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International Centre for Mechanical Sciences

Variational Models and Methods in Solid and Fluid Mechanics

CISM Courses and Lectures, vol. 535

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INTERNATIONAL CENTRE FOR MECHANICAL SCIENCES

COURSES AND LECTURES - No. 535

VARIATIONAL MODELS AND METHODS IN SOLID AND FLUID MECHANICS

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SpringerWienNewYork

This volume contains 63 illustrations

Printed in Italy
SPIN 80124210

All contributions have been typeset by the authors.

ISBN 978-3-7091-0982-3 SpringerWienNewYork

PREFACE

For this would be agreed by all:
that Nature does nothing in vain
nor labours in vain

Olympiodorus, Commentary on Aristotle's Meteora translated by Ivor Thomas in the Greek Mathematica Works Loeb Classical Library

La nature, dans la production de
ses effets, agit toujours par les
voies les plus simples

Pierre de Fermat

The CISM course C-1006 "Variational models and methods in solid and fluid mechanics" was held July 12-16, 2010 in Udine, Italy. There were about forty five participants from different european countries. The papers included in this volume correspond to the content of five mini-courses of 6 hours each which have been delivered during this week.

Variational formulation of the governing equations of solid and fluid mechanics is a classical but a very challenging topic. Variational methods give an efficient and elegant way to formulate and solve mathematical problems that are of interest for scientists and engineers. This formulation allows for an easier justification of the well-posedness of mathematical problems, the study of stability of particular solutions, a simpler implementation of numerical methods. Often, mechanical problems are more naturally posed by means of a variational method. Hamilton's principle of stationary (or least) action is the conceptual basis of practically all models in physics. The variational formulation is also useful for obtaining simpler approximate asymptotical models as done in the theory of homogeneization. In many problems of mechanics and physics, the functionals being minimized depend on parameters which can be considered as random *variables. Variational structure of such problems always brings considerable simplifications in their study.*

In this course, three fundamental aspects of the variational formulation of mechanics will be presented: physical, mathematical and applicative ones.

The first aspect concerns the investigation of the nature of real physical problems with the aim of finding the best variational formulation suitable to those problems. A deep knowledge of the physical problems is needed to determine the Lagrangian of the system and the nature of the variations of its motions which may be considered admissible. Actually one could say that all knowledge which is available about a system is resumed by the choice of:

- *a configuration space used to describe mathematically the system*
- *a set of admissible motions used to describe the different ways in which the system may evolve*
- *a Lagrangian functional which once minimized supplies evolution equations and boundary conditions*

The second aspect is the study of the well-posedeness of those mathematical problems which need to be solved in order to draw previsions from the formulated models. It is relatively simple to conjecture properties to be required to the Lagrangian functional in order to be assured the well-posedness of the corresponding evolution system. Much more complex is to get such results of well-posedness studying some evolution equations which are obtained with euristic schemes different from those based on Hamilton's principle. In fact always, when one needs to study mathematically a set of evolution equations, the first move is to try to put them in a variational form. It is then advisable and wiser to try to use a variational principle at the beginning of the formulation of a mathematical model.

The third aspect is related to the direct application of variational analysis to solve real engineering problems. Variational principles supply very powerful tools for getting qualitative previsions about the behaviour of the studied systems, but also for formulating effective numerical methods to get quantitative previsions.

The following problems have been presented and studied during the course :

- *Rayleigh-Hamilton's Principle for establishing governing equations and boundary conditions for second gradient models for heterogeneous deformable bodies ;*

- *A variational approach to multiphase flow problems and description of diffuse solid-fluid interfaces;*
- *New variational models of brittle fracture mechanics and some related problems ;*
- *The methods of stochastic calculus of variations and their applications to the homogenization problems and modeling of microstructures and their evolution ;*
- *Dynamical problems in damping generation and control in the situations where the energy initially conferred to a system undergoes a principle of irreversible energy confinement into a small region ;*

We are extremely grateful to all participants of the course for creating a nice atmosphere for scientific discussions. We would like also to express our thanks to the CISM staff for their assistance in the running of the course.

Francesco dell'Isola, University of Rome "La Sapienza"

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CONTENTS

Variational principles are a powerful tool also for formulating field theories

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Abstract Variational principles and calculus of variations have always been an important tool for formulating mathematical models for physical phenomena. Variational methods give an efficient and elegant way to formulate and solve mathematical problems that are of interest for scientists and engineers and are the main tool for the axiomatization of physical theories.

1 Introduction and historical background

1.1 Metrodoron and his followers

The ideas we want to evocate in this lecture are very old and were put forward already in the hellenistic period: for a detailed discussion about this point the reader is referred to the beautiful book by Lucio Russo (2003). In that book it is established that "modern" science actually was born in the hellenistic era, when Metrodoron lived. Metrodoron was a pupil of a famous greek philosopher, Epicurus, and, in our opinion, the following Metrodoron's sentence is a statement (the first?) belonging to the modern philosophy of science:

«Μέμνεσο ὅτι θνητὸς ὢν τῇ φύσει καὶ λαβὼν χρόνον ὡρισμένον ἀνέβης τοῖς περὶ φύσεως διαλογισμοῖς ἐπὶ τὴν ἀπειρίαν καὶ τὸν αἰῶνα καὶ κατεῖδες "τὰ τ' ἐόντα τά τ' ἐσσόμενα πρό τ' ἐόντα"».

Metrodoron,

"Always remember that you were born mortal and such is your nature and you were given a limited time: but by means of your reasonings about Nature you could rise to infinity and to eternity

> *and you indeed contemplate* ***"the things that were, and that were to be, and that had been before"***. Metrodoron
>
> Gnomologium Epicureum Vaticanum X (fr.37 Alfred Körte, Metrodori Epicurei Fragmenta, " Jahrbücher für classiche Philologie", Suppl. XVII, 1890, p. 557).

This dictum, following Körte, comes from a lost letter or book by Metrodoron (the Epicurean philosopher) addressed to Menestratos who was presumably one of his pupils. The words quoted in bold are a citation from Iliad, I 70 (the translation into English of the sentence in boldface is ours; except for this citation the translation has been taken from Homer by Murray (1924), see the ref. (14)).

