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The Mathematical Legacy of Leon Ehrenpreis

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Springer Proceedings in Mathematics

Volume 16

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The Mathematical Legacy of Leon Ehrenpreis

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ISSN 2190-5614 ISSN 2190-5622 (electronic)
Springer Proceedings in Mathematics
ISBN 978-88-470-1946-1 ISBN 978-88-470-1947-8 (eBook)
DOI 10.1007/978-88-470-1947-8
Springer Milan Heidelberg New York Dordrecht London

Library of Congress Control Number: 2011945461

Mathematics Subject Classification: 11-06, 32-06, 42-06, 46-06

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Leon Ehrenpreis at Stockholm

Preface

Like many other mathematicians around the world, we were saddened and shocked when news reached us that Leon Ehrenpreis had passed away on 16 August 2010. Our first instinct was to collect a volume of mathematical contributions by his many friends and collaborators as well as by many mathematicians whose mathematical career has been influenced by Leon's work. We are very appreciative for the immediate support that Springer and Dr. Francesca Bonadei have given to our idea and for the enthusiastic response of the many authors who have agreed to participate in what we consider as an act of respect, friendship, and affection for Leon. We are also indebted to Leon's daugther, Yael Ehrenpreis Meyer, who has shared with us the beautiful picture of Leon in Stockholm, a picture that so perfectly reflects Leon's zest for life. Finally, we are grateful to Professor Malgrange for sharing with us a personal letter that Leon wrote to him in June 1960 and which is appended to this volume.

As a way of introduction to the volume, we include, in the next few pages, three short essays that focus on three different periods of Leon Ehrenpreis' mathematical life.

Milan, Italy — Irene Sabadini
Orange, USA — Daniele C. Struppa

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Part I
Introduction to the Volume

Leon Ehrenpreis: Some Old Souvenirs

Bernard Malgrange

In the years 1952–1953, I had finished my studies at École Normale Supérieure, and I had a position of research in CNRS, under the supervision of Laurent Schwartz. His book on the theory of distributions had been recently published; this book and his paper on mean periodic functions were full of open problems on linear differential equations, especially with constant coefficients and convolution equations. I was mainly interested in the problem of "elementary solutions": given a differential polynomial P with constant coefficients, does there exist a distribution f on $\mathbb{R}^n$ verifying $Pf = \delta$, δ the Dirac measure?

Schwartz suggested to solve this problem by finding a "tempered" f: by Fourier transform, this is equivalent to the problem of "division of a distribution" by a polynomial. I tried this method, but unsuccessfully (the problem was solved several years later, independently by Hörmander and Łojasiewicz). But I found that one can bypass the division of distributions: by duality, one is reduced to the following problem: if a family $\{P\varphi_\alpha\}$ (φ_α, functions $\mathcal{C}^\infty$ with compact support) tend to zero in a suitable sense, then the $\{\varphi_\alpha\}$ tend also to zero. Now, by Fourier transform P is transformed into a polynomial, and φ_α into an entire function with some growth conditions at infinity described by the Paley–Wiener theorem. And a simple argument of maximum modulus gave the required result.

There are a lot of convergence conditions which can be chosen. The simplest is perhaps the following one: if the φ_α's have a bounded support and if the $P\varphi_\alpha$ tend to zero in L^2, then the φ_α tend also to zero in L^2.

The same method, with a little more work, gives also the following results:

(i) Let f be a $\mathcal{C}^\infty$ function (resp. a distribution of finite order) in $\mathbb{R}^n$; then there exists another one g with $Pg = f$.
(ii) The exponential-polynomial solutions of $Pf = 0$ are dense in the $\mathcal{C}^\infty$, or in the distributions solutions.

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I. Sabadini, D.C. Struppa (eds.), *The Mathematical Legacy of Leon Ehrenpreis*, Springer Proceedings in Mathematics 16,
DOI 10.1007/978-88-470-1947-8_1,

Furthermore, the same results are true for $\mathbb{R}^n$ replaced by an open convex set.

I published notes in *Comptes Rendus de l'Académie des Sciences* on these results. A short time later somebody, I think J. Dieudonné, told me that a young American mathematician, named Leon Ehrenpreis, had obtained also the same results. They were published in *American Journal* in 1954 under the title "Solution of some problems of division *I*".

This was the beginning of a kind of emulation, although this was essentially the only one time where we obtained independently similar results. To this emulation, I can perhaps add the name of Lars Hörmander, who namely reproved the existence theorem in his thesis by proving the required L^2 inequality directly with energy integrals, without Fourier transform; this permitted to him to get by the same method existence theorems for some equations with variable coefficients ("equations of principal type"), which could not be obtained by our complex methods.

Concerning the next period, I will mention mainly the series of papers by Ehrenpreis "Solution of some problem of division", especially the numbers III and IV. Let me indicate briefly the main results of these papers.

In III, he solves a problem left open by the preceding works: given a differential polynomial P with constant coefficients, and a distribution f in $\mathbb{R}^n$ (not necessarily of finite order), there exists another one g with $Pg = f$. The proof consists in a very precise analysis in terms of Fourier transform of the topology of the space $\mathcal{D}$ of Schwartz (i.e. the space of $\mathcal{C}^\infty$ functions with compact support). Later, I interpreted this analysis as giving a theorem of propagation of regularity for the solutions of equations with constant coefficients. For a more systematic study of this point of view, I refer to the book "Linear partial differential operators" by Hörmander.

