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Vibration Problems ICOVP 2011

The 10th International Conference on Vibration Problems

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Preface

The tenth International Conference on Vibration Problems (ICOVP 2011) in Prague represents a continuation of a tradition which started at the A. C. College, Jalpaiguri, India in 1990. We well remember the further fruitful meetings in Jalpaiguri (1993), then at the University of North Bengal (1996) and at Jadavpur University in West Bengal (1999). This successful series continued in Moscow (2001), then at the Technical University of Liberec, Czech Republic (2003) and in Istanbul (2005). The last two assemblies of experts in dynamics took place at Bengal Engineering and Science University, Shibpur (2007) and at the Indian Institute of Technology, Kharagpur (2009). The ICOVP Conferences have a long tradition, and attract a large number of highly qualified participants. This, together with the top-quality papers presented and excellent organization, have established them as a high-level forum where engineers, researchers, university teachers, students and other professionals can present recent developments and discuss the scientific, technical and experimental results and ideas in various areas of rational, experimental and applied Dynamics.

Dynamics as a scientific discipline draws inspiration from a large variety of engineering areas, such as Mechanical and Civil Engineering, Aero and Space Technology, Wind and Earthquake Engineering and Transport and Building Machinery. Moreover, the basic research in Dynamics nowadays includes many fields of theoretical physics and various interdisciplinary subject areas. It is encouraging that the ICOVP Conferences have matured into a reference platform reflecting the state of the art of Dynamics in the broadest sense of the term. Indeed, the most recent ICOVP Conference, held in Prague this year, covered all branches of Dynamics and offered the most up-to-date results of development to participants from 40 countries.

The ICOVP 2011 concentrated numerous papers of a very high scientific and technical level. It has shown that the ICOVP Conferences have become increasingly attractive for participants from the global scientific community. The International Scientific Committee has had an extremely difficult task choosing the best contributions from the 280 submitted abstracts for oral presentation and for publication in the Proceedings. Almost 200 papers were tentatively selected and their full text version submitted. Afterwards, two or more reviewers carefully assessed these individual papers and decided on their final acceptance and inclusion in the Conference Proceedings.

As the second step, the International Scientific Committee was approached to select the best papers. After very strict evaluation, the respective authors were invited to prepare revised versions for submission to the special volume *Springer Proceedings in Physics – Vibration Problems ICOVP 2011*. Following the next review process, 110 manuscripts were included in the final set of papers and sorted into nine chapters. We would like to express our deep gratitude to all authors and reviewers for their enormous effort and patience related to the preparation of this volume.

Looking through the volume, highly promising trends are noticeable. Clearly many challenging topics are being investigated using extremely sophisticated combinations of analytic and numeric procedures. Many qualitatively new phenomena have been identified theoretically and verified experimentally. Non-conventional methods based on various aspects of topology are being developed and applied directly in relevant exploration. Research in Dynamics is adopting more and more discrete models inspired by principles of digital technology itself. Uncertainty modeling is making good progress, offering various new approaches in the realm of Dynamics and Vibration Control. The formulation of problems of Stochastic Dynamics reflects better natural conditions and provides realistic results which have broad applications. Progress is being made in optimal random filtering, which provides realistic models of various physical phenomena on the basis of large data processing or data mining. Non-linear problems of every type related to flow-induced vibrations are widely discussed, providing applicable results not only in basic but also in applied research and industry.

Taking these aspects into account, we are firmly convinced that this volume, published by the highly reputed publishing house Springer Science+Business Media, represents a well-balanced overview of theoretical, numerical and experimental work on fundamental and applied studies performed in Dynamics and related branches during recent years. It is a great privilege to show recognition to all the authors for their invaluable contributions and their willingness to share their knowledge with other readers. Their contribution has become the backbone of the scientific success of the Conference and ensures the high quality of this special volume.

Concluding the introductory remarks, our warmest gratitude should be expressed primarily to the founders of the excellent ICOVP tradition, namely Professor M. M. Banerjee and Professor P. Biswas. At the same time, we cannot complete the preface to this special volume without expressing our sincerest thanks to a number of individuals whose invaluable help made possible the organization of the ICOVP 2011 Conference in Prague. These are the members of the International Scientific Committee and the highly efficient Local Organizing Committee under the chairmanship of Professor B. Marvalová. Her far-sighted management and enormous effort were essential for the success of the Conference. Our genuine thanks should be addressed to our other colleagues at the Technical University of

Liberec, who supported us during the entire period of the Conference and during the preparation of this volume.