In different words, Metrodoron states that by using (the right!) equations you can forecast future behavior of physical systems.

1.2 Why Variational Principles and Calculus of Variation?

In recent time, a lost Archimedes' book (19) has been rediscovered. Some authors claim that Archimedes seems to have solved, in this book and using a variational principle, the technological problem of finding the optimal shape of a boat. Archimedes seems to have chosen, as optimality criterion, that the vertical position must be a "very" stable configuration (see Rorres (2004)). In the book of Russo (21) it is demonstrated in even a more convincing way that many optimization techniques were well-known in hellenistic science. In particular Russo proves that the problem of the determination of the regular polygon having maximal area has been solved in that period. Thus, the use of a variational principle and optimization methods to solve technological problems is less recent than it is usually believed. In general, variational formulation of the governing equations of solid and fluid mechanics is a classical but very challenging topic. This kind of formulation allows for an easier proof of the well-posedness of mathematical problems, for an easier investigation of the study of stability of particular solutions, and for a simpler implementation of numerical methods. Often (but one who believes in Russo's vision about the birth of science could say instead "always"), mechanical problems are more naturally posed by means of variational methods. Hamilton's principle of stationary (or least) Action is the conceptual basis of practically all models in physics. The variational formulation is also useful for obtaining simpler approximate asymptotical models as it is done in the theory of homogenization.

We want simply to state here that the Principle of Virtual Works and the Principle of Least Action have roots much deeper than many scientists believe (see Vailati, 1897). Although many histories of science claim differently, most likely the majority of physical theories were first formulated in terms of these Principles, and only subsequently they were reconsidered from other points of view. In our opinion the Principle of Least Action, which supplies a "geometric" version of mechanics was indeed the tool used by the true founders of mechanics (i.e. the scientists of the hellenistic period) to establish it. As argued also by Colonnetti (5) and Netz and Noel (19)) surely also Archimedes and ancient greek scientists were accepting such a point of view.

The epigones of the hellenistic science, who were not able to understand the delicate mathematical arguments used by the first scientists, however could understand the minimality conditions obtained by their "maitres" (i.e. the conditions corresponding to those which we call now Euler-Lagrange equations and boundary conditions) and could grasp the "physical" arguments used to interpret them. This phenomenon is perfectly clear to everyone who is ready to consider carefully -for instance- the evolution of the theory of Euler-Bernoulli Beam (a useful reference about this point is the book of Benvenuto (1981)). Euler postulated a Principle of Least Action for the Elastica, and gets the celebrated equilibrium differential equation and boundary conditions for the equilibrium of beams by using the calculation procedure due to Lagrange (which is the departing idea of Calculus of Variations). Then Navier prepared his lectures for the Ecole Polytechnique and resumed the results obtained by Euler deciding to "spare" to the (engineering) students the difficulties of the calculus of variations. He started directly from the equilibrium equation, obtained by means of an "ad hoc" principle of balance of force and couple, and imposed boundary conditions based on "physical" assumptions. As a consequence, for a long while, generations of engineers believed that the beam equations were to be obtained in this way. Only when numerical simulations became popular, then they (actually, some of them) became aware of variational "principles". However these principle were proven as theorems starting from "balance postulates" and were considered simply as a mathematical (rather abstruse) tool and not as a fundamental heuristic concept. And this attitude is not changed even when it became clear that every serious advancement of mechanical science has been obtained using variational principles. Indeed the so called "physical sense" (a gift that many claim to posses but which nobody can claim to be able to master or to teach) is not very useful to postulate the right "balance principles" when one is in "terra incognita". For instance, when Lagrange and Sophie Germain wanted to find the plate equations they needed to employ a variational principle (and they could find the (right!) natural boundary conditions). Again when Cosserat brothers wanted to improve Cauchy Continuum Mechanics they "rediscovered" the right technique: i.e. the Principle of Least Action. Also Quantum Mechanics has been developed starting from a Variational Principle (see e.g. the references of Feynman (11), Lagrange (15) and Lanczos (16)).

Therefore an important warning is due to young researcher: refrain from trying to extend available models by means of "ad hoc" adaptations of available theories: always look for the right Action functional to be minimized!

1.3 The problem of including dissipation

One useful tool for handling complicated situations is used in Continuum Mechanics by Paul Germain when formulating second gradient theories: the Principle of Virtual Powers. Again, as remarked always in the history of the development of ideas, when this history can be reconstructed, the effective way to be used to proceed is that which starts from a Principle of Least Action, eventually generalized into a Principle of Virtual Powers. For a long time the opponents to Second Gradient Theories argued about its lack of consistency, due to the difficulties they claim to find in "getting" boundary conditions. This is a really odd statement. Indeed variational principles easily produce mathematically correct boundary conditions. So maybe what those opponents want to say is that as they are not so clever as Navier, they do not manage to interpret physically the boundary conditions found via a (correct and meaningful) variational principle. Of course if one refuses to use the Principle of Least Action he can find very difficult the job of determining some set of boundary conditions which are compatible with the (independently postulated!) bulk evolution equations. If instead one accepts the Archimedean (the reader will allow us to dream, without definitive evidence that such was the point of view of Archimedes) approach to mechanics then all these problems of well-posedness of mathematical models completely disappear.

Variational Principles always produce intrinsically well-posed mathematical problems, if the Action functional is well behaving. Of course passing from Lagragian systems (the evolution of which are governed by a Least Action functional) to non-Lagragian systems (for which such a functional may not exist) maybe difficult. This problem is related (but is not completely equivalent) to the problem of modelling dissipative phenomena. It is often stated that dissipation cannot be described by means of a Least Action Principle. This is not exactly true, as it is possible to find some Action functionals for a large class of dissipative systems. However it is true that not every conceived system can be regarded as a Lagragian one. This point is delicate and will be only evocated here. In general a non-Lagragian system can be regarded as Lagragian in two different ways: i) because it is an "approximation" of a Lagrangian system (see the case of Cattaneo equation for heat propagation), and this approximation leads to "cancel" the lacking part of the "true" Action Functional ii) because the considered system is simply a subsystem of a larger one which is truly Lagrangian. The assumption that variational principle can be used only for non-dissipative systems is contradicted by, e.g., the work presented in this book by Prof. Frankfort (12), where you find variational principles modelling dissipative systems. Indeed it is often stated that a limit of the modelling procedure based on variational principles consists in their impossibility of encompassing "nonconservative" phenomena. We do not believe that this is the case: however in order to avoid to be involved in a problem which is very difficult to treat, when dealing with dissipative systems, we will assume a slightly different point of view, usually attributed to Hamilton and Rayleigh.

