Xueli Wang
Dingyi Pei

Modular Forms with Integral and Half-Integral Weights

Science Press
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With 2 figures

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ISBN 978-7-03-033079-6
Science Press Beijing

ISBN 978-3-642-29301-6 ISBN 978-3-642-29302-3 (eBook)
Springer Heidelberg New York Dordrecht London

Library of Congress Control Number: 2012934552

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

The theory of modular forms is an important subject of number theory. Also it has very important applications to other areas of number theory such as elliptic curves, quadratic forms, etc. Its contents is vast. So any book on it must necessarily make a rather limited selection from the fascinating array of possible topics. Our focus is on topics which deal with the fundamental theory of modular forms of one variable with integral and half-integral weight. Even for such a selection we have to make further limitations on the themes discussed in this book. The leading theme of the book is the development of the theory of Eisenstein series.

A fundamental problem is the construction of a basis of the space of modular forms. It is well known that, for any weight $\geqslant 2$ and the weight 1, the orthogonal complement of the space of cusp forms is spanned by Eisenstein series. Does this conclusion hold for the half-integral weight < 2? The problem for weight $1/2$ was solved by J.P.Serre and H.M.Stark. Then one of the authors of this book, Dingyi Pei, proved that the conclusion holds for weight $3/2$ by constructing explicitly a basis of the orthogonal complement of the space of cusp forms. To introduce this result and some of its applications is our motivation for writing this book, which is a large extension version of the book "Modular forms and ternary quadratic forms" (in Chinese) written by Dingyi Pei.

Chapter 1 can be viewed as an introduction to the themes discussed in the book. Starting from the problem of representing integers by quadratic forms we introduce the concept of modular forms. In Chapter 2, we discuss the analytic continuation of Eisenstein series with integral and half-integral weight, which prepares the construction of Eisenstein series in Chapter 7.

In Chapters 3-5, some fundamental concepts, notations and results about modular forms are introduced which are necessary for understanding later chapters. More specifically, we introduce in Chapter 3 the modular group and its congruence subgroups and the Riemannian surface associated with a discrete subgroup of $SL_2(\mathbb{R})$. Furthermore, the concept of cusp points for a congruence subgroup is presented. In Chapter 4, we define modular forms with integral and half-integral weight, calculate the dimension of the space of modular forms using the theorem of Riemann-Roch. Chapter 5 is dedicated to define Hecke rings and discuss some of their fundamental properties. Also in this chapter the Zeta function of a modular form with integral or half-integral weight is described. In particular, we deduce the functional equation of the Zeta function of a modular form, and discuss Weil's Theorem.

In Chapter 6, the definitions of new forms and old forms with integral and half-integral weight are given. In particular the Atkin-Lehner's theory and the Kohnen's theory, with respect to new forms for integral and half-integral weight, are discussed at length respectively.

In Chapter 7, we construct Eisenstein series. The first objective is to construct Eisenstein series with half-integral weight $\geqslant 5/2$. The second objective is the construction of Eisenstein series with weight $1/2$ according to Serre and Stark. Then the method of the construction for Eisenstein series of weight $3/2$ is introduced, followed by the construction of Cohen-Eisenstein series. For completeness, the construction of Eisenstein series with integral weight, which is due to Hecke, is also given in the last section of the chapter.

The Shimura lifting is the main objective of Chapter 8 where we follow the way depicted by Shintani. Weil representation is introduced first and some elementary properties of Weil representation are discussed. Then the Shimura lifting from cusp forms with half-integral weight to ones with integral weight is constructed. Also the Shimura lifting for Eisenstein spaces is deduced in this chapter.

In Chapter 9, we discuss the Eichler-Selberg trace formula for the space of modular forms with integral and half-integral weight. The simplest case of the Eichler-Selberg trace formula on $SL_2(\mathbb{Z})$ is deduced in terms of Zagier's method. Then the trace formula on a Fuchsian group is obtained by Selberg's method. Finally the Niwa's and Koknen's trace formulae are obtained for the space of modular forms with half-integral weight and the group $\Gamma_0(N)$.