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Jiří Náprstek
On behalf of the editors

Contents

Part I
Keynote Lectures

Bifurcation and Chaos of Multi-body Dynamical Systems

Jan Awrejcewicz and G. Kudra

Abstract Triple physical pendulum in a form of three connected rods with the first link subjected to an action of constant torque and with a horizontal barrier is used as an example of plane mechanical system with rigid limiters of motion. Special transition rules for solutions of linearized equations at impact instances (Aizerman-Gantmakher theory) are used in order to apply classical tools for Lyapunov exponents computation as well as for stability analysis of periodic orbits (used in seeking for stable and unstable periodic orbits and bifurcations of periodic solutions analysis). Few examples of extremely rich bifurcational dynamics of triple pendulum are presented.

Keywords Pendulum • Impact • Bifurcation • Periodic orbit • Quasi-periodic orbit • Chaotic attractor • Lyapunov exponents • Non-smooth dynamics

1 Introduction

A single or a multiple pendulum (in their different forms) are very often studied theoretically or experimentally [1–3]. A single pendulum plays an important role in mechanics since many interesting non-linear dynamical behavior can be illustrated and analyzed using this simple system. But a single degree-of-freedom models are only the first step to understand a real behavior of either natural or engineering systems. Many physical objects are modeled by a few degrees of freedom and an attempt to investigate coupled pendulums is recently observed.

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On the other hand, it is well known that impact and friction accompanies almost all real behavior, leading to non-smooth dynamics. The example of modeling of the piston – connecting rod – crankshaft system by the use of triple physical pendulum with rigid limiters of motion is presented in the work [4].

The non-smooth dynamical systems can be modeled as the so-called piece-wise smooth systems (PWS) and they are also interesting from a point of view of their bifurcational behavior, since they can exhibit certain non-classical phenomena of non-linear dynamics [5, 6]. One of the important tools of non-linear dynamics is the linear stability theory, useful among others in the analysis of bifurcations of periodic solutions and in the identification of attractors through Lyapunov exponents. These tools are well-developed and known in the case of smooth systems. However the same tools with small modifications [6, 7] can be also used for the PWS systems. The modifications consist in the suitable transformation of the perturbation in the point of discontinuity, accordingly to the so called Aizerman-Gantmakher theory [8, 9].

In the present paper some examples of identification of attractors in the system of triple physical pendulum with the horizontal barrier are given. The system used is a special case of the more general model of triple pendulum investigated in earlier works of the authors.

2 Event Driven Model of Mechanical System with Limiters of Motion

Let us assume firstly more general case of mechanical system of n-degrees-of-freedom with vector of generalized coordinates $\mathbf{q}(t) = [q_1(t), \ldots, q_n(t)]^T$, symmetric $n \times n$ mass matrix $\mathbf{M}(\mathbf{q}, t)$ and $n \times 1$ force vector $\mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}, t)$. The system is subjected to m rigid unilateral constraints $\mathbf{h}(\mathbf{q}, t) = [h_1(\mathbf{q}, t), \ldots, h_m(\mathbf{q}, t)]^T \geq 0$. We define a set $I = \{1, 2, \ldots, m\}$ of indices of all defined unilateral constraints h_i and the set $I_{act} = \{i_1, i_2, \ldots, i_s\}$ of indices of s constraints permanently active on a certain time interval $[t_i, t_{i+1}]$. Physically it means that the system slides along obstacles with indices from the set I_{act}.

In the case of frictionless constraints, the system on time interval $[t_i, t_{i+1}]$ is governed by the following set of differential and algebraic equations (DAEs)

$$\mathbf{M}(\mathbf{q}, t)\,\ddot{\mathbf{q}} = \mathbf{f}_{\mathbf{q}}(\mathbf{q}, \dot{\mathbf{q}}, t) + \left(\frac{\partial \mathbf{h}_{act}(\mathbf{q}, t)}{\partial \mathbf{q}^T}\right)^T \boldsymbol{\lambda}_{act},$$
$$0 = \mathbf{h}_{act}(\mathbf{q}, t), \quad 0 = \dot{\mathbf{h}}_{act}(\mathbf{q}, t) = \frac{\partial \mathbf{h}_{act}(\mathbf{q}, t)}{\partial \mathbf{q}^T}\dot{\mathbf{q}} + \frac{\partial \mathbf{h}_{act}(\mathbf{q}, t)}{\partial t} \tag{1}$$

with the following event functions determining the time instances t_{i+1}

$$\boldsymbol{\lambda}_{act} = [\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_s}]^T > 0,$$