I was much impressed by this paper. But I was even more impressed by the next one, number IV. This paper is devoted to convolutions equations $\mu * f = g$, μ a given distribution with compact support, f and g $\mathcal{C}^\infty$ functions or distributions. The main results are the following:

(i) A necessary and sufficient condition for $\mu*$ to be surjective in the space of $\mathcal{C}^\infty$ functions, or distributions, in $\mathbb{R}^n$. The condition, called by Ehrenpreis "slowly decreasing", is as follows:

If $\hat{\mu}$ is the Fourier transform of μ (which is an entire function in $\mathbb{C}^n$), there exists $a > 0$ such that for each real z, there exists another z' with $|z' - z| \leq a\log(1 + |z|)$ and $|\hat{\mu}(z')| \geq (a + |z|)^{-a}$ (here $|z|$ is any norm in $\mathbb{C}^n$).

If we replace "distribution" by "distribution of finite order", one needs a stronger condition: the first inequality should be replaced by $|z' - z| \leq a$.

(ii) A necessary and sufficient condition for "hypoellipticity" (called "ellipticity" after Schwartz): all distributions f verifying $\mu * f = 0$ are $\mathcal{C}^\infty$ functions.

The condition generalizes the one obtained for differential polynomials by Hörmander in his thesis; but Ehrenpreis says that his own result was obtained independently.

The condition is the following: first, $\hat{\mu}$ should be slowly decreasing; furthermore, on the variety of zeros of $\hat{\mu}$, one has an inequality $|\operatorname{Im} z| \geq a\log(1 + |z|)$.

But Ehrenpreis was soon after that work interested by a much more general situation: the overdetermined linear systems with constant coefficients. In 1960, he announces general results in this context.

For simplicity, I limit myself to systems with one unknown (the general case is similar). Also, I consider only the case of $\mathcal{C}^\infty$ functions in $\mathbb{R}^n$; in the case of distributions and convex open sets, the results are similar.

We give $P_1, \dots, P_m$, linear differential operators with constant coefficients. Then the results are as follows:

(i) Given functions $f_1, \dots, f_m$ $\mathcal{C}^\infty$, there exists a g $\mathcal{C}^\infty$ verifying $P_i g = f_i$ if and only if the f_i's verify the "trivial compatibility conditions": if $Q_1, \dots, Q_m$ are differential polynomials with constant coefficients satisfying $\Sigma Q_i P_i = 0$, then one has $\Sigma Q_i f_i = 0$

(ii) The exponential-polynomial solutions of $P_1 g = \cdots = P_m g = 0$ are dense in all solutions.

Actually, Ehrenpreis gives a much more precise statement, called "fundamental principle": roughly speaking, the $\mathcal{C}^\infty$ solutions of the system are Fourier transforms or "integrals" (in a suitable sense) of measures with support the complex variety of zeros of $\widehat{P_1}(z) = \cdots = \widehat{P_m}(z) = 0$, $\widehat{P_i}$ the polynomial associated to P_i.

I will just explain roughly how one can get (i) and (ii) (the fundamental principle requires some more work). By duality and Fourier transform, the problem is reduced to the following:

Let μ be a distribution with compact support, and $\hat{\mu}$ its Fourier transform. According to Paley–Wiener theorem, $\hat{\mu}$ is an entire function of exponential type with polynomial growth in any strip $|\operatorname{Im} z| \leq a$, $a \in \mathbb{R}$, and conversely.

Now, suppose that, at every point $z_0 \in \mathbb{C}^n$, $\hat{\mu}$ is in the ideal of formal series in $(z - z_0)$ generated by $\widehat{P_1}, \dots, \widehat{P_m}$. Then, one has $\hat{\mu} = \Sigma \widehat{P_i} \hat{\nu}_i$, where ν_i are Fourier transforms of distributions with compact support (or, to abbreviate, entire functions with Paley–Wiener growth).

It is classical that, with these hypotheses, one has locally in $\mathbb{C}^n$, $\hat{\mu} = \Sigma \widehat{P_i} f_i$, f_i germs of holomorphic functions. Now the theory of Cartan–Oka proves that, in fact, one has a global result, i.e. $\hat{\mu} = \Sigma \widehat{P_i} f_i$, f_i entire. The problem is to prove that one can choose the f_i with Paley–Wiener growth.

The idea is to copy more or less the method of Cartan: first, get local bounds. Then, to globalize the result, use a theorem of vanishing of cohomology "with Paley–Wiener bounds". Note that, at this time, the idea of cohomology with bound was absolutely new.

As I said, these results were announced in 1960, in the paper "The fundamental principle for linear constant coefficients partial differential equations". A little more details were given in some monographed notes of lectures at Stanford. But it takes about 3 years to have a complete manuscript; and the final book "Fourier analysis in several complex variables" was not published before 1970. Needless to say that the book contains many more results on ellipticity, Cauchy problem, quasi-analytic classes, etc.