In Chapter 10, some applications of modular forms and Eisenstein series to the arithmetic of quadratic forms are described. We first present the Schulze-Pillot's proof of Siegel theorem. Then some results of representation of integers by ternary quadratic forms are explained. We also give an upper bound of the minimal positive integer represented by a positive definite even quadratic form with level 1 or 2.

Although many modern results on modular forms with half-integral weight are contained in this book, it is written as elementarily as possible and it's content is self-contained. We hope it can be used as a reference book for researchers and as a textbook for graduate students.

The authors would like to thank Ms. Yuzhuo Chen for her many helps. Also many thanks should be given to Dr. Junwu Dong for his helpful suggestions and carefully typesetting the draft of this book. We especially wish to thank Dr. Wolfgang Happle Happle for carefully reading the draft of this book and correcting some errors in the draft. The author Xueli Wang wishes to thank Prof. Dr. Gerhard Frey for stimulating discussions and providing the environment of I.E.M in Essen University, where part

of the draft has been done. Xueli Wang hope to give deepest gratitude for his lovely and beautiful wife, Dr. Dongping Xu, who assumed all of the housework over the years. Finally, the author Xueli Wang would like to dedicate this book to the 80th birthday of his father.

Xueli Wang Dingyi Pei
Guangzhou
September, 2011

Contents

Chapter 1

Theta Functions and Their Transformation Formulae

In this chapter, we introduce theta functions of positive definite quadratic forms and study their transformation properties under the action of the modular group.

Let a, b, c and n be positive integers with $(a, b, c) = 1$. Denote by $N(a, b, c; n)$ the number of integral solutions $(x, y, z) \in \mathbb{Z}^3$ of the following equation:

$$ax^2 + by^2 + cz^2 = n.$$

Define the theta function by

$$\theta(z) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{2\pi \mathrm{i} n^2 z}, \quad z \in \mathbb{H},$$

where $\mathbb{H}$ is the upper half of the complex plane, i.e., $\mathbb{H} = \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}$. It is clear that $\theta(z)$ is holomorphic on $\mathbb{H}$. Put

$$f(z) = \theta(az)\theta(bz)\theta(cz),$$

then

$$f(z) = 1 + \sum_{n=1}^{\infty} N(a, b, c; n)\mathrm{e}^{2\pi \mathrm{i} nz}.$$

Hence the number $N(a, b, c; n)$ is the n-th Fourier coefficient of the function. This shows that we know the number $N(a, b, c; n)$ if the Fourier coefficients of f can be computed explicitly. It is clear that there is a close relationship between $f(z)$ and the θ function. We shall see later that $f(z)$ is a modular form of weight $3/2$ from the transformation properties of θ under the action of linear fractional transformations. After having studied some properties of modular forms, we shall resume this topic later. Firstly, we shall consider some more general problems.

Now let t be a positive real number, put

$$\varphi(x) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{-\pi t(n+x)^2}.$$

The series satisfies $\varphi(x+1)=\varphi(x)$. Hence it has the following Fourier expansion:

$$\varphi(x)=\sum_{m=-\infty}^{\infty} c_m \mathrm{e}^{2\pi \mathrm{i} m x},$$

where

$$c_m=\int_0^1 \varphi(x)\mathrm{e}^{-2\pi \mathrm{i} m x}\mathrm{d}x=\int_{-\infty}^{\infty}\mathrm{e}^{-\pi t x^2-2\pi \mathrm{i} m x}\mathrm{d}x=t^{-1/2}\mathrm{e}^{-\pi m^2/t}.$$

Hence

$$\varphi(x)=t^{-1/2}\sum_{m=-\infty}^{\infty}\mathrm{e}^{-\pi m^2+2\pi \mathrm{i} m x}. \tag{1.1}$$