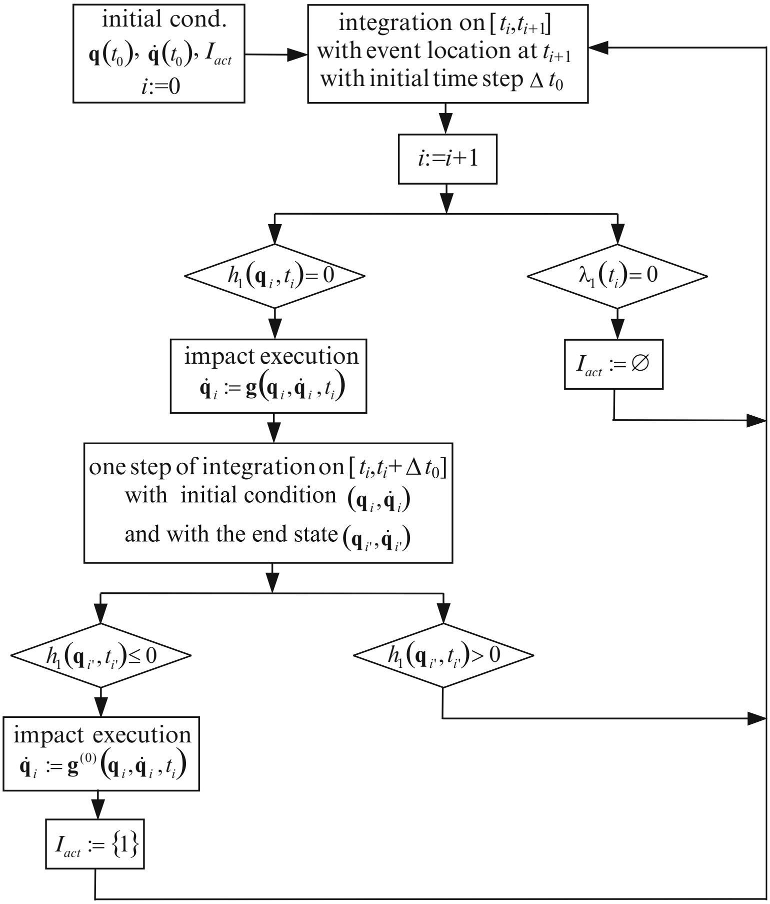

Fig. 1 Scheme for the numerical simulation of the system

$$\mathbf{h}_{inact}(\mathbf{q},t) = \left[h_{j_1}(\mathbf{q},t), h_{j_2}(\mathbf{q},t), \ldots, h_{j_{m-s}}(\mathbf{q},t)\right]^T > 0, \tag{2}$$

where $\mathbf{h}_{act}(\mathbf{q},t) = [h_{i_1}(\mathbf{q},t), h_{i_2}(\mathbf{q},t), \ldots, h_{i_s}(\mathbf{q},t)]^T$ is the vector of s constraints permanently active on $[t_i, t_{i+1}]$, $\boldsymbol{\lambda}_{act}$ is the vector of non-negative Lagrange multipliers and $\mathbf{h}_{inact}$ is the vector of $m-s$ inactive constraints, i.e. constraints which indices belong to the set $I \backslash I_{act} = \{j_1, j_2, \ldots, j_{m-s}\}$. The event t_{i+1} is determined by the use detection of zero-crossing of any component of $\boldsymbol{\lambda}_{act}$ or $\mathbf{h}_{inact}$. At time instance t_{i+1} the suitable changes in initial conditions (due to the impact) and in the set I_{act} take place and the next piece of solution $[t_{i+1}, t_{i+2}]$ is governed by the new DAEs. In this way the system has been modeled as a piece-wise smooth (PWS) DAEs.