2 Finding a mathematical model for natural phenomena

2.1 Principle of Least Action

We want to discuss here about the problem of finding a mathematical model for natural phenomena. We start with an epistemological Principle:

> "The Principle of Least Action tells us how to construct a mathematical model to be used for describing the world and for predicting the evolution of the phenomena occurring in it".

In the following modeling scheme, we give the right heuristic strategy to be used for finding an effective model using the Principle of Least Action. The recipe includes the following ingredients:

1. Establish the right kinematics needed to describe the physical phenomena of interest, i.e. the kinematical descriptors modeling the state of considered physical systems.
2. Establish the set of admissible motions for the system under description, i.e. establish the correct model for the admissible evolution of the system.
3. Employ the "*physical intuition*" to find the right *Action functional* to be minimized, i.e. modeling what Nature wants to minimize.

We start by finding the kinematical descriptors, because of the need of modeling the states of the considered system. Then we introduce motion, in such a way we model the evolution of the system to be described. Finally we ask Nature what is the quantity to minimize. Keeping this quantity in mind, we introduce the Action functional. To start with, it is necessary to focus the attention on a specific class of systems and on phenomena occurring to them. A configuration is the mathematical object used to model the state of considered systems: the set of possible configurations will be denoted by C. The motion is the mathematical model describing the evolution of considered systems: it is a C-valued function defined on time interval (t_0, t_f); the set of all admissible motions will be denoted by M. The Action is a real-valued function, defined on M, which models the "preferences" of nature.

Finally, to use the Principle of Least Action one needs three steps further,

4. Find the Euler-Lagrange conditions which are consequence of the postulated Least Action Principle
5. Interpret these condition on a physical ground
6. Determine, in terms of the postulated Action functional, the numerical integration scheme to be used to get the previsions needed to drive, by means of our theory, our experimental, technological or engineering activity.

2.2 The Rayleigh-Hamilton principle

When postulating an extended Rayleigh-Hamilton principle, the point 4 of the previously presented heuristic strategy will be further divided into two steps as follows:

- 4a. Once the quantities which expend power on the considered velocity fields are known in terms of postulated Action, introduce a suitable definite positive Rayleigh dissipation functional
- 4b. Equate the first variation of Action functional to the Rayleigh dissipation functional and get the evolution equations (including boundary conditions) which govern the motion of the system

Although in the literature the choice of including a Rayleigh-Hamilton principle in the class of variational principles is sometimes considered inappropriate, we will follow what seems to us the preference of the majority of the authors: therefore we do call "variational" also the strategy which we just described, not limiting the use of this adjective to the models using exclusively the Least Action Principle.

2.3 La Cinèmatique d'Abord !

According to Metrodoron, mathematical and physical objects are two different concepts. Indeed, equations are necessary for modeling physical systems but they refer to mathematical objects. When one solves the equations formulated in the framework of his model then he has to transform the obtained equations into previsions valid for the physical system he is studying. A good modeling procedure uses mathematics for finding the motion which minimizes Action. If this mathematics gives a reasonable forecasting of the observed evolution, then the model is valid. However, not everything is described by a given model. A model is always focused on a set of phenomena.

The set of phenomena that are focused by a model is established by the kinematics:

La Cinèmatique d'Abord !

In the previous scheme it is clear that the most "fundamental" step concerns the choice of the set of configurations used for characterizing the "accessible" states of the system. When constructing a mathematical model using the discussed epistemological principle, one must start with a precise and clear determination of the set C. The second step concerns the determination of admissible motions which clearly depend on the evolutionary phenomena one wants to model. A correct modeling process always starts specifying "admissible" kinematics.

2.4 La Nature agit toujours par les voies les plus simples

After having specified the admissible kinematics, one can wonder about the *desire* of Nature. The *utility* of Nature is a real-valued function defined on M. Following Maupertuis, we will call Action this "utility". Also Nature must consider which is the contingent situation: not all admissible motions are accessible by a physical system under given specific conditions. Therefore we must specify a subset M_A of the set M: the set of accessible motions. The *real* motion will be chosen by the system minimizing the Action in the subset M_A. Indeed:

La Nature agit toujours par les voies les plus simples.

2.5 Two possible choices for the set of admissible and accessible motions

In the famous textbook of Arnold (1) the author, following the tradition, does not "try" to explain Maupertuis' Principle of Least Action. We instead dare to try to deal with this. In the process of minimization of the Action, we need to specify the set of motions among which we look for minima. The choice of Lagrange is that of isochronous motions. Two motions are isochronous when they both start, at the given instant t_0, from a given configuration C_0 and arrive, at the same instant t_f, at the same final configuration C_f. On the other hand, the choice of Maupertuis is to focus on the set of motions with a "fixed energy content" and which are starting from the same configuration C_0 and ending (the instants of start and stop are not specified!) at the configuration C_f. In the set of admissible motions an "energy" functional must then be introduced: i.e. a functional which associates an energy content to any motion and any time instant t. The set of accessible motions is constituted by all motions from C_0 to C_f which have a constant energy content. The choice of Maupertuis, if not suitably modified, seems to limit the range of applicability of variational principles to non dissipative phenomena.

2.6 Further famous quotes

Many books in Calculus of Variations and/or Variational Principles, see e.g. that of Lanczos (1970), start with a preface, introduction or introductory chapter dealing with historical prolegomena and sometimes end with a philosophical chapter. In presenting this lecture notes, we did not dare to break with tradition.

> "For this would be agreed by all: that Nature does nothing in vain nor labours in vain". Olympiodorus, Commentary on Aristotle's ™Meteora translated by Ivor Thomas in the Greek Mathematical Works Loeb Classical Library
>
> "La nature, dans la production de ses effets, agit toujours par les voies les plus simples". Pierre de Fermat.

