequency a beam with a concentrated mass and when a two degree-of-freedom system can be used to estimate the lowest natural frequency of a beam with a concentrated mass to which a single degree-of-freedom system is attached.

In Chapter 5, the Timoshenko theory is introduced, which gives improved estimates for the natural frequency. One of the objectives of this chapter is to numerically show under what conditions one can use the Euler-Bernoulli beam theory and when one should use the Timoshenko beam theory. Therefore, many of the same systems that are examined in Chapter 3 are re-examined in this chapter and the results from each theory are compared and regions of applicability are determined.

The transverse and extensional vibrations of thin rectangular and annular circular plates are presented in Chapter 6. The results of extensional vibrations of circular plates have applicability to MEMS resonators for RF devices. In the last chapter, Chapter 7, the Donnell and Flügge shell theories are introduced and used to obtain approximate natural frequencies and mode shapes of single-wall and double-wall carbon nanotubes. The results from these shell theories are compared to those predicted by the Euler-Bernoulli and Timoshenko beam theories.

I would like to thank my colleagues Dr. Balakumar Balachandran for his encouragement to undertake this project and his continued support to its completion and Dr. Amr Baz for his assistance with some of the material on beam energy harvesters. I would also like to acknowledge the students in my 2011 spring semester graduate class where much of this material was “field-tested.” Their comments and feedback led to several improvements.

College Park, Maryland Edward B. Magrab

Contents

Chapter 1
Introduction

1.1 A Brief Historical Perspective

It is likely that the early interest in vibrations was due to the development of musical instruments such as whistles and drums. In was in modern times, starting around 1583, when Galilei Galileo made his observations about the period of a pendulum, that the subject of vibrations attracted scientific scrutiny. In the 1600's, strings were analyzed by Marin Mersenne and John Wallis; in the 1700's, beams were analyzed by Leonhard Euler and Daniel Bernoulli and plates were analyzed by Sophie Germain; in the 1800's, plates were analyzed by Gustav Kirchhoff and Simeon Poisson, and shells by D. Codazzi and A. E. H. Love. A complete historical development of the subject can be found in (Love, 1927). Lord Rayleigh's book *Theory of Sound*, which was first published in 1877, is one of the early comprehensive publications on the subject of vibrations. Since the publication of his book, there has been considerable growth in the diversity of devices and systems that are designed with vibrations in mind: mechanical, electromechanical, biomechanical and biomedical, ships and submarines, and civil structures. Along with this explosion of interest in quantifying the vibrations of systems, came great advances in the computational and analytical tools available to analyze them.

1.2 Importance of Vibrations

Vibrations occur all around us. In the human body, where there are low-frequency oscillations of the lungs and the heart and high-frequency oscillations of the larynx as one speaks. In man-made systems, where any unbalance in machines with rotating parts such as fans, washing machines, centrifugal pumps, rotary presses, and turbines, can cause vibrations. In buildings and structures, where passing vehicular, air, and rail traffic or natural phenomena such as earthquakes and wind can cause oscillations.

In some cases, oscillations are undesirable. In structural systems, the fluctuating stresses due to vibrations can result in fatigue failure. When performing precision measurements such as with an electron microscope externally caused oscillations

E.B. Magrab, *Vibrations of Elastic Systems*, Solid Mechanics and Its Applications 184,
DOI 10.1007/978-94-007-2672-7_1, © Springer Science+Business Media B.V. 2012

must be substantially minimized. In air, roadway, and railway vehicles, oscillatory input to the passenger compartments must be reduced. In machinery, vibrations can cause excessive wear or cause situations that make a device difficult to control. Vibrating systems can also produce unwanted audible acoustic energy that is annoying or harmful.

On the other hand, vibrations also have many beneficial uses in such widely diverse applications as vibratory parts feeders, paint mixers, transducers and sensors, ultrasonic devices used in medicine and dentistry, sirens and alarms for warnings, determining fundamental properties of materials, and stimulating bone growth.