The algorithm for the execution of changes in the system state and changes in the set I_{act} at each event time t_j, used in our numerical simulation, is presented in Fig. 1. Because of the limited space, we restrict this scheme to the simplified case, where only one constraint $h_1(\mathbf{q}, t)$ is defined ($I = \{1\}$). In the Fig. 1 the following notations are used: $\mathbf{q}_j = \mathbf{q}(t_j)$, $\dot{\mathbf{q}}_j = \dot{\mathbf{q}}(t_j)$, $t_{j'} = t_j + \Delta t_0$ and the function $\mathbf{g}(\mathbf{q}, \dot{\mathbf{q}}, t)$ represents the impact law with the restitution coefficient e while the function $\mathbf{g}^{(0)}(\mathbf{q}, \dot{\mathbf{q}}, t)$ represents impact with the restitution coefficient equal to zero independently from the system parameters.

The applied impact model is the generalized Newton's (restitution coefficient) impact law based on the reference [5], and has the following final form for the impact with the obstacle defined by $h_i(\mathbf{q}, t) = 0$:

$$\mathbf{g}(\mathbf{q}, \dot{\mathbf{q}}, t) = \begin{bmatrix} \left(\nabla_{\mathbf{q}} h_i(\mathbf{q}, t)\right)^T \\ \begin{bmatrix} \mathbf{t}_1^T \\ \cdots \\ \mathbf{t}_{n-1}^T \end{bmatrix} \cdot \mathbf{M}(\mathbf{q}, t) \end{bmatrix}^{-1} \cdot \left(\begin{bmatrix} -e\left(\nabla_{\mathbf{q}} h_i(\mathbf{q}, t)\right)^T \\ \begin{bmatrix} \mathbf{t}_1^T \\ \cdots \\ \mathbf{t}_{n-1}^T \end{bmatrix} \cdot \mathbf{M}(\mathbf{q}, t) \end{bmatrix} \dot{\mathbf{q}} + \begin{Bmatrix} -(e+1)\dfrac{\partial h_i(\mathbf{q}, t)}{\partial t} \\ 0 \\ \cdots \\ 0 \end{Bmatrix} \right), \tag{3}$$

where $\mathbf{t}_j$ are the base vectors of the subspace of the configuration space $\mathbf{q}$, tangent to the impact surface $h_i(\mathbf{q}, t)$ at the impact point. For more details on the impact model see works [3, 4, 7].

3 Linear Stability Model

For the dynamical system in the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t), \tag{4}$$

where $\mathbf{x} = [\mathbf{q}^T, \dot{\mathbf{q}}^T]^T$, the small perturbation of the solution is governed by the following linear equations

$$\delta \dot{\boldsymbol{x}} = \frac{\partial \mathbf{f}(\mathbf{x}, t)}{\partial \mathbf{x}^T} \delta \boldsymbol{x}(t), \tag{5}$$

where we have assumed $\delta t = 0$ since the perturbation in time is independent from the perturbation $\delta \boldsymbol{x}$ ($\delta \dot{t} = 0$). The Eq. 5 are useful among others in the stability and bifurcation analysis of periodic solutions, as well as in the Lyapunov exponents calculation.

In the case of non-smooth dynamical system we cannot apply directly the linear stability theory since the Jacobian in (5) is not determined. But in the case of the PWS system the function $\mathbf{f}(\mathbf{x}) = \mathbf{f}_i(\mathbf{x})$ is sufficiently smooth on each time interval $[t_i, t_{i+1}]$ between two successive discontinuity points and the linear stability can applied using variational Eq. 5 on intervals $[t_i, t_{i+1}]$, and applying at each discontinuity point t_i special transformation rules accordingly to the Aizerman-Gantmakher theory (for $\delta t = 0$):

$$\delta \boldsymbol{x}_i^+ = \frac{\partial \mathbf{g}_i\left(\mathbf{x}_i^-, t_i\right)}{\partial \mathbf{x}^T} \delta \boldsymbol{x}_i^- + \left[\frac{\partial \mathbf{g}_i\left(\mathbf{x}_i^-, t_i\right)}{\partial \mathbf{x}^T} \mathbf{f}_i\left(\mathbf{x}_i^-, t_i\right) + \frac{\partial \mathbf{g}_i\left(\mathbf{x}_i^-, t_i\right)}{\partial t} - \mathbf{f}_{i+1}\left(\mathbf{x}_i^+, t_i\right) \right] \delta t_e \tag{6}$$