Now, the problem is:

> What is utility?

3 In other words: How to find "Real Motions"?

Up to now no mathematical structure has been assumed for M_A. Indeed, Action functional is simply a real-valued map defined on M_A. "Practical" problems require the *calculation* of *real motions* by means of introduced model. Following Lagrange (15), we introduce a particular class of Action functionals in terms of a Lagrangian Action density function: so constructing in a particular way Action functional to obtain so called "Lagrangian functionals".

We need to introduce a topological structure in M_A, i.e. we need to clearly define what we mean when we say that "two motions are close". If we want to find minima of a real-valued function, then we need to estimate derivatives and equate these derivatives to zero. Action is a function defined in the set of motions (not real numbers!). Thus, we need

- to understand what is an infinitesimal variation of motion,
- to find a differential of a functional and
- to estimate the order of infinitesimal of its remainder.

In other words, we need to learn how to find a first order Taylor expansion for a Lagrangian functional by establishing the meaning of the expressions :

- Infinitesimal variation of motion.
- Differential of a functional.
- Order of infinitesimals for remainders.

This implies the need of Frechèt and Gateaux derivatives in manifolds with charts in Banach spaces. This is the right mathematical frame for studying this subject. However, Lagrange did not know that he was using such a mathematical frame and did not know anything about Frechèt and Gateaux derivatives. Thus, in this notes we try to go around the related mathematical difficulties and follow the original approach of Lagrange.

The motion minimizing Action will be searched among the motions for which the first variation of Action vanishes.

For Lagrangian functionals this condition is equivalent to a partial differential equation which is called Euler-Lagrange condition relative to the given Action functional. This procedure generalizes the corresponding one used for real-valued functions of several real variables. One serious problem with papers that start from balance equations and "play" with forces is that they do not "find" boundary conditions. In these references ((7; 8; 9)) one can find examples of modelling procedures in which one finds simultaneously bulk and boundary conditions.

From an historical point of view, in the theory of beam we deal with contact actions (normal and shear forces and momenta) because Navier has written lecture notes for l'Ecole Polytechique, trying to produce a text for students that was as simple as possible. He wrote final equations and explain not only bulk but also boundary conditions with the aid of "physical sense". However, it is very difficult in general to **find** evolution equations and boundary conditions with physical sense. On the other hand, variational principles give boundary conditions automatically and without the help of any physical sense.

Thus, Variational Principles allow Science to unveil Nature and for unveiling Nature you need a Lagrangian functional.

4 Lagrangian Action Functionals: technical details

We follow Landau and Lifshitz (1977) and Moiseiwitsch (1966).

Let $\Psi_\sigma(x_\mu)$ be any set of n tensor fields defined on $\mathbb{R}^m$, (σ being a multi-index and $\mu = 1, 2, ..., m$). We define the Lagrangian density as:

$$\mathfrak{L}\left(x_\mu, \Psi_\sigma, \frac{\partial \Psi_\sigma}{\partial x_\mu}\right). \tag{1}$$

We can then introduce the Action functional as

$$\mathfrak{A} = \int_T \mathfrak{L}\left(x_\mu, \Psi_\sigma, \frac{\partial \Psi_\sigma}{\partial x_\mu}\right) \tag{2}$$

Where T is a hyper-volume in the $m-$th dimensional space determined by the coordinates x_μ. When we will want to derive the theory of second gradient materials, this approach will not be appropriate, because we would need to add the dependence on the second gradient of Ψ_σ in (1).

4.1 Variation of the Action Functional

We now consider small variations $\varepsilon\eta_\sigma(x_\mu)$ of the considered fields $\Psi_\sigma(x_\mu)$:

$$\tilde{\Psi}_\sigma(x_\mu) = \Psi_\sigma(x_\mu) + \varepsilon\eta_\sigma(x_\mu), \tag{3}$$

where the $\eta_\sigma(x_\mu)$ are any set of linearly independent functions of the x_μ which vanish on the part $\partial_d T$ ($\partial_d T \subseteq \partial T$) of the boundary ∂T of the hyper-volume T, on which the kinematical condition are prescribed. The variation of the Action functional can then be computed as:

$$\Delta\mathfrak{A} = \int_T \mathfrak{L}\left(x_\mu, \tilde{\Psi}_\sigma, \frac{\partial \tilde{\Psi}_\sigma}{\partial x_\mu}\right) - \int_T \mathfrak{L}\left(x_\mu, \Psi_\sigma, \frac{\partial \Psi_\sigma}{\partial x_\mu}\right), \tag{4}$$

where T is a hyper-volume in the m-th dimensional space determined by the x_μ. The computation of the variation of the Action functional now proceeds as follows:

$$\Delta\mathfrak{A} = \int_T \mathfrak{L}\left(x_\mu, \Psi_\sigma + \varepsilon\eta_\sigma, \frac{\partial \Psi_\sigma}{\partial x_\mu} + \varepsilon\frac{\partial \eta_\sigma}{\partial x_\mu}\right) - \int_T \mathfrak{L}\left(x_\mu, \Psi_\sigma, \frac{\partial \Psi_\sigma}{\partial x_\mu}\right) + O\left(\varepsilon^2\right) \tag{5}$$

which, with a slight abuse of notations, can be written at the first order in ε as:

$$\delta\mathfrak{A} = \varepsilon \int_T \sum\nolimits_\sigma \left(\frac{\partial \mathfrak{L}}{\partial \Psi_\sigma}\eta_\sigma + \sum_{\mu=1}^{m} \frac{\partial \mathfrak{L}}{\partial\left(\partial \Psi_\sigma / \partial x_\mu\right)} \frac{\partial \eta_\sigma}{\partial x_\mu}\right) \tag{6}$$