The 1960 announcement interested very much the (few) experts of the subject. At the first time, I was extremely surprised, may be a little bit sceptical. But, after two years, in the absence of complete proofs, I tried to give my own version. It differs from that of Ehrenpreis in two points:

First, I use a Dolbeault cohomology with bounds, instead of Čech cohomology as Ehrenpreis. In fact, a theorem of vanishing of Dolbeault cohomology "with Paley–Wiener growth" is surprisingly simple, much more that Čech cohomology with the same bound. Some time later, Hörmander got practically definitive results on Dolbeault cohomology with growth condition given by any plurisubharmonic function; he gave also an exposition of Ehrenpreis theory using this theorem.

The second difference, less important, is that I used (local) estimates on $\mathcal{C}^\infty$ functions, instead of holomorphic ones (the use of Dolbeault cohomology permits it). These estimates come from a development of the theory of division of distributions.

I note also that Palamodov gave also a version of the theory (his version is more close to that of Ehrenpreis).

These works finish essentially the subject. One could think of an extension to general systems of convolution equations, but this seems very difficult, or even almost impossible. The only one reasonable result to be expected was the density of exponential polynomials for general systems of homogeneous equations, a result obtained for one variable by Schwartz in his theory of mean periodic functions. But, in 1974 Gurevitch proved that the result is not true for several variables.

Let me finish by a few words about our personal relations. Actually, we met for the first time in Paris, in 1958 (if my memory is correct); this was a rather long time after our first works. Before that meeting, I thought of Ehrenpreis, with a little bit of tension, as a rather abstract person with whom I was more one less in competition. But, at our first meeting, he was so open and friendly that all tension disappeared totally. We became friends, although we did not meet so often. I remember especially a visit he made in Tunis, in 1970, where I stayed for one year. I think he was very pleased with this visit, except that the Jewish Tunisian food seemed not to fit him. Later, I met him several times in New-York, where I come often for familial reasons. He came to some lectures I gave to Courant Institute.

More recently, not a long time before his death, I had the surprise and pleasure to see him at a lecture I gave at Kolchin Seminar, in CUNY. I was especially happy, since I had not seen him since a rather long time, and we took the opportunity to remember old souvenirs. When leaving him, I could not imagine that it was our last meeting.

Leon Ehrenpreis, a Unique Mathematician

Daniele C. Struppa

1 Introduction

What made Ehrenpreis' mathematics so unique was his bold approach to classical problems, and his interest in finding an overarching and unifying framework for a variety of apparently unrelated problems. In this note I will try to highlight this characteristic, by looking at some of Ehrenpreis' papers which are not, strictly speaking, connected with either the Fundamental Principle or the Radon Transform.

Malgrange's section on the work that he, Hörmander, and Ehrenpreis accomplished in the context of systems of linear constant coefficient partial differential operators has illuminated a particularly intense period in the history of modern analysis: in this context, the contribution of Ehrenpreis is almost completely summarized in his first full length book [8].

The section authored by Kuchment, on the other hand, gives a beautiful picture of Ehrenpreis' involvement with integral geometry and its far reaching work on the Radon transform, as described in his pioneering work [15].[1]

My own involvement with Ehrenpreis stemmed from me being (from 1978 to 1981) a doctoral student of Carlos Berenstein, who himself was a former student of Leon. As such I came to meet Ehrenpreis many times during his frequent visits to College Park, Maryland. What I remember most from our conversations, and from his talks, was his overarching belief that one should consider the theory of holomorphic functions (in several complex variables) as a special case of a more

[1] As a somewhat amusing and personal note, I should mention that in the late 1980s I had founded a small publishing company in southern Italy, Mediterranean Press was its name; at that time Ehrenpreis was visiting my department, and he had accepted my invitation to write a book on the Radon transform for my company. During the next several years, I therefore saw several preliminary versions of the book, but by the mid-1990s I had left Italy, sold my equity in the company, and Ehrenpreis had found a much more appropriate outlet for his work.

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I. Sabadini, D.C. Struppa (eds.), *The Mathematical Legacy of Leon Ehrenpreis*, Springer Proceedings in Mathematics 16,
DOI 10.1007/978-88-470-1947-8_2,

general theory of overdetermined systems. As for the theory of holomorphic functions in one variable, one should be trying to think of it as a special case of a more general theory of mean-periodic functions. His belief in this general approach was illustrated by some of his most original and beautiful work. In this short note, I would like to focus on three specific instances in which his worldview allowed him to recreate classical theorems in a much more general setting, thus opening the way to fruitful and unexpected generalizations.