During the last decade or so, there has been a increase in the development of electromechanical devices and systems at the micrometer and nanometer scale. The introduction of these artifacts at this sub millimeter scale has created renewed interest in the vibrations of elastic systems. These developments have lead to new families of devices and sensors such as vibrating cantilever beam mass sensors, piezoelectric beam energy harvesters, carbon nanotube oscillators, and vibrating cantilever beam sensors for atomic force microscopes. Along with these devices come additional effects that are important at this scale such as viscous air damping, squeeze film damping, electrostatic attraction, and the size and location of a proof mass. Thus, the range of applications that the vibration of elastic systems has to consider has been increased.

1.3 Analysis of Vibrating Systems

The analyses of systems subject to vibrations or designed to vibrate have many aspects. Typically, a system is designed to meet a set of vibration performance criteria such as to oscillate at a specific frequency, avoid a system resonance, operate at or below specific amplitude levels, have its response controlled, and be isolated from its surroundings. These criteria may involve the entire system or only specific portions of it. To determine if the performance criteria have been met, experiments are performed to determine the characteristics of the input to the system, the output from the system, and the system itself. Some of the characteristics of interest could be whether the input is harmonic, periodic, transient, or random and its respective frequency content and magnitude. Some of the characteristics of the output of the system could be the magnitude and frequency content of the force, velocity, displacement, acceleration, or stress at one or more locations. Some of the characteristics of the system itself could be its natural frequencies and mode shapes and its response to a specific input quantity.

To design a system to meet its performance criteria, it is often necessary to model the system and then to analyze it in the context of these criteria. The type of model one uses may be a function of its size: the sub micrometer scale, micrometer scale, millimeter scale, or the centimeter scale and greater, which we denote as the macro scale. The model will also be a function of its shape, the way in which it is expected to oscillate, the way it is supported, and how it is constrained. If shape can be ignored, then the system can be modeled as a spring-mass system. If geometry is important, then one must choose an appropriate representation such as a beam, plate, or shell and decide if the geometry can be treated as a constant geometry or if it must be treated as a system with variable geometry. The system's environment, in conjunction with its size, will determine which type of damping is important and if it must be taken into account. The model may also have to include the effects of any attachments to its interior and to its boundaries and may have to account for externally applied constraints and forces such as an elastic foundation, in-plane forces, and coupling to other elastic systems. Thus, there are many decisions that must be made with regard to what should be included in the model so that it adequately represents the actual system.

1.4 About the Book

The main goal of the book is to take the large body of material that has been traditionally applied to modeling and analyzing vibrating elastic systems at the macro scale and apply it to vibrating systems at the micrometer and nanometer scale. The models of the vibrating elastic systems that will be discussed include single and two degree-of-freedom spring-mass systems, Euler-Bernoulli and Timoshenko beams, thin rectangular and annular plates, and cylindrical shells. A second goal is to present the material in such a manner that one is able to select the least complex model that can be used to capture the essential features of the system being investigated. The essential features of the system could include such effects as in-plane forces, elastic foundations, an appropriate form of damping, in-span attachments and attachments to the boundaries, and such complicating factors as electrostatic attraction, piezoelectric elements, and elastic coupling to another system. To assist in the model selection, a very large amount of numerical results has been generated so that one is able compare the various models to determine how changes to boundary conditions, system parameters, and complicating factors affect the natural frequencies and mode shapes and the response to externally applied displacements and forces.

In order to be able to cover the wide range of models and complicating factors in sufficient detail, an efficient means of presenting the material is required. The approach employed here has been to obtain an expression for the total energy of each model and then to use the extended Hamilton's principle to derive the governing equations and boundary conditions. The expression for the total energy of the system includes the effects of any complicating factors. In addition to providing an efficient and consistent way in which to obtain the governing equations and boundary conditions, the expression for the total energy of the system can be used directly as the starting point for the Rayleigh-Ritz method. Another advantage of the energy approach is that the results given here can be extended to systems that include other effects by modifying the expression for the total energy. The expressions used to

Table 1.1 The elastic systems considered in this book. Typical MEMS and NEMS applications of these systems are described in Table 1.2