where

$$\delta t_e = - \frac{\dfrac{\partial event_i\left(\mathbf{x}_i^-, t_i\right)}{\partial \mathbf{x}^T} \delta \boldsymbol{x}_i^-}{\dfrac{\partial event_i\left(\mathbf{x}_i^-, t_i\right)}{\partial \mathbf{x}^T} \mathbf{f}_i\left(\mathbf{x}_i^-, t_i\right) + \dfrac{\partial event_i\left(\mathbf{x}_i^-, t_i\right)}{\partial t}},$$

and where $\mathbf{x}_i^- = \lim_{t \to t_i^-} \mathbf{x}(t)$, $\mathbf{x}_i^+ = \lim_{t \to t_i^+} \mathbf{x}(t)$, $\delta \boldsymbol{x}_i^+ = \lim_{t \to t_i^+} \delta \boldsymbol{x}(t)$, $\delta \boldsymbol{x}_i^- = \lim_{t \to t_i^-} \delta \boldsymbol{x}(t)$, $\mathbf{g}_i(\mathbf{x})$ is the function representing jump in the system state $\mathbf{x}_i^+ = \mathbf{g}_i\left(\mathbf{x}_i^-\right)$ in the discontinuity point and $event_i(\mathbf{x}, t)$ is the scalar function used for detection of the discontinuity instance at t_i ($event_i\left(\mathbf{x}_i^-, t_i\right) = 0$).

The linearized differential-algebraic equations of the system (1) are

$$\mathbf{M}(\mathbf{q}, t)\ \delta \ddot{\mathbf{q}} = \frac{\partial \mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}, t)}{\partial \mathbf{q}^T} \delta \mathbf{q} + \frac{\partial \mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}, t)}{\partial \dot{\mathbf{q}}^T} \delta \dot{\mathbf{q}} + \frac{\partial}{\partial \mathbf{q}^T} \left(\left(\frac{\partial \mathbf{h}_{act}(\mathbf{q})}{\partial \mathbf{q}^T} \right)^T \boldsymbol{\lambda}_{act} \right) \delta \mathbf{q}$$

$$+ \left(\frac{\partial \mathbf{h}_{act}(\mathbf{q})}{\partial \mathbf{q}^T} \right)^T \delta \boldsymbol{\lambda}_{act} - \left(\frac{\partial \mathbf{M}(\mathbf{q}, t)}{\partial \mathbf{q}^T} \delta \boldsymbol{q} \right) \ddot{\mathbf{q}},\ 0 = \frac{\partial \mathbf{h}_{act}(\mathbf{q}, t)}{\partial \mathbf{q}^T} \delta \boldsymbol{q}$$

$$0 = \dot{\mathbf{q}}^T \frac{\partial^2 \mathbf{h}_{act}(\mathbf{q}, t)}{\partial \mathbf{q} \partial \mathbf{q}^T} \delta \boldsymbol{q} + \frac{\partial \mathbf{h}_{act}(\mathbf{q}, t)}{\partial \mathbf{q}^T} \delta \dot{\mathbf{q}}, \tag{7}$$

where

$$\ddot{\mathbf{q}} = \mathbf{M}(\mathbf{q}, t)^{-1} \left(\mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}, t) + \left(\frac{\partial \mathbf{h}_{act}(\mathbf{q}, t)}{\partial \mathbf{q}^T} \right)^T \boldsymbol{\lambda}_{act} \right)$$

and where we have also assumed $\delta t = 0$.

We have applied Eq. 7 together with the transformation rules (6) in the Lyapunov exponents calculation for the mechanical system presented in the Sect. 2. Note that

Eq. 6 with the impact law $\mathbf{g}_i(\mathbf{x}, t) = \mathbf{g}_i^{(0)}(\mathbf{x}, t)$ with the restitution coefficient equal to zero applied in the case where the sliding motions starts (see Fig. 1), gives the perturbation $(\boldsymbol{\delta}\mathbf{q}, \boldsymbol{\delta}\dot{\mathbf{q}})$ consistent with the algebraic equations in (7) and the perturbation vector $\boldsymbol{\delta}\mathbf{x}^+$ lies in the (2n-2)-dimensional subspace (in the case of only one constraint permanently active).

In the well-known algorithm of Lyapunov exponents computation the Gram-Schmidt reorthonormalization procedure is applied after some time of integration of variational equations. After use of this procedure to the vector of perturbations $\boldsymbol{\delta x}$ fulfilling 2s algebraic equations in (7) (in the case of s constraints permanently active), we obtain the new set of perturbation vectors, from which 2n-2s satisfy the algebraic equations and 2s of them do not. Then in our procedure we simply set that 2s vectors to zero vectors, obtaining the new "degenerated" set of orthonormal vectors, satisfying algebraic equations.