2 The Hartogs' Theorem

I will begin with the beautiful proof that Ehrenpreis gave in 1961 for the well-known Hartogs' theorem on the removability of compact singularities for holomorphic functions of more than one complex variable. The theorem states that if K is a compact set in $\mathbb{C}^n$, with $n \geq 2$, every holomorphic function outside of K can be extended (in a unique way) to a holomorphic function inside of K. This result, which was originally proved by Hartogs [17] in 1906 and was probably the first to demonstrate the unique flavor of complex analysis in several variables, has been given many different proofs and has been generalized to many settings [26, 28]. But it was only with Ehrenpreis' surprising [7] that it became clear that the result has little to do with holomorphic functions, but it is rather a consequence of an essentially algebraic property of the Cauchy–Riemann system. The actual statement of Ehrenpreis is as follows:

Theorem 1 *Let K be a compact set in $\mathbb{R}^n$, and let $P_1, \ldots, P_r$ be r polynomials in n complex variables with no common factors. Denote by $P_i(D)$ the differential operator that is obtained by replacing the complex variable $z = (z_1, \ldots, z_n)$ in $\mathbf{P}$ by the formal differential operator $D = (-i\partial/\partial x_1, \ldots, -i\partial/\partial x_n)$. Then every infinitely differentiable function on $\mathbb{R}^n \setminus K$ which is a solution, in $\mathbb{R}^n \setminus K$, of the system $\mathbf{P}(D)f = 0$, namely $P_1(D)f = \cdots = P_r(D)f = 0$, can be extended uniquely to an infinitely differentiable function on $\mathbb{R}^n$, solution everywhere of the same system. The new solution coincides with f on $\mathbb{R}^n \setminus K_\varepsilon$, where K_ε indicates a small ε neighborhood of K.*

The proof of the result is a brilliant (and early) example of the use of cohomology vanishing arguments. Essentially, one extends f in some arbitrary way to an infinitely differentiable function g and then notices that the collection $\{P_i(D)g\}$ is a compactly supported 1-cocycle with coefficients in the sheaf $\mathcal{R}$ of solutions of the system $\mathbf{P}(D)f = 0$. Using the Ehrenpreis–Malgrange division theorem [6, 23] (which essentially states that an entire quotient between a holomorphic function and a polynomial has the same growth order as the original holomorphic function), Ehrenpreis shows that the first cohomology group with compact support and with coefficients in the sheaf $\mathcal{R}$ vanishes, and therefore the 1-cocycle is a 1-coboundary, and the correction that this provides is sufficient to modify the original extension g into a global solution of the system.

This proof is beautiful on several counts: it is very simple (in fact, it can be given in full detail in just a few lines), it takes advantage of the equally beautiful Ehrenpreis–Malgrange lemma (in itself a powerful statement on the nature of polynomials), and finally it uncovers the fundamentally algebraic nature of the problem. The extension to the case of a rectangular system is technically more complicated and was given in detail, for example, in [25], as well as sketched in Ehrenpreis' own [8]. But the structure of the proof is so easy that it is in fact possible to generalize it further to infinite-order differential operators (see, for example, [20]) as well as to convolution equations as in [24]. We refer the reader to [26, 28] for a rather complete history of the various developments surrounding the various proofs of the Hartogs' theorem, and where more complete references (including the works of Kawai concerning the case of systems of variable coefficient differential equations) are given.

3 The Edge-of-the-Wedge Theorem

A second instance in which the theory of several complex variables is reinterpreted in a larger context is offered by Ehrenpreis' interest in a general approach to the question of extension of holomorphic functions. Clearly Hartogs' theorem is an example of such an interest, but Ehrenpreis was interested in a more general issue, in which the extension was not necessarily across a compact set. To this problem Ehrenpreis devoted a series of papers, [9–11, 13, 14, 16], whose focus, in a sense, is on the extension of the edge-of-the-wedge theorem, from the case of holomorphic functions to the case of more general solutions to overdetermined systems of linear constant coefficient differential equations.

This is not the place for a full discussion of the problem, but it is probably worth sketching at least the fundamental setting, which Ehrenpreis considered in his papers beginning with [9], but whose intellectual origins can once again be traced back to [8]. Consider s (not necessarily) different open sets $\Omega_1, \ldots, \Omega_s$ in $\mathbb{R}^n$ and r differential operators (once again not necessarily different) $\mathbf{D}_j = (D_{j1}, \ldots, D_{js})$ with constant coefficients. Suppose, furthermore, that there is a set X contained in every closed set $\overline{\Omega_j}$, which can be used to parameterize the solutions (in some suitable Analytically Uniform space) of $\mathbf{D}_j f_j = 0$ on each Ω_j. We use this term to indicate, in accordance with Chap. IX of [8], that a suitable Cauchy problem (determined by the operators $\mathbf{D}_j$ and initial values on X) is well posed. Suppose now that the solutions f_j satisfy on X some differential relations

$$\sum_{ij} a_{ij} \partial_i f_j = 0$$

generated by suitable constant coefficient differential operators ∂_i. Then one may ask what kind of consequences can be derived regarding the f_j. In particular, is it possible to extend them to being solutions of those same operators $\mathbf{D}_j$ on larger sets (this is a removability of singularities problem, of a very different nature from the one we examined in the section on the Hartogs' theorem, since the singularities are not confined, in this case, to compact sets)?

The study of such general Cauchy problems, and more precisely the conditions under which the problem is well posed (conditions on the spaces of functions involved, on the geometrical properties of the varieties associated to the operators, and on the specific geometry of X) are discussed in [8], but while the results of Ehrenpreis are extremely general, they are somewhat difficult to apply to specific conditions.