System		Additional factors	Cross section	Boundary attachments	In-Span attachments
Spring-Mass	Single degree-of-freedom	Damping: structural, viscous, squeeze film, viscous fluid Electrostatic force van der Waals force Magnetic force Piezoelectric element	–	–	–
	Two degree-of-freedom	Piezoelectric element	–	–	–
Beams	Euler-Bernoulli theory	Damping: structural, viscous, squeeze film, viscous fluid, viscous air Axial force Elastic foundation Electrostatic force Elastic coupling to another beam Layered beams	Constant Continuously variable Constant with abrupt change in properties	Translation spring Torsion spring Concentrated mass Extended mass	Translation spring Concentrated mass Single degree-of-freedom system Finite-length rigid mass
	Timoshenko theory	Axial force Elastic foundation Elastic coupling to another beam	Constant Continuously variable Constant with abrupt change in properties	Translation spring Torsion spring Concentrated mass	Translation spring Torsion spring Concentrated mass Single degree-of-freedom system
Thin Plates	Rectangular	–	Constant	–	Concentrated mass
	Circular	In-plane force Elastic foundation Extensional oscillations	Constant	Translation spring Torsion spring Concentrated mass	Concentrated mass
Thin Cylindrical Shells	Donnell's theory Flügge's theory	Elastic coupling to another shell	Constant	–	–

arrive at the governing equations and boundary conditions will be the same. A list of the elastic systems and their additional factors that are considered in this book to model microelectromechanical and nano electromechanical systems are given in Table 1.1 and the corresponding specific applications associated with these elastic systems are given in Table 1.2.

To make the application of the energy approach more efficient, an appendix, Appendix B, is provided with a general derivation of the extended Hamilton's principle for systems with one or more dependent variables and it is shown there the conditions required in order for one to be able to generate orthogonal functions. Since a primary solution method employed in this book is the separable of variables, the generation and use of orthogonal functions is very important. Consequently, the use of energy approach, the application of the extended Hamilton's principle, and the results of Appendix B provide the basis for a consistent approach to deriving the governing equations and boundary conditions and the basis for two very powerful solution techniques: the generation of orthogonal functions and the separation of variables and the Rayleigh-Ritz method. It will be seen that a major advantage of the use of the extended Hamilton's principle is that the boundary conditions are a natural consequence of the method. This will prove to be very important when the Timoshenko beam theory, thin plate theory, and thin cylindrical shell theories are considered. In these cases, obtaining the boundary conditions can be quite involved if the force balance and moment balance methods are used.

To determine the effects that various parameters and complicating factors have on a system, the following procedure is employed. For each elastic system, a solution for a very general set of boundary conditions and complicating factors as is practical is obtained. Once the general solution has been obtained, many of its special cases are examined in a direct and straightforward manner. This approach, while

Table 1.2 Typical MEMS and NEMS application areas of the elastic systems described in Table 1.1

System		Typical MEMS and NEMS Applications
Spring-Mass	Single degree-of-freedom	Piezoelectric and magnetic energy harvesters Atomic force microscopy
	Two degree-of-freedom	Enhanced piezoelectric energy harvester Filters Atomic force microscopy
Beams	Euler-Bernoulli theory	Biosensors Effects of proof mass Piezoelectric energy harvester Atomic force microscopy Electrostatic devices
	Timoshenko theory	Single- and double-wall carbon nanotubes
Thin Plates	Rectangular	–
	Circular	RF devices
Thin Cylindrical Shells	Donnell's theory Flügge's theory	Single- and double-wall carbon nanotubes

introducing a little more algebraic complexity at the outset, is a very efficient way of obtaining a solution to a class of systems and greatly reduces the need to re-solve and/or re-derive the equations each time another combination of factors is examined. In most cases, many of the systems' special cases are listed in tables. As a consequence, in several cases, new analytical results have been obtained.

In order to be able to use the least complex model to represent a system, each subsequent system is compared to a simpler model. For example, the conditions under which a beam with a concentrated mass can be modeled as a single degree-of-freedom system are determined. Other examples are the determination of the conditions when a beam can be used to model a narrow thin plate and when the Euler-Bernoulli beam theory can be used instead of the Timoshenko beam theory.

An underlying aspect that allows one to present the large amount of material given in this book is the availability of the modern computer environments such as MATLAB® and Mathematica®. These programs permit one to devote less space to presenting special numerical solution techniques and more space to the development of the governing equations and boundary conditions, obtaining the general solutions, and presenting and discussing the numerical results. Consequently, virtually all solutions that are derived in this book have been numerically evaluated. This has produced a substantial amount of annotated graphical and tabular results that illustrate the influence that the various system parameters have on their respective responses. Many of these numerical results are new. In addition, the numerical results are presented in terms of non dimensional quantities making them applicable to a wide range of systems.