Integrating by parts and recalling that η_σ vanish on $\partial_d T$ it is easy to get:

$$\delta\mathfrak{A} = \varepsilon \int_T \sum\nolimits_\sigma \eta_\sigma \left(\frac{\partial \mathfrak{L}}{\partial \Psi_\sigma} - \sum_{\mu=1}^{m} \frac{\partial}{\partial x_\mu} \left(\frac{\partial \mathfrak{L}}{\partial \left(\partial \Psi_\sigma / \partial x_\mu \right)} \right) \right) \tag{7}$$
$$+\varepsilon \int_{\partial T/\partial_d T} \sum\nolimits_\sigma \eta_\sigma \sum_{\mu=1}^{m} \frac{\partial \mathfrak{L}}{\partial \left(\partial \Psi_\sigma / \partial x_\mu \right)} N_\mu,$$

where $\partial T/\partial_d T$ is the difference between ∂T and $\partial_d T$ and N_μ is the external unit normal of $\partial T/\partial_d T$. Imposing $\delta\mathfrak{A} = 0$, the arbitrariness of η_σ gives, for any σ:

$$\frac{\partial \mathfrak{L}}{\partial \Psi_\sigma} - \sum_{\mu=1}^{m} \frac{\partial}{\partial x_\mu} \left(\frac{\partial \mathfrak{L}}{\partial \left(\partial \Psi_\sigma / \partial x_\mu \right)} \right) = 0, \quad \forall x_\mu \in T, \tag{8}$$

$$\sum_{\mu=1}^{m} \frac{\partial \mathfrak{L}}{\partial \left(\partial \Psi_\sigma / \partial x_\mu \right)} N_\mu = 0, \quad \forall x_\mu \in \partial T/\partial_d T. \tag{9}$$

In the case of a discontinuity material surface Σ (with unit normal N_μ) the (9) have to be completed by

$$\sum_{\mu=1}^{m} [| \frac{\partial \mathfrak{L}}{\partial \left(\partial \Psi_\sigma / \partial x_\mu \right)} |] N_\mu = 0, \quad \forall x_\mu \in \Sigma, \tag{10}$$

where $[| (\cdot) |]$ is the jump of $(\cdot)$ across the surface Σ. These equations are known as the Euler-Lagrange equations corresponding to the considered Lagrangian density.

4.2 The Space-Time Case ($m = 4$)

Let us now consider the particular case in which $m = 4$. This case corresponds, for instance, to the case $x_\mu = (x_1, x_2, x_3, t)$. We have that $\eta_\sigma(x_\mu)$ are any set of linearly independent functions of the x_μ which vanish on the boundary of time type domain,

$$\eta_\sigma(x_1, x_2, x_3, t_0) = \eta_\sigma(x_1, x_2, x_3, t_1) = 0$$

and on the part $\partial_d V$ of the boundary ∂V of the volume V, on which the kinematical conditions are prescribed,

$$\eta_\sigma(x_1, x_2, x_3, t) = 0, \quad \forall (x_1, x_2, x_3) \in \partial_d V, \quad \forall t \in [t_0, t_1].$$

It is easy to show that in this particular case eq. (7) yields

$$\delta\mathfrak{A} = \varepsilon \int_{t_0}^{t_1} dt \int_V \sum\nolimits_\sigma \eta_\sigma [\frac{\partial \mathfrak{L}}{\partial \Psi_\sigma} - \sum_{k=1}^{3} \frac{\partial}{\partial x_k}\left(\frac{\partial \mathfrak{L}}{\partial\left(\partial \Psi_\sigma/\partial x_k\right)}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathfrak{L}}{\partial\left(\partial \Psi_\sigma/\partial t\right)}\right)] + \varepsilon \int_{\partial V/\partial_d V} \sum\nolimits_\sigma \eta_\sigma \sum_{k=1}^{3} \frac{\partial \mathfrak{L}}{\partial\left(\partial \Psi_\sigma/\partial x_k\right)} N_k + \int_\Sigma \sum\nolimits_\sigma \left[\left|\eta_\sigma \sum_{k=1}^{3} \frac{\partial \mathfrak{L}}{\partial\left(\partial \Psi_\sigma/\partial x_k\right)}\right|\right] N_k \tag{11}$$

The stationarity $\delta\mathfrak{A} = 0$ of the Action implies, for any $\sigma = 1, 2, ..., n$,

$$\frac{\partial \mathfrak{L}}{\partial \Psi_\sigma} - \sum_{k=1}^{3} \frac{\partial}{\partial x_k}\left(\frac{\partial \mathfrak{L}}{\partial\left(\partial \Psi_\sigma/\partial x_k\right)}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathfrak{L}}{\partial\left(\partial \Psi_\sigma/\partial t\right)}\right) = 0, \quad \forall x_k \in V, \tag{12}$$

$$\sum_{k=1}^{3} \frac{\partial \mathfrak{L}}{\partial\left(\partial \Psi_\sigma/\partial x_k\right)} N_k = 0, \quad \forall x_\mu \in \partial V/\partial_d V, \tag{13}$$

$$\sum_{k=1}^{3} \left[\left|\frac{\partial \mathfrak{L}}{\partial\left(\partial \Psi_\sigma/\partial x_k\right)}\right|\right] N_k = 0, \quad \forall x_\mu \in \Sigma. \tag{14}$$

Which are the standard Euler-Lagrange equations. We will see in the next chapters of this book how to generalize (14) when Σ can move freely.

5 Principle of Virtual Power and Principle of Least Action

The principle of least Action, when formulated for Action functionals admitting first differentials, can be regarded as a particular form of the principle of virtual powers. Indeed, if

$$\mathfrak{A} = \mathfrak{A}^{int} + \mathfrak{A}^{ext} + \mathfrak{A}^{ine} \tag{15}$$

then

$$\delta\mathfrak{A} = 0 \quad \Longleftrightarrow \quad \delta\mathfrak{A}^{int} + \delta\mathfrak{A}^{ext} + \delta\mathfrak{A}^{ine} = 0. \tag{16}$$

Identifying

$$\delta\mathfrak{A}^{int} = \mathfrak{P}^{int} \quad \delta\mathfrak{A}^{ext} = \mathfrak{P}^{ext}, \quad \delta\mathfrak{A}^{ine} = \mathfrak{P}^{ine}, \tag{17}$$

we get

$$\mathfrak{P}^{int} + \mathfrak{P}^{ext} + \mathfrak{P}^{ine} = 0. \tag{18}$$

Which is the standard form of principle of virtual powers.