In the papers that begin with [9], however, Ehrenpreis fixes his attention on the way in which these results are far reaching generalizations of well-known function theory theorems. For example, if $s = 1$, $\Omega = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 > 0\}$, and if $r = 1$, with D being now the Laplacian, then one can consider a very special differential relation, say

$$\frac{\partial f}{\partial x_1} = 0,$$

and then any general theorem will end up being a generalization of what is known as the *reflection theorem*, namely the theorem that states that harmonic functions in the half-space $\{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 > 0\}$, which satisfy $\partial f / \partial x_1 = 0$, can be extended to harmonic functions on all of $\mathbb{R}^n$.

By the same token, the edge-of-the-wedge theorem can be seen in this context. Take all the differential operators to be the Cauchy–Riemann systems in n variables, and take two open sets Ω_1, Ω_2 to be complex tubes over two convex cones in the real space. Then the differential relation is actually the request that the two functions f_1, f_2, holomorphic respectively on Ω_1 and Ω_2, coincide on the real boundary of the two tubes. The conclusion of the edge-of-the-wedge theorem then is that there is a holomorphic function f which extends the $f_j's$ to the convex hull of $\Omega_1 \bigcup \Omega_2$.

Once again, Ehrenpreis shows us here a very general approach to a variety of different problems in which holomorphicity (or harmonicity) are just special cases of functions which are solutions to more general systems of differential equations. I can only leave it to the reader to further explore these ideas in the articles cited in the references.

4 Infinite-Order Differential Operators and the Fabry Gap Theorem

Finally, I want to go back to the interest of Ehrenpreis in convolution equations, and in the role they could play in understanding some classical properties of holomorphic functions in one complex variable. As Malgrange has observed in his note, a full extension of the Ehrenpreis–Palamodov Fundamental Principle to (systems of) convolution equations is not possible, essentially because of the example of Gurevich to which Malgrange makes reference. This said, Ehrenpreis never abandoned the possibility that at least for some classes of convolutors, it may be possible to prove what is essentially a version of the Fundamental Principle. He first showed how to obtain a weak version of the Fundamental Principle in Chap. 11 of [8], but his result was somewhat hard to apply, and the restrictions on the convolutors are hard to decipher. But his intuition was in fact correct. That this was the case was shown first by Berenstein and Dostal [1] in a very special case, and later on by Berenstein and Taylor [5], at least for the case of systems of convolution equations with one unknown function. What Berenstein and Taylor show in [5] is that it is possible to construct a class of convolutors (which they call *slowly decreasing*, following a terminology already used by Ehrenpreis to indicate the condition that is necessary to establish surjectivity in suitable spaces) for which a reasonable analogue of the Fundamental Principle holds. Their theory was further extended to the case of rectangular systems of convolution equations in all LAU spaces in my dissertation [27]. It is worth pointing out (and in fact it is necessary in view of what will follow) that infinite-order differential operators on the space of holomorphic functions offer an example of slowly decreasing convolutors, and therefore the theory developed in [5, 27] can be applied to solutions of (systems of) such operators. One of the consequences of these extensions of the Fundamental Principle consists in the fact that convergent exponential sums, both in one and in several variables, can be considered as solutions to systems of slowly decreasing convolution equations and in particular (when holomorphic functions are considered) to systems of infinite-order differential equations.

This leads us to one of the most intriguing contributions of Ehrenpreis to classical complex analysis. In Chap. 12 of his monograph [8], as well as in [12], Ehrenpreis reconsiders the classical Fabry gap theorem. In brief, the theorem can be stated as follows: let z denote the complex variable, and, for complex numbers c_j and real numbers a_j, consider the series

$$\sum_{j=1}^{+\infty} c_j e^{i a_j z}.$$

Assume that, in the strip $|\mathrm{Im}\, z < 1|$, the series converges, uniformly on compact sets, to a function $f(z)$ which can be analytically continued to a neighborhood of some point z_0 on the boundary of the strip itself. Then, if the sequence $\{a_j\}$ is *lacunary* in the sense that

$$\frac{n}{a_n} \to 0 \quad \text{as } n \to \infty$$

and there exists a positive constant c such that

$$|a_n - a_m| \geq c|n - m|,$$

the function f can actually be continued analytically to an entire strip containing z_0, and, on the compact subsets of this new strip, the series $\sum_{j=1}^{+\infty} c_j e^{i a_j z}$ converges to the continuation of $f(z)$. There are several ways to look at this theorem, and maybe the most important classical reference is Levinson's important [22]. But Ehrenpreis offers in Chap. 12 of [8] two very unconventional approaches. One consists in noticing that every exponential $e^{i a_j z}$ is itself a solution of the particular differential equation

$$\frac{df}{dz} - i a_j f = 0,$$

and therefore it is not unreasonable to think of the series $\sum_{j=1}^{+\infty} c_j e^{ia_j z}$ as a series of solutions to different differential equations. Since exponentials, in several variables, appear as solutions to overdetermined systems of differential equations, this offers Ehrenpreis a way toward a very powerful generalization. More precisely, Ehrenpreis considers a sequence $\{\mathbf{D}_j\}$ of differential operators, with $\mathbf{D}_j = (D_{j1}, \ldots, D_{jn})$, and then seeks conditions on such operators that allow us to study overconvergence properties of the series $\sum f_j$, where the summands in the series are solutions to $\mathbf{D}_j f_j = 0$. The results that Ehrenpreis obtained in this direction are somewhat technical and probably ripe for further analysis. As far as I know, they have not yet been explored with the depth they seem to deserve.