Reference

Love AEH (1927) A treatise of the mathematical theory of elasticity, 4th edn. Dover, New York, NY, pp 1–31

Chapter 2
Spring-Mass Systems

The single degree-of-freedom system subject to mass and base excitation is used to model an elastic system to determine the frequency-domain effects of squeeze film air damping and viscous fluid damping. This model is also used to determine the important response characteristics of electrostatic attraction and van der Waals forces, the maximum average power from piezoelectric and electromagnetic coupling, and to illustrate the fundamental working principle of an atomic force microscope. The two degree-of-freedom system is introduced to examine microelectromechanical filters, atomic force microscope specimen control devices, and as a means to increase the input to piezoelectric energy harvesters. An appendix gives the details of the derivation of a hydrodynamic function that expresses the effects of a viscous fluid on a vibrating cylinder.

2.1 Introduction

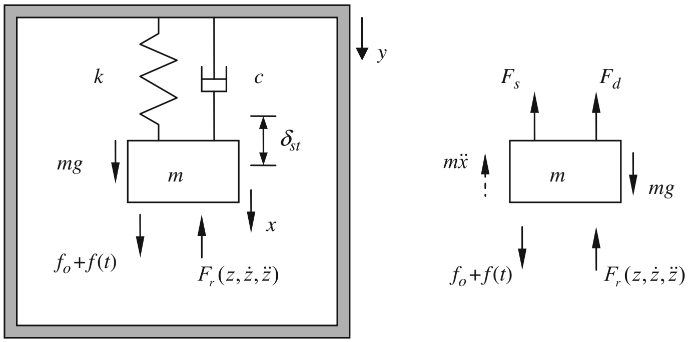
In determining the response of structural systems in the subsequent chapters, it will be seen that the different models frequently reduce to that of a set of single degree of freedom systems. Thus, a basic understanding of the response of single degree-of-freedom systems in general and its response when the system is subjected to various complicating factors such as squeeze film damping, viscous fluid loading, electrostatic attraction, and piezoelectric and electromagnetic coupling is required. In this chapter, we shall analyze such systems in the absence of the structural aspects; in the subsequent chapters, the structure will be taken into account.

2.2 Some Preliminaries

2.2.1 A Brief Review of Single Degree-of-Freedom Systems

A single degree-of-freedom system is shown in Fig. 2.1. The static displacement of the mass is δ_{st}. The mass undergoes a displacement $x(t)$ and the rigid container a

E.B. Magrab, *Vibrations of Elastic Systems*, Solid Mechanics and Its Applications 184,
DOI 10.1007/978-94-007-2672-7_2,



(a) (b)

Fig. 2.1 (**a**) Vertical vibrations of a spring-mass-damper system (**b**) Free-body diagram

known displacement $y(t)$. Both of these displacements are with respect to an inertial frame. The relationship between these two displacements is

$$z(t) = x(t) - y(t)\,. \tag{2.1}$$

The mass is subjected to an externally applied constant force f_o, a time-varying force $f(t)$, and a reaction force $F_r(z, \dot{z}, \ddot{z})$. This reaction force has been introduced so that forces that are produced by such phenomena as squeeze film damping, electrostatic attraction, and viscous fluids can be straightforwardly incorporated. When the rigid container is stationary, $y(t) = 0$ and $z(t) = x(t)$.

Referring to Fig. 2.1b, a summation of forces on the mass m in the vertical direction gives

$$m\frac{d^2x}{dt^2} + F_s + F_d + F_r(z, \dot{z}, \ddot{z}) = mg + f_o + f(t) \tag{2.2}$$

where the over dot indicates the derivative with respect to the time t and $g = 9.81$ m/s^2 is the acceleration of gravity. The constant force f_o can be caused, for example, by an electrostatic attraction (see Section 2.5.2), by a pressure difference between the top and bottom surfaces of the mass, and by a magnetic force if the mass were composed of a magnetic material.