4 Triple Pendulum Model

Three joined stiff links coupled with viscous damping and moving on the plane are presented in Fig. 2. The system position is defined by three angles ψ_i ($i = 1,2,3$), and each of the first body is under action of constant torque q_1. The set of possible

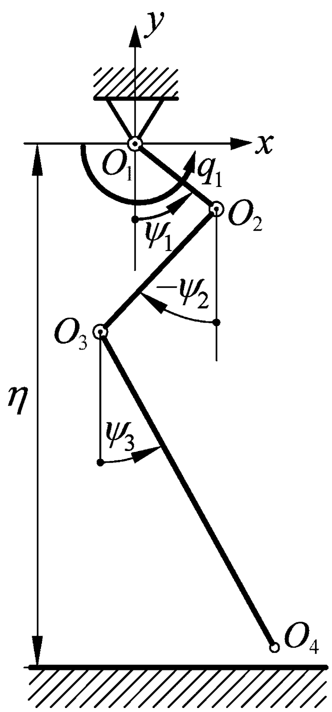

Fig. 2 Mechanical system

configurations of the system is bounded by the horizontally situated rigid and frictionless barrier.A vector of generalized coordinates is the vector of three angles $\mathbf{q} = \boldsymbol{\psi} = [\psi_1, \psi_2, \psi_3]^T$. The mass matrix, force vector and the set of algebraic equations defining rigid obstacle are as follows

$$\mathbf{M}(\mathbf{q}, t) = \mathbf{M}(\boldsymbol{\psi}) = \begin{bmatrix} 1 & \nu_{12}\cos\ (\psi_1 - \psi_2) & \nu_{13}\cos\ (\psi_1 - \psi_3) \\ \nu_{12}\cos\ (\psi_1 - \psi_2) & \beta_2 & \nu_{23}\cos\ (\psi_2 - \psi_3) \\ \nu_{13}\cos\ (\psi_1 - \psi_3) & \nu_{23}\cos\ (\psi_2 - \psi_3) & \beta_3 \end{bmatrix},$$

$$\mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}, t) = \mathbf{f}\left(\boldsymbol{\psi}, \dot{\boldsymbol{\psi}}, t\right) = -\mathbf{N}(\boldsymbol{\psi})\ \dot{\boldsymbol{\psi}}^{\mathbf{2}} - \mathbf{C}\,\dot{\boldsymbol{\psi}} - \mathbf{p}(\psi) + \mathbf{f}_e\left(\boldsymbol{\psi}, \dot{\boldsymbol{\psi}}, t\right), \quad (8)$$

$$h_1(\boldsymbol{\psi}) = \eta - l_1\cos\psi_1,\ h_2(\boldsymbol{\psi}) = \eta - \sum_{i=1}^{2} l_i\cos\psi_i,\ h_3(\boldsymbol{\psi}) = \eta - \sum_{i=1}^{3} l_i\cos\psi_i$$

where

$$\mathbf{N}(\boldsymbol{\psi}) = \begin{bmatrix} 0 & \nu_{12}\sin(\psi_1 - \psi_2) & \nu_{13}\sin(\psi_1 - \psi_3) \\ -\nu_{12}\sin(\psi_1 - \psi_2) & 0 & \nu_{23}\sin(\psi_2 - \psi_3) \\ -\nu_{13}\sin(\psi_1 - \psi_3) & -\nu_{23}\sin(\psi_2 - \psi_3) & 0 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix}, \quad \mathbf{p}(\boldsymbol{\psi}) = \begin{Bmatrix} \sin\psi_1 \\ \mu_2\sin\psi_2 \\ \mu_3\sin\psi_3 \end{Bmatrix}, \quad \mathbf{f}_e = \begin{Bmatrix} q_1 \\ 0 \\ 0 \end{Bmatrix},$$

and where $\dot{\boldsymbol{\psi}}^2 = \left[\dot{\psi}_1^2, \dot{\psi}_2^2, \dot{\psi}_3^2\right]^T$, where l_i is non-dimensional length of i-th link, c_i is non-dimensional damping coefficient in the i-th joint while ν_{ij} and μ_i are other non-dimensional parameters of the system.