But Ehrenpreis also offers another way of interpreting the series $\sum_{j=1}^{+\infty} c_j e^{ia_j z}$; specifically he points out that if $f(z) = \sum_{j=1}^{+\infty} c_j e^{ia_j z}$, then f can be thought of as a solution of the convolution equation $S * f = 0$, where S is the convolutor whose Fourier transform is, up to some converging factor, the entire function $\Pi(1 - z/a_j)$. It was this beautiful intuition that proved to be most fruitful and opened the way for a variety of interesting generalization. Most notable is probably Kawai's work [18, 19] on what he called the *Fabry–Ehrenpreis gap theorem*, and which stemmed from the interpretation of S as an infinite-order differential operator. Kawai's work is also extremely beautiful and brings into the picture the theory of hyperfunctions, as the natural environment for the study of infinite-order differential operators. As it often happens, new results open new doors, and Berenstein and the author pushed further some of these ideas and applied them to what they called now the *Fabry–Ehrenpreis–Kawai gap theorem* in a series of papers, which exploited the original intuition of Ehrenpreis and found its most general formulation in [2, 3, 21] and finally in [4]. In those papers, we believe that the original vision of Ehrenpreis on the role that convolution equations can play in understanding the overconvergence behavior of Dirichlet series (and generalized Dirichlet series) is carried out to a great extent.

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Leon Ehrenpreis, Recollections from the Recent Decades

Peter Kuchment

Leon Ehrenpreis was an outstanding world-class mathematician and a wonderful, warm person. I had a privilege to consider myself his friend for the last two decades. It is hard to do justice to his manifold mathematics and personality, but I will try to at least add some recollections to this tribute volume.[1]

Leon Ehrenpreis has been one of my mathematical heroes for about 40 years. I first encountered his, Lars Hörmander's, Bernard Malgrange's, and Victor Palamodov's fundamental and beautiful works on systems of linear constant coefficient PDEs in early 1970s, when I was an undergraduate student and then a PhD candidate. They have had a profound impact on me, in particular when working on the Floquet theory of periodic PDEs, which we with Leonid Zelenko started developing in a few years. I am sure that Bernard Malgrange and Daniele Struppa have described this part of Leon's legacy much better than I ever could. I will only address some of the research Leon pursued in the last two decades of his life, which I was lucky to witness.

Some time around 1988, a medical industry contract forced me to learn the basics of a fascinating topic that I had never heard of before, the so-called computed tomography. This turned out to be fateful. Our research group in Voronezh found the mathematics of tomography so challenging and exciting that in the following decades several of us have being devoting a significant part of time working on tomographic problems. Appearance in the 1980s of the Russian translation of the cornerstone book on this topic by Frank Natterer [52] also helped. Interestingly enough, I discovered that several mathematicians whom I admired for their work in completely different areas (e.g., Carlos Berenstein, Simon Gindikin, and Victor Palamodov) had already been working on tomography-related issues. This is an instance of a strange effect that I have observed several times in my life, when several

[1] One can also read the AMS Notices article [30] for recollections of several Leon's friends and colleagues. A volume on tomography [10] is also dedicated to Leon's memory.

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I. Sabadini, D.C. Struppa (eds.), *The Mathematical Legacy of Leon Ehrenpreis*, Springer Proceedings in Mathematics 16,
DOI 10.1007/978-88-470-1947-8_3, © Springer-Verlag Italia 2012

people working in closely related areas suddenly and independently make a sharp turn to the same new direction.

The end of the 1980s was a fascinating time in the former Soviet Union, when contacts with the West have started to become somewhat possible. In particular, the existence of some of famous mathematicians could be checked experimentally (before that, the names like L. Ehrenpreis, L. Hörmander, P. Lax, L. Nierenberg, and many others seemed to me to belong to some deities rather than real people). In 1989 I had my first chance to travel abroad, and I spent about a month in the USA going to various universities and to an AMS tomography conference in Arcata, California. This is where I saw for the first time some of my scientific heroes (e.g., L. Ehrenpreis, S. Helgason, F. Natterer) in flesh.[2]

Meeting Leon in Arcata was a big surprise to me, since I had no clue that he had become interested in integral geometry or tomography. This was another instance of the simultaneous change of direction. He showed a polite interest in what I told him about my PDE work related to his, but it was clear that he was thinking in somewhat different (although not orthogonal) direction now. This was the first time when I heard Leon mentioning his book on Radon transform, which was "nearly finished." It did appear ... in 2003 [28]. In the 13–14 years in between, Leon had been sending generously the nth versions of his manuscript to anyone interested, and the ideas and problems contained in these texts have influenced many of us.

After emigrating later in 1989 to the USA, I found employment at the Wichita State University in Kansas. The year 1990 was a tough time for finding employment for a middle-age emigree mathematician with mediocre, at best, command of English. Having recommendation letters from colleagues such as L. Ehrenpreis was crucial, and I am indebted forever to them and many other mathematicians who supported me in various ways in these difficult times.