When the spring is linear, $F_s = k(z + \delta_{st})$, where k is the spring constant (N/m). The spring constant k is sometimes referred to as the derivative of the spring force since $dF_s/dz = k$. When the damper is a linear viscous damper, $F_d = c\dot{z}$, where c is the damper constant (Ns/m). For this case, Eq. (2.2) becomes

$$m\frac{d^2x}{dt^2} + c\frac{dz}{dt} + k(z + \delta_{st}) + F_r(z, \dot{z}, \ddot{z}) = mg + f_o + f(t). \tag{2.3}$$

At the MEMS and NEMS scale, viscous damping arises from different phenomena that are functions of ambient pressure and temperature, amplitude and frequency of oscillation, viscosity, and geometric characteristics. Consideration of these effects and the computation of c can be found in (Martin and Houston 2007; Bhiladvala and Wang 2004; Keskar et al. 2008; Li et al. 2006).

It is seen from Eq. (2.3) that

$$\delta_{st} = \frac{1}{k}\left(mg + f_o\right) \text{ m}. \tag{2.4}$$

From Eq. (2.1), $x(t) = z(t) + y(t)$ and, therefore, using Eq. (2.4), Eq. (2.3) can be written as

$$m\frac{d^2z}{dt^2} + c\frac{dz}{dt} + kz + F_r\left(z, \dot{z}, \ddot{z}\right) = f(t) - m\frac{d^2y}{dt^2}. \tag{2.5}$$

This equation represents the motion of the mass about the static equilibrium position.

When $y(t) = 0$, Eq. (2.5) becomes

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx + F_r\left(x, \dot{x}, \ddot{x}\right) = f(t) \tag{2.6}$$

and when the reaction force is not present, $F_r = 0$ and Eq. (2.6) simplifies to

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = f(t). \tag{2.7}$$

Equation (2.7) can be used to model torsional oscillations. If k_t is the torsional spring constant (Nm/rad), c_t the torsional viscous damping constant (Nsm/rad), θ the angular rotation of the mass (rad), J the mass moment of inertia (kg m^2), and $M(t)$ the applied external moment (Nm), then Eq. (2.7) can be written as

$$J\frac{d^2\theta}{dt^2} + c_t\frac{d\theta}{dt} + k_t\theta = M(t). \tag{2.8}$$

Before proceeding, the following definitions are introduced.

Natural Frequency—ω_n

For translating systems

$$\omega_n = 2\pi f_n = \sqrt{\frac{k}{m}} \text{ rad/s} \tag{2.9}$$

where f_n is the natural frequency in Hz. For torsional oscillations

$$\omega_n = 2\pi f_n = \sqrt{\frac{k_t}{J}} \text{ rad/s}. \tag{2.10}$$

Damping Factor—ζ

For translating systems

$$\zeta = \frac{c}{2m\omega_n} = \frac{c}{2\sqrt{km}} = \frac{c\omega_n}{2\,k}. \tag{2.11}$$

When $0 < \zeta < 1$ the system is called an underdamped system, when $\zeta = 1$ it is critically damped, and when $\zeta > 1$ it is overdamped system. When $\zeta = 0$, the system is undamped.

For torsional oscillations

$$\zeta = \frac{c_t}{2\,J\omega_n} = \frac{c_t}{2\sqrt{k_t J}}. \tag{2.12}$$

Period of Undamped Oscillations—T

$$T = \frac{1}{f_n} = \frac{2\pi}{\omega_n} \quad \text{s}. \tag{2.13}$$

We return to Eq. (2.5) and set $F_r = 0$ to obtain

$$m\frac{d^2z}{dt^2} + c\frac{dz}{dt} + kz = f(t) - m\frac{d^2y}{dt^2} \tag{2.14}$$

We now introduce Eqs. (2.9) and (2.11) into Eq. (2.14) to arrive at the following governing equation of motion in terms of the natural frequency and damping factor

$$\frac{d^2z}{dt^2} + 2\zeta\omega_n\frac{dz}{dt} + \omega_n^2 z = \frac{f(t)}{m} - \frac{d^2y}{dt^2}. \tag{2.15}$$

If we let $\tau = \omega_n t$, then Eq. (2.15) becomes

$$\frac{d^2z}{d\tau^2} + 2\zeta\frac{dz}{d\tau} + z = \frac{f(\tau)}{k} - \frac{d^2y}{d\tau^2}. \tag{2.16}$$

It is mentioned that when $f(\tau) = 0$, Eq. (2.16) can be used to describe the motion of an accelerometer, where $d^2y/d\tau^2$ is the acceleration of the base (Balachandran and Magrab 2009, p. 237).