The system response is obtained numerically by the use of the Runge-Kutta integration method of the differential equations between each two successive discontinuity points (where the activity of the obstacles changes: the impact takes place or the time interval of sliding begins or ends). These points are detected by halving integration step until obtaining assumed precision. After the simulation of the system, the next step was the stability analysis of the solution in the investigated model, which in fact is piece-wise smooth (PWS) one. The classical methods and algorithms basing on the linear perturbation equations are used with the modifications taking into account the perturbations jump in the discontinuity points [9]. The numerical software for Lyapunov exponents calculation and periodic orbit stability analysis (seeking for periodic orbits and their bifurcations analysis) was developed.

For more details on modeling, relations between real and non-dimensional parameters, numerical algorithms, etc., see works [3, 4, 7].

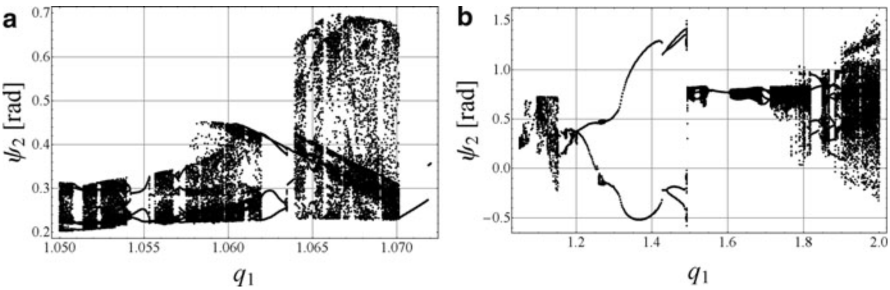

Fig. 3 Bifurcational diagrams

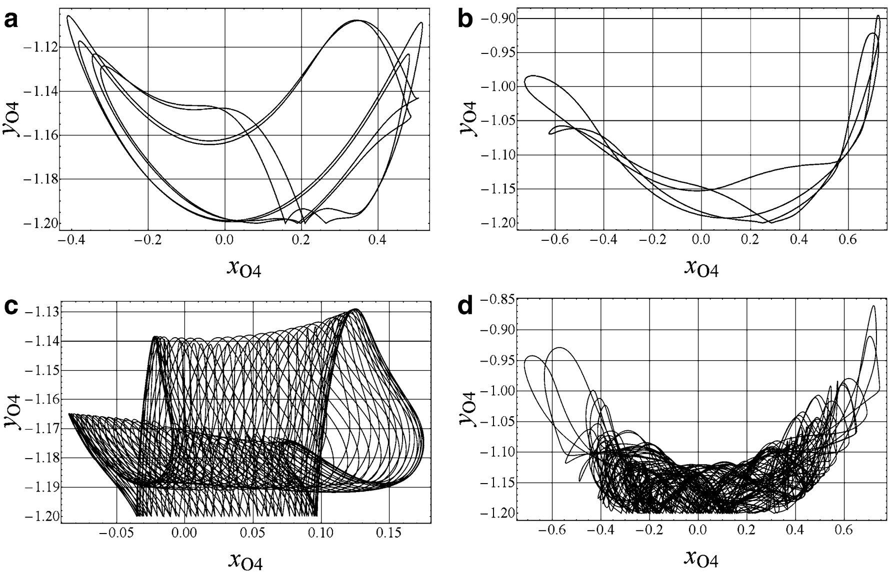

Fig. 4 Projections of periodic (**a**, $q_1 = 1.063$; **b**, $q_1 = 1.59$), quasi-periodic (**c**, $q_1 = 1.63$) and chaotic (**d**, $q_1 = 2$) attractors

5 Numerical Examples

The examples of extremely rich bifurcational dynamics of the modeled system is presented for the following non-dimensional parameters: $l_1 = O_1\ O_2 = 0.05$, $l_2 = O_2\ O_3 = 0.02$, $l_3 = O_3\ O_4 = 1$, $\eta = 1.2$ and $c_1 = c_2 = c_3 = 0.8$. The restitution coefficient is $e = 0.8$ and contact between links and obstacles is assumed to be frictionless. The externally applied torque q_1 is used as bifurcational parameter.

In Fig. 3 one can find two bifurcational diagrams where the parameter q_1 is increasing quasi-statically. In Fig. 3a the relative change of the torque is very small (about 2%) but the richness and number of bifurcational phenomena observed is