Is the principle of virtual power more general than principle of least Action? First answer: the principle of virtual powers involves differentials which are not exact, in general. Therefore, once fixed the kinematics, the principle of virtual power is actually more general. In both $\mathfrak{P}^{int}$ and $\mathfrak{P}^{ext}$ one can include dissipative terms, which cannot, in general, be derived from an Action functional. However, it is not clear if, suitably extending the space of configurations and the set of admissible motions, one can introduce an Action functional also for systems which, in a restricted kinematics, appear as dissipative. Controversies in the literature about this subject are not yet solved.

6 Hamilton-Rayleigh Approach

We propose to use the Hamilton-Rayleigh compromise. We introduce an Action functional and a Dissipation Rayleigh functional and, by means of them, we formulate the Principle of Virtual Work. Rayleigh dissipation functional $\mathfrak{R}$ is defined as a linear functional on the set of velocities, not on the set of motions as $\mathfrak{A}$. Therefore, $\delta\mathfrak{R}$ is defined as a linear functional of the variation $\delta\dot{m}$. The principle of virtual works formulated following Hamilton-Rayleigh takes the form: (the lack of the upper dot on RHS is not a mistake!)

$$\delta\mathfrak{A}\left(\delta m\right) = \mathfrak{R}\left(\delta m\right). \tag{19}$$

7 Conclusions

We recall an ancient and useful recipe for building theories for describing effectively physical phenomena:

“In Nomina est Natura Rerum”. Anonymous

This statement (passed to us by the middle age tradition) is formulated for defending mathematical formalism. This sentence claims that it is impossible to talk about any mathematical model without using the appropriate language. So, for instance, it is impossible to say clearly what is the first variation of Action using simply “words” from natural language, i.e. without writing integrals on T and, to proceed, we need to give "precisely" names to things. Therefore, to specify precisely how our models are constructed we need to introduce symbols and formulas.

However, we can also say that

"Nomina sunt Consequentia Rerum." Iustinianus, Institutiones Liber II,7,3

This because we are not blindly building our mathematical model. We get informations about physics and from these informations we actually formulate our models.

We can finally state that the "old" method of basing the formulation of mathematical models on the variational approach works: indeed it works very well.

Bibliography

[1] Arnold, V.I., Mathematical Methods of Classical Mechanics, Springer, 2nd edition, May 16, 1989.

[2] Benvenuto, E., La scienza delle costruzioni e il suo sviluppo storico, Sansoni, Firenze, 1981.

[3] Berdichevsky V., Variational Principles of Continuum Mechanics, Springer, 2009.

[4] Bourdin, B., Francfort, G.A., Marigo, J.-J., The variational approach to fracture, J. Elasticity, 91, 1-3, 1-148, 2008. (The paper also appeared as a Springer book: ISBN: 978-1-4020-6394-7).

[5] Colonnetti, G., Scienza delle costruzioni, Torino, Edizioni scientifiche Einaudi, 3^{rd} ed., 1953-57.

[6] Cosserat, E., Cosserat, F., Sur la Théorie des Corps Déformables, Herman, Paris, 1909.

[7] dell'Isola, F., Seppecher, P., The relationship between edge contact forces, double force and interstitial working allowed by the principle of virtual power, Comptes Rendus de l'Academie de Sciences - Serie IIb: Mecanique, Physique, Chimie, Astronomie, vol. 321, pp. 303-308, 1995.

[8] dell'Isola, F., Seppecher, P., Edge Contact Forces and Quasi-Balanced Power, Meccanica, vol. 32, pp. 33-52, 1997.

[9] dell'Isola, F., Madeo, A., Seppecher, P., Boundary Conditions in Porous Media: A Variational Approach, Int. Journal of Solids and Structures, Vol. 46, 3150-3164, 2009.

[10] dell'Isola, F., Madeo, A., Seppecher, P., "Beyond Euler-Cauchy continua". In this book.

[11] Feynman, R.P., Leighton, R., Sands, M., The Feynman Lectures on Physics: The Definitive and Extended Edition, (Feynman, Leighton, Sands), 3 vv., Addison Wesley, Reading (MA) 2^{nd} ed., 2005.

[12] Francfort, G.A., The chapter in this book by Prof. Gilles Frankfort.

[13] Germain, P., La méthode des puissances virtuelles en mécanique des milieux continus. Premi‘ere partie. Théorie du second gradient, J. Mécanique, 12, 235-274, 1973.

[14] Homer. The Iliad. Translated by Murray, A. T., Loeb Classical Library Volumes1. Cambridge, MA, Harvard University Press; London,William Heinemann Ltd., 1924.

[15] Lagrange, J.L., Mécanique Analytique, Editions Jaques Gabay, Sceaux, 1788.

[16] Lanczos, C., The Variational Principles of Mechanics. Toronto: University of Toronto, 1970.

[17] Landau, L.D., Lifshitz, E.M., Quantum Mechanics: Non-Relativistic Theory. Vol. 3 (3^{rd} ed.), Pergamon Press, 1977.

[18] Moiseiwitsch, B.L., Variational Principles, Dover, 1966.

[19] Netz, R., Noel., W., The Archimedes Codex: How a Medieval Prayer Book Is Revealing the True Genius of Antiquity's Greatest Scientist, Da Capo press, 2009. See also the website http://www.archimedespalimpsest.org/

[20] Rorres, C., Completing Book II of Archimedes's On Floating Bodies, The mathematical intelligencer, 26,3. Pages 32-42, 2004.

[21] Russo, L., The Forgotten Revolution, Springer Verlag, 2003.

[22] Vailati, G., Il principio dei lavori virtuali da Aristotele a Erone d'Alessandria, Scritti (Bologna, Forni, 1987), vol. II, pp. 113-128. Atti della R. Accademia delle Scienze di Torino, vol. XXXII, adunanza del 13 giugno 1897, quaderno IG (091) 75 I - III. 1987.

Beyond Euler-Cauchy Continua: The structure of contact actions in N-th gradient generalized continua: a generalization of the Cauchy tetrahedron argument.