2.2.2 General Solution: Harmonically Varying Forcing

We assume that $\zeta < 1$, the initial conditions are zero, and the applied force and base displacement are of the form

$$\begin{aligned} f(t) &= F_o \cos(\Omega\tau) \\ y(t) &= Y_o \cos(\Omega\tau) \end{aligned} \tag{2.17}$$

where $\Omega = \omega/\omega_n$. It is seen that when $\omega = \omega_n, \Omega = 1$,. To obtain a solution to Eq. (2.16), we assume

$$\begin{aligned} x(\tau) &= X_o \cos(\Omega\tau) \\ z(\tau) &= Z_o \cos(\Omega\tau) \end{aligned} \tag{2.18}$$

and find that (Balachandran and Magrab 2009, pp. 671–673)

$$z(\tau) = H(\Omega)\left(\frac{F_o}{k} + \Omega^2 Y_o\right)\cos(\Omega\tau - \theta(\Omega)) \tag{2.19}$$

where

$$\begin{aligned} H(\Omega) &= \frac{1}{\sqrt{\left(1-\Omega^2\right)^2 + (2\zeta\Omega)^2}} \\ \theta(\Omega) &= \tan^{-1}\frac{2\zeta\Omega}{1-\Omega^2}. \end{aligned} \tag{2.20}$$

The quantity $H(\Omega)$ is the amplitude response and the quantity $\theta(\Omega)$ is the phase response.

It is seen from Eq. (2.20) that the frequency at which the maximum value of the amplitude response occurs is a function of ζ, as will be demonstrated subsequently. A plot of $H(\Omega)$ and $\theta(\Omega)$ is shown in Fig. 2.2. It is seen that for viscous damping, the phase angle is 90° when $\Omega = 1$, irrespective of the value of ζ.

When $Y_o = 0$, there are three frequency regions of interest based on Eq. (2.20). The first region is when $\Omega << 1$, where $H(\Omega) \cong 1$ and, from Eq. (2.19), $z(\tau) \sim 1/k$. This region is denoted the stiffness controlled region and is important in sensor design. The second region is when $\Omega = 1$, where $H(\Omega) = 1/(2\zeta)$ and $z(\tau) \sim 1/c$.

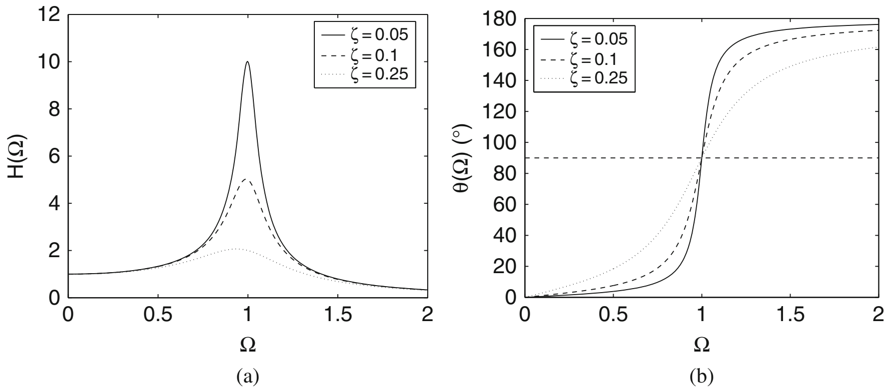


Fig. 2.2 Response of a single degree-of-freedom system with viscous damping (**a**) Amplitude response and (**b**) Phase response

This region is called the damping controlled region and is important in the design of energy harvesters. The third region is when $\Omega >> 1$, where $H(\Omega) \cong 1/\Omega^2$ and $z(\tau) \sim 1/m$. This region is called the mass controlled region and is important in the design of vibration isolators.

We now use Eq. (2.20) to define the quality factor Q.