Settling down in Wichita was rather pleasant. My family loved the city. The mathematics department was quite good, including several prominent people in the areas of my interest, in particular in inverse problems (Victor Isakov and Ziqi Sun). When I started bringing in speakers and collaborators, Leon Ehrenpreis was one of the first invitees, and since then he had become a constant visitor of our department and then of the Mathematics department of Texas A&M, where I moved in 2001. His lectures and discussions that I and my graduate students had with him were extremely interesting, scientifically rewarding, and personally enjoyable.

I will skip some personal recollections, which one can find in [30] and concentrate rather on mathematics. One of the first topics that we discussed was a strange byproduct of the papers [48, 49] published a couple of years before. There we with S. Lvin described the range of the so-called exponential Radon transform, which

[2]The Arcata meeting was also the place where I met for the first time other colleagues, whom I now consider as long-time friends (J. Boman, D. Finch, A. Markoe, E. T. Quinto, G. Uhlmann, and many others). I could not even imagine that twelve years later I would have a privilege to work at the same department at Texas A&M with another group of researchers whose work I studied and admired as a young mathematician in Russia, such as Ron Douglas, Ciprian Foias, Carl Pearcy, and Gilles Pisier.

arises in the Single Photon Emission Computed Tomography (SPECT), an important medical imaging method [52]. I will not burden the reader with technicalities and just describe the result on a hand-waving level. It is known [34–36, 42, 43, 52] that the ranges of Radon-type transforms are usually of infinite codimension in natural function spaces. Knowing the description of the range plays an important role in integral geometry and tomography. After range conditions are found, it is usually straightforward to go back and check their necessity,[3] while a proof of their completeness is usually technical. Thus, when the conditions of [48] were found, we expected that reproving their necessity should be a piece of cake: just plug the transform of a function into these conditions and see immediately that they are satisfied. However, when we did this, we discovered an infinite and totally nonobvious to us set of nonlinear differential identities for the standard sine function: for any odd natural n,

$$\sum_{k=0}^{n} \binom{n}{k} \left(\frac{d}{dx} - \sin x\right) \circ \left(\frac{d}{dx} - \sin x + i\right) \circ \cdots \circ \left(\frac{d}{dx} - \sin x + (k-1)i\right) \times \left((\sin x)^{n-k}\right) = 0, \tag{1}$$

where i is the imaginary unit, and $\circ$ denotes the composition of differential operators. The attempt to prove these identities directly (i.e., without any integral geometry and Fourier analysis) succeeded [49] but took a significant time. We are still puzzled by the meaning of these identities [50]. Several integral geometry and tomography experts devoted their time and effort to trying to understand better the meaning of these range conditions. This is also what we set out to do with my PhD student Valentina Aguilar and Leon Ehrenpreis. We succeeded in the following sense: we showed, in particular, that these identities are equivalent to an interesting theorem of separate analyticity type.

Theorem 1 ([8]) *Let D be a disk in $\mathbb{R}^2$, and f be a function in the exterior of D. Suppose that when restricted to any tangent line L to D, the function $f|_L$, as a function of one real variable, extends to an entire function on the complexification of L. Then f, as a function on $\mathbb{R}^2 \setminus D$, extends to an entire function on $\mathbb{C}^2$.*

Well, this fact also did not look obvious to us. Analyticity of f in a complex neighborhood of $\mathbb{R}^2 \setminus D$ follows from the old (and not that well-known) separate analyticity theorem by S. Bernstein (see [9]); however this theorem cannot produce statement about f being an entire function. Thus, since proving the above theorem, a couple of things about it kept bothering us for several years. First of all, this is a fact of several complex variables, while our proof did not look like an SCV argument at all. Is there a truly complex analysis proof? Another, related, question is whether such a theorem can be proven for a different convex body instead of a

[3] For instance, when the so-called *moment conditions* [35, 42] for the standard Radon transform are written, checking their necessity boils down to noticing that the kth power $(x \cdot \omega)^k$ of the inner product of two vectors is a homogeneous polynomial of degree k with respect to each of them.

disk D? An SCV proof was later provided in [54], although it was rather complicated and was not generalizable (at least, easily) to other convex curves. Leon has worked out some other examples of convex algebraic curves (unpublished), but general picture remained unclear to us. Finally, A. Tumanov presented recently [59] a beautiful short proof based on attachment of analytic disks (where Tumanov is a great expert), which works for any strictly convex body D with a mild condition on the smoothness of its boundary.

Another issue that we addressed with Leon and my Master student Alex Panchenko also originated from emission tomography. The exponential Radon transform in SPECT depends upon an "attenuation" parameter $\mu \geq 0$. In [29], we introduced and studied a "mother" exponential Radon transform, which had no free parameters, but by different restrictions of which one can obtain the exponential Radon transforms corresponding to all possible values of the attenuation. We also obtained the range description there, which was based upon the F. John's differential equations. In this particular case, the (ultrahyperbolic) John's equation could be recast as a boundary Cauchy–Riemann equation.

Although we have not done any joint research since 2000, we kept discussing (in person and by e-mail) various integral-geometric and PDE issues. One was the fascinating and surprisingly hard "strip problem" [1, 2, 4, 6, 37, 38, 57, 58], to which Leon has contributed [27, 28] and which he extended to a more general PDE setting (see, e.g., [6, 27, 28]). It was eventually resolved due to efforts of several mathematicians, including M. Agranovsky, J. Globevnik, and A. Tumanov (see the reference above).