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This work is dedicated to Professor Antonio Romano in occasion of his 70-th birthday.

Abstract The most general and elegant axiomatic framework on which continuum mechanics can be based starts from the Principle of Virtual Works (or Virtual Power). This Principle, which was most likely used already at the very beginning of the development of mechanics (see e.g. Benvenuto (1981), Vailati (1897), Colonnetti (1953), Russo (2003)), became after D'Alembert the main tool for an efficient formulation of physical theories. Also in continuum mechanics it has been adopted soon (see e.g. Benvenuto (1981), Salençon (1988), Germain (1973), Berdichevsky (2009), Maugin (1980), Forest (2006)). Indeed the Principle of Virtual Works becomes applicable in continuum mechanics once one recognizes that to estimate the work expended on regular virtual displacement fields of a continuous body one needs a distribution (in the sense of Schwartz). Indeed in the present paper we prove, also by using concepts from differential geometry of embedded Riemanniam manifolds, that the Representation Theorem for Distributions allows for an effective characterization of the contact actions which may arise in $N-th$ order strain-gradient multipolar continua (as defined by Green and Rivlin (1964)), by univocally distinguishing them in actions (forces and $n-th$ order forces) concentrated on contact surfaces, lines (edges) and points (wedges). The used approach reconsiders the results found in the pioneering papers by Green and Rivlin (1964)-(1965) , Toupin (1962), Mindlin (1964)-(1965) and Casal (1961) as systematized, for second gradient models, by Paul Germain (1973). Finally, by recalling the results found in dell'Isola and Seppecher (1995)-(1997), we indicate how Euler-Cauchy approach to contact actions and the celebrated tetrahedron argument may be adapted to $N-th$ order strain-gradient multipolar continua.

1 Introduction

In a forthcoming review paper the authors will try to describe why, how and when many theories were conceived to go beyond the conceptual framework established for continuum mechanics in the Euler and Cauchy era. In this introduction are formulated only few comments about some papers which seem to be the starting point of the most modern studies in continuum mechanics. In this field -among the many available in the literature- the textbooks which we have found more instructive are those of Paul Germain and Jean Salençon. In them, without any loss of mathematical rigour, what nowadays seems the most effective approach to the axiomatization of mechanics is presented to the students of the École polytechnique. This approach is based on the Principle of Virtual Works. Paul Germain research paper on second gradient continua (1972) shows how fruitful is the aforementioned approach. Most likely one of the most illuminating paper in modern continuum mechanics is due to Green and Rivlin (1964): it is entitled "Simple force and stress multipoles" and formulates what the authors call "multipolar continuum mechanics". Indeed Green and Rivlin start there the foundation of what has been later called also the theory of generalized continua. They also address the problem of establishing simultaneously the bulk evolution equations and the correct boundary conditions for generalized continua: equations and boundary conditions which they find by postulating the Principle of Virtual Work. In this aspect their theory is perfectly orthodox with respect to the paradigm put forward, many years earlier, by Cosserat brothers, in their fundamental textbooks (1908)-(1909). Of great importance for understanding the relationship between Cosserat continua and higher order gradient continua studied by Green and Rivlin is the short but very clear paper by Bleustein (1967), where the boundary conditions found by Toupin in a previous work are interpreted also from a physical point of view. We must also cite here the papers by Mindlin, who also contributed greatly to the development of important generalizations of Euler-Cauchy continuum models. In particular in Mindlin (1965) is started the study of third gradient continua, which is developed in a great extent.

However in all cited papers the Cauchy cuts considered are very regular: therefore the cited authors refrain from the consideration of contact actions concentrated on edges and wedges. Instead Germain considers Cauchy cuts in which the normal can suffer discontinuities of the first kind: he therefore needs to consider contact actions concentrated on edges. However Germain limits his treatment to second gradient continua: in his theory there are not wedge contact actions. Also in Toupin's strain-gradient theory (1962) the consideration is limited to second gradient continua. In cited paper Toupin limits himself to the consideration of a particular class of second gradient continua: those in which only a particular class of contact double-forces (using the nomenclature by Germain) can be exerted: the class constituted by "couple-stresses".

Those which are called by many authors (see e.g. Maugin (2010) and Forest (2005)-(2006)) generalized continua actually strictly include higher gradient continua which we consider here. In generalized continua together with the placement field one can introduce many other kinematical descriptors, which are other fields defined in the material configuration of considered continuum. The first example of such a set further kinematical descriptors is given by Cosserats who add to the placement field also a field of "changes of attitude", i.e. a field of rotations, which describe a large class of "microscopically structured" continua. As clarified by Bleustein (1967) Toupin's continua are a class of Cosserat continua in which an internal constraint has been introduced. In the following sections, while commenting some papers recently published in the field, it is discussed how the approach used by Germain can be reconciled with an approach which parallels more strictly the one used, for first gradient continua, by Cauchy.

It is not easy (but this analysis will be attempted in the aforementioned review paper) to explain why the foundation of continuum mechanics "à la Cauchy" has been considered "more physically grounded" than the axiomatization based on the Principle of Virtual Powers. In the present work we prove that .at least for higher gradient continuum theories, the two approaches are completely equivalent.

Variational Principles and Calculus of Variations have always been an important tool in formulating mathematical models for physical phenomena. Among many others the textbook by Berdichevsky (2009) clearly shows that this statement holds, in particular, for Continuum Mechanics.

We are sure that the Principle of Virtual Works and the Principle of Least Action have roots much deeper than many scientists believe. (see e.g. Vailati (1897)). One can conjecture that the majority of physical theories were first formulated in terms of these Principles, and only subsequently reconsidered from other points of view. The Principle of Least Action, which supplies a "geometric" (see Russo (2003)) version of mechanics, is likely to be indeed the tool used by the true founders of mechanics (i.e. the scientists of the Hellenistic period) to establish it. As conjectured also by Colonnetti (1953) and Rorres (2004) Archimedes himself was basing his mechanical investigations on the Principle of Virtual Works. More recently, as already stated above in a more detailed way, Green, Rivlin, Toupin, Mindlin, Casal, and Germain formalized various versions of the theory of generalized continua basing them on the Principle of Virtual Powers: however the most illuminating treatises in this subject remain those due to Cosserat brothers (1908)-(1909).