Quality Factor—*Q*

A quantity that is often used to define the band pass portion of H(Ω) when ζ is small is the quality factor Q, which is given by

$$Q = \frac{\Omega_c}{B_w}. \tag{2.21}$$

The quantity Ω_c is the center frequency and is defined as the geometric mean frequency

$$\Omega_c = \sqrt{\Omega_{cu}\Omega_{cl}} \tag{2.22}$$

and B_w is the bandwidth given by

$$B_w = \Omega_{cu} - \Omega_{cl}. \tag{2.23}$$

where Ω_{cu} and Ω_{cl}, respectively, are the upper and lower cutoff frequencies that satisfy

$$H(\Omega_{cl}) = H(\Omega_{cu}) = \frac{H_{\max}}{\sqrt{2}}. \tag{2.24}$$

The quantity $H_{\max}$ is given by (Balachandran and Magrab, 2009, p. 211)

$$H_{\max} = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \quad \zeta \le 1\big/\sqrt{2} \tag{2.25}$$

and occurs at

$$\Omega_{\max} = \sqrt{1-2\zeta^2} \quad \zeta \le 1\big/\sqrt{2}. \tag{2.26}$$

When an explicit relation for determining the value of $H_{\max}$ does not exist, one determines its value numerically.

Using Eqs. (2.24) and (2.25), it can be shown that (Balachandran and Magrab, 2009, p. 212)

$$\begin{aligned}\Omega_{cu} &= \sqrt{1-2\zeta^2+2\zeta\sqrt{1-\zeta^2}}\\ \Omega_{cl} &= \sqrt{1-2\zeta^2-2\zeta\sqrt{1-\zeta^2}}.\end{aligned} \tag{2.27}$$

When $\zeta < 0.1$, Eq. (2.27) can be approximated by

$$\begin{aligned}\Omega_{cu} &\approx \sqrt{1+2\zeta} \approx 1+\zeta \\ \Omega_{cl} &\approx \sqrt{1-2\zeta} \approx 1-\zeta.\end{aligned} \tag{2.28}$$

Thus,

$$\begin{aligned}\Omega_c &\approx \sqrt{1-\zeta^2} \approx 1 \\ B_w &\approx 2\zeta\end{aligned}$$

and Eqs. (2.21) and (2.25) give

$$Q \approx \frac{1}{2\zeta} \approx H_{\max} \tag{2.29}$$

which overestimates the value of Q. The error made in using Eq. (2.29) relative to Eq. (2.21) is less than 3% for $\zeta < 0.1$ and when $\zeta < 0.01$ the error is less than 0.03%.

The quality factor has been shown to be of fundamental importance in the determination of the noise floor in MEMS sensors and plays a role in determining the sensitivity of certain MEMS devices (Gabrielson 1993; Levinzon 2004).

2.2.3 Power Dissipated by a Viscous Damper

The average power that is dissipated in the viscous damper per period of oscillation $T = 2\pi/\omega = 2\pi/(\omega_n\Omega)$ is

$$P_{avg} = \frac{1}{T}\int_0^T P_i dt = \frac{\Omega}{2\pi}\int_0^{2\pi/\Omega} P_i d\tau \tag{2.30}$$

where P_i is the instantaneous power given by

$$\begin{aligned}P_i &= F_d\dot{z} = c\left(\frac{dz}{dt}\right)^2 = c\omega_n^2\left(\frac{dz}{d\tau}\right)^2 \\ &= \frac{2\zeta k}{\omega_n}H^2(\Omega)\Omega^2\left(\frac{\omega_n F_o}{k} + \Omega^2\omega_n Y_o\right)^2 \sin^2(\Omega\tau - \theta(\Omega)) \ \text{W}.\end{aligned} \tag{2.31}$$

In obtaining Eq. (2.31), we have used Eqs. (2.11) and (2.19). Upon substituting Eq. (2.31) into Eq. (2.30) and performing the integration, we obtain

$$P_{avg} = \frac{\zeta k}{\omega_n}H^2(\Omega)\Omega^2\left(\frac{\omega_n F_o}{k} + \Omega^2\omega_n Y_o\right)^2 \ \text{W}. \tag{2.32}$$