Leon was also very much interested in the activity concerning the "restricted spherical means" operator, i.e., a version of Radon transform that integrates a given function over spheres of arbitrary radii, but with the centers restricted to a hypersurface S. The study of such operators was very active since the beginning of the 1990s, due first to needs of approximation theory , then self-sustained just due to the beauty and complexity of arising problems (see [3, 7] and references therein), and finally it received a huge boost in the last decade, due to the discovered relations to a newly developing method of medical imaging, the so-called thermo-/photo-acoustic tomography (see the surveys [5, 31–33, 47, 60] and references there).

The restricted spherical mean problem happens to be a very particular case of one of the questions raised by Leon in his book [28]. This brings us from the "small" problems discussed above to the much more general thinking Leon has been doing on transforms of Radon type and their very wide generalizations. This was reflected in his papers of the period and in the monograph [28]. The title of this book,"*The Universality of the Radon Transform*," and the wealth of topics and ideas covered and variety of open problems suggested shows how deeply Leon believed in wide range importance of this approach. He was not the first to realize such widespread applicability of transforms of Radon type, although probably the first to give such a bold name to a book. Fritz John in his book [46] showed how important this circle of ideas is for PDEs. Israel Gelfand, Simon Gindikin, Sigurdur Helgason, Victor Palamodov, and many other mathematicians studied in detail applications to PDEs, harmonic analysis, group representation theory, special functions, mathematical physics, etc. (e.g., [34–36, 42–45, 55]). Still, Leon's book is rather unique in terms of many nonstandard issues raised there. Leon also was unique in his writing style, introducing new notations and names for well-known objects, which does not help a reader. However, after getting through these hurdles, one opens a treasure chest of ideas.

The variety of things that Leon addressed in the book [28] and his other publications of the time [11–27], and which he considered inter-related, is enormous: "exotic" boundary-value problems for PDEs, Poisson summation formulas, Eisenstein and Poincare series on $SL(2,\mathbb{R})$ and $SL(3,\mathbb{R})$, various number-theoretic problems, Hartogs–Lewy extension, FBI transform (although it carries an unrecognizable name in [28], being an instance of what he called "nonlinear Fourier transform"), edge-of-the-wedge theorems, Phragmén–Lindelöf type theorems for PDEs, special functions, among others.

Notwithstanding the overarching title, a wide variety of topics covered, and large volume, [28] is neither a textbook on the "usual" Radon transform nor a comprehensive historical survey or reference manual; it is not designed for reading by an uninitiated; it does not cover many important developments, techniques, and results that one can find in [34–36, 39–45, 55, 56], such as curved manifolds case, κ-operator approach, Radon transforms of differential forms and tensors, projective geometry setting, most of the group representation relations, etc. At Leon's request, Todd Quinto and I contributed the appendix [51] to [28] devoted to a brief survey of some tomographic applications. Due to the natural size limitations, it also cannot be considered comprehensive. One can find a thorough discussion of tomographic issues in [52, 53].

In spite of all these omissions, this unique book [28] should occupy a space on the bookshelf of anyone working on PDEs, Fourier analysis, several complex variables, and integral geometry. I am sure it will be a source of inspiration for many mathematicians, who will take their time to get through the text.

The memory of Leon Ehrenpreis will stay with all who encountered his amazing mathematics and experienced his friendship. I am grateful to the fate for giving me the chance and privilege to meet Leon and to collaborate with him.

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Part II
Invited Papers

Analyticity on Curves

Mark Agranovsky and Lawrence Zalcman

Abstract Under what conditions can one conclude that a continuous function on a plane domain Ω is holomorphic, given that its restrictions to a collection of Jordan curves in Ω which cover Ω admit holomorphic extensions? We survey progress on this problem over the past 40 years, with an emphasis on recent results.

1 Introduction

The circle of ideas discussed in this paper originates with the following:

Question *Let $f \in C(\mathbb{R}^2)$ and suppose that for each circle γ of (fixed) radius $r > 0$ in the plane, the restriction of f to γ has a continuous extension to the closed disc $\overline{D}_\gamma$ bounded by γ which is analytic in the open disc D_γ. Must f be an entire function?*

It is well known (and easy to see) that f extends from $\gamma_r(w) = \{z : |z - w| = r\}$ continuously to a function analytic on $D_{\gamma_r(w)}$ if and only if

$$\int_{\gamma_r(w)} f(z) z^n \, dz = 0, \quad n = 0, 1, 2, \ldots, \tag{1}$$

We dedicate this paper to the memory of our friend Leon Ehrenpreis. Leon was fascinated by the strip problem, contributed to its solution [13], and led the way in generalizing it from a result concerning analytic functions to solutions of elliptic equations [14]. Indeed, one of his last major addresses, the opening lecture of the conference Integral Geometry and Tomography, delivered at Stockholm University on August 12, 2008, was entitled "The Strip Theorem for PDE"; see [15, II–IV].

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I. Sabadini, D.C. Struppa (eds.), *The Mathematical Legacy of Leon Ehrenpreis*,
Springer Proceedings in Mathematics 16,
DOI 10.1007/978-88-470-1947-8_4,