For a long time some opponents to second gradient theories argued about its "lack of consistency", due to the difficulties in "interpreting" boundary conditions. However it has to be remarked that if one refuses to use the Principle of Virtual Powers he can find very difficult the job of finding some set of boundary conditions which are compatible with the (independently postulated!) bulk evolution equations. Actually it happens that many epigones, after having initially refused to use this principle also in continuum mechanics, have later rephrased with different notations many of the results already available in the literature.

If instead one accepts the *D'Alembertian* approach to mechanics all these problems of well-posedness of mathematical models completely disappear.

2 Second and Higher Gradient Continuum Theories

In the last fifty years it has been widely recognized that in order to describe a wealth of physical phenomena it is needed to introduce mechanical theories which take into account contact actions more complex than those considered in the format given by Cauchy to continuum mechanics. Some well-known contributions in this regard are given in the papers listed in the references by Toupin, Mindlin, Green, Rivlin, Maugin, Forest, Germain, Suiker, Sokolowski, Triantafyllidis among many others.

More recently it has been recognized that second or even higher gradient models are needed when continuum models are introduced for describing systems in which strong inhomogeneities of physical properties are present at eventually different length scales (see e.g. Abu et al. (2008), Alibert et al. (2003), Polizzotto (2007), Pideri and Seppecher (1997), Trianttafyllidis et al. (1986)-(1998), Yang and Misra (2010), Yang et al. (2011)), and may be of great importance also in continuum systems in which some "microscopical" degrees of freedom can "capture" a relevant amont of deformation energy (see e.g. Carcaterra (2005) or Carcaterra et al. (2006)).

Actually, immediately after the development of the Cauchy format of continuum mechanics, a first relevant generalization in the aforementioned direction was conceived by Eugène and François Cosserat, but their efforts were not continued until late in XX century. Cosserat described continuum bodies in which contact actions were to be modelled not only by means of surface forces, but also by means of surface couples. The conceptual differences between Cauchy-type continuum mechanics and Cosserat-type continuum mechanics were relevant, and the second one could not be obtained by means of simple modifications of the first one. The remarkable mathematical difficulties confronted by Cosserat rendered their work difficult to be accepted, and for a long period their results were nearly completely ignored. This circumstance can be easily understood: the structure of Cosserat contact actions is complex. Indeed in Cosserat continua one needs, together with Cauchy stress tensor also a Couple stress tensor, for representing contact Couples.

2.1 A first method for extending Cauchy model for continuous bodies

In order to develop continuum mechanics by going beyond the Cauchy format it is possible to use at least two different approaches.

The most simple of them, used also by Cosserats, starts by postulating how the power expended by internal actions in a body depends on the "virtual" velocity field and its gradients. Starting from this postulate one can deduce, by means of a successive application of the theorem of divergence, i.e. by means of several iterative integrations by parts, which are the contact actions which can be exerted at the boundary of the considered body. Hence, this method starts from the notion of stress tensors and deduces from it the concept of contact actions. It is based on the D'Alembert Principle of Virtual Work and has been resumed by Green and Rivlin, Mindlin, Casal and subsequently by Paul Germain, in his enlightening papers (1972-1973). This Principle is undoubtedly a great tool in Mechanics which has not been improved since its original first and "standard" formulation, differently to what stated in Fried and Gurtin (2006)-(2008) and in Podio-Guidugli (2009). It is not clear why these last authors consider as "non-standard" a formulation of the Principle of Virtual Powers which can be found stated "word-for-word" for instance in the textbooks of Jean Salençon..

Indeed other authors (e.g. the paper by Degiovanni, Marzocchi, Musesti (1999)-(2010) in the references) stated that:

In particular, the approach by means of the theory of distributions, mentioned by Germain himself but not fully developed, is here adopted from the beginning. Clearly, in order to obtain deeper results such as the Cauchy

Stress Theorem, some extra regularity has to be assumed. Note that a power depends in general from two variables, the velocity field and the subbody. So it is a bit more complex than a mere distribution.

In the same spirit in dell'Isola and Seppecher (1995)-(1997) the starting assumptions concerning contact actions are: i) for every subbody of considered body the power expended by contact actions on a generic velocity field is a distribution (i.e. a linear and continuous functional on velocity fields) ii) the power expended by contact actions is quasi-balanced (generalizing the assumption used in Noll and Virga (1990)). Then in aforementioned papers by using different polynomial test velocity fields and different families of subbodies, the Cauchy construction for stress tensors is obtained.

The works of Green and Rivlin, Mindlin and Germain have been taken up again and again, (e.g. in Fried and Gurtin (2006)-(2008)) often rephrasing them without introducing any notable amelioration and often second gradient continua are somehow confused with Cosserat continua.

Paul Germain, following a tradition set in France by André Lichnerowicz, uses the original version (and more efficient) absolute notation due to Levi-Civita. This version, at least in this context, is the most adapted, as many objects of different tensorial order are to be simultaneously handled. Sometimes those who are refraining from using the most sophisticated version of Levi-Civita absolute Calculus are lead to refer to the needed stress tensors and the related contact actions indistinctly using the names "hyperstresses" and "hypertractions". On the contrary Germain (following the spirit of Green and Rivlin) tries to convey through the nomenclature chosen the physical meaning to be attached to the new mathematical objects which he is introducing: for instance he calls "double forces" the actions which are expending powers on the velocity gradient in the directions which are normal to the surfaces of Cauchy cuts. Germain then decomposes these "double forces" into "couples" and "symmetric double forces" recognizing (following Bleustein) that couples were already introduced by Cosserats. Germain's notation supports the mechanical and physical intuition contrarily to what does a generic nomenclature based on some "hyper" prefixes.

2.2 A second method for extending Cauchy model and its relationship with the first

The second method starts by postulating the type of contact action which can be exerted on the boundary of every "regular" part of a body and then proceeds by proving a "representation" theorem for the considered class of contact actions: the existence of stress tensors is then deduced from the postulated form of contact actions with the addition of a "balance-type"