The average dissipated power is a maximum at the value of $\Omega = \Omega_{\max}$ that makes

$$\frac{dP_{avg}}{d\Omega} = 0. \tag{2.33}$$

We shall determine the maximum dissipated power for two separate cases: $F_o \neq 0$ and $Y_o = 0$ and $Y_o \neq 0$ and $F_o = 0$. For the first case, we perform the operation indicated by Eq. (2.33) and find that $\Omega_{\max} = 1$. Thus, the maximum average power dissipated into the viscous damper by the external force is

$$\begin{aligned} P_{avg,\max} &= \frac{\zeta k}{\omega_n} H^2(\Omega_{\max})\,\Omega_{\max}^2 \left(\frac{\omega_n F_o}{k}\right)^2 \\ &= \frac{P_F}{4\zeta} = \frac{P_F Q}{2} \text{ W} \end{aligned} \tag{2.34}$$

where

$$P_F = \frac{\omega_n F_o^2}{k} \text{ W}. \tag{2.35}$$

For the second case, we employ Eq. (2.33) and find

$$\Omega_{\max\,2,1} = \sqrt{2\left(1 - 2\zeta^2\right) \pm \sqrt{4\left(1 - 2\zeta^2\right)^2 - 3}} \quad \zeta < 0.2588. \tag{2.36}$$

Thus, the maximum average power dissipated into the viscous damper by the rigid container's displacement is

$$P_{avg,\max} = \zeta P_Y H^2(\Omega_{\max\,1})\,\Omega_{\max\,1}^6 \text{ W} \tag{2.37}$$

where

$$P_Y = k\omega_n Y_o^2 \text{ W}. \tag{2.38}$$

When $\zeta << 1$, we see from Eq. (2.36) that $\Omega_{\max\ 1} \approx 1$. In this case, the maximum average dissipated power simplifies to

$$P_{avg,\max} = \frac{P_Y}{4\zeta} = \frac{P_Y Q}{2} \text{ W}. \tag{2.39}$$

The difference between Eqs. (2.39) and (2.37) is less than 1% when $\zeta < 0.1$ and as ζ decreases this difference deceases.

We see from Eqs. (2.34) and (2.39) that for lightly damped systems, when the input powers are equal ($P_F = P_Y$), the maximum average dissipated powers are equal and vary inversely with the damping factor. In addition, these maximum values occur very close to the system's natural frequency.

2.2.4 Structural Damping

Structural damping is a model that assumes that the dissipation in the system is due to losses in the material that provides the stiffness for the system. One structural damping model is to assume that the structural damping is independent of frequency. A model that satisfies this criterion is (Balachandran and Magrab, 2009, p. 249)

$$F_s = kz + k\frac{2\eta}{\omega}\frac{\partial z}{\partial t} \text{ N} \tag{2.40}$$

where η is an empirically determined constant. This model is restricted to systems undergoing harmonic oscillations.

To obtain the governing equation of motion, we substitute Eq. (2.40) in Eq. (2.2), set $F_r = 0$, and employ the assumptions used to arrive at Eq. (2.5). These operations yield

$$m\frac{d^2z}{dt^2} + \left(c + k\frac{2\eta}{\omega}\right)\frac{dz}{dt} + kz = f(t) - m\frac{d^2y}{dt^2}. \tag{2.41}$$

We shall limit our discussion to the case where $f(t) = 0$ and, because of the restrictions on Eq. (2.40), it is assumed that

$$y = Y_o \cos(\omega t).$$

Then Eq. (2.41) becomes

$$m\frac{d^2z}{dt^2} + \left(c + k\frac{2\eta}{\omega}\right)\frac{dz}{dt} + kz = m\omega^2 Y_o \cos(\omega t) \tag{2.42}$$

or in terms of the non dimensional parameters,

$$\frac{d^2z}{d\tau^2} + \left(2\zeta + \frac{2\eta}{\Omega}\right)\frac{dz}{d\tau} + z = Y_o\Omega^2 \cos(\Omega\tau). \tag{2.43}$$

The solution to Eq. (2.43) is

$$z(\tau) = H_{\text{v+s}}(\Omega)\cos(\Omega\tau - \theta_{\text{v+s}}(\Omega)) \tag{2.44}$$

where the amplitude response and phase response, respectively, are

$$\begin{aligned} H_{\text{v+s}}(\Omega) &= \frac{1}{\sqrt{\left(1-\Omega^2\right)^2 + (2\zeta\Omega + 2\eta)^2}} \\ \theta_{\text{v+s}}(\Omega) &= \tan^{-1}\frac{2\zeta\Omega + 2\eta}{1-\Omega^2}. \end{aligned} \tag{2.45}$$