Taking $x=0$ in equation (1.1) we get

$$\tilde{\theta}(\mathrm{i}t)=t^{-1/2}\tilde{\theta}(-1/(\mathrm{i}t)),$$

where $\tilde{\theta}(z)=\theta(z/2)$. Because $\tilde{\theta}(z)$ is a holomorphic function on the upper half plane, we have that

$$\tilde{\theta}(-1/z)=(-\mathrm{i}z)^{1/2}\tilde{\theta}(z), \quad \forall z\in\mathbb{H}. \tag{1.2}$$

For the multi-valued function $z^{1/2}$, we choose $\arg(z^{1/2})$ such that $-\pi/2<\arg(z^{1/2})\leqslant \pi/2$. In general, we have that $(z_1z_2)^{1/2}=\pm z_1^{1/2}z_2^{1/2}$ where we take "$-$" if one of the following conditions is satisfied:

(1) $\mathrm{Im}(z_1)<0$, $\mathrm{Im}(z_2)<0$, $\mathrm{Im}(z_1z_2)>0$;

(2) $\mathrm{Im}(z_1)<0$, $\mathrm{Im}(z_2)>0$, $\mathrm{Im}(z_1z_2)<0$;

(3) z_1 and z_2 are both negative, or one of them is negative and the imaginary of the other one is positive.

Otherwise we take "+".

Let $f(x_1,\cdots,x_k)$ be an integral positive definite quadratic form in k variables. Define the matrix

$$A=\left(\frac{\partial^2 f}{\partial x_i\partial x_j}\right).$$

Then A is a positive definite symmetric integral matrix with even entries on the diagonal. It is clear that

$$f(x_1,\cdots,x_k)=\frac{1}{2}xAx^{\mathrm{T}},$$

where $x=(x_1,\cdots,x_k)\in\mathbb{Z}^k$ is a row vector, x^{T} is the transposal of x. We now define the θ function of f as

$$\theta_f(z)=\sum_{x\in\mathbb{Z}^k}\mathrm{e}^{2\pi \mathrm{i} f(x)z} \quad \text{for all } z\in\mathbb{H}.$$

It is clear that

$$\theta_f(z) = \sum_{x\in\mathbb{Z}^k} \mathrm{e}^{\pi \mathrm{i} x A x^{\mathrm{T}} z} = \sum_{n=0}^{\infty} r(f,n)\mathrm{e}^{2\pi \mathrm{i} n z},$$

where $r(f,n)$ is the number of the solutions of $f(x) = n$ with $x \in \mathbb{Z}^k$. $\theta_f(z)$ is absolutely and uniformly convergent in any bounded domain of $\mathbb{H}$, so it is holomorphic on the whole of $\mathbb{H}$.

Let N be the least positive integer such that all the entries of the matrix NA^{-1} are integers and the entries on the diagonal are even. This implies that $\det A$ is a divisor of N^k. Hence the prime divisors of $\det A$ are also prime divisors of N. But it is clear that $N|2\det A$. So all the odd prime divisors of N are certainly prime divisors of $\det A$.

If we consider A as a matrix on the ring $\mathbb{Z}_2$ of 2-adic integers, it can be proved that there exists an inverse matrix S on $\mathbb{Z}_2$ such that

$$SAS^{\mathrm{T}} = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & & \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_r \end{pmatrix},$$

where A_i is either an integer of $2\mathbb{Z}_2$ or a symmetric matrix $\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$ with $a, b, c \in \mathbb{Z}_2$.

It is clear that there is at least one A_i which is a 1×1 matrix if k is odd. So we get the following

Lemma 1.1 *If k is odd, then $2|\det A$ and $4|N$; if k is even, then $N|\det A$. If $4|k$, then $\det A \equiv 0$ or $1 \mod 4$; if $k \equiv 2 \mod 4$, then $\det A \equiv 0$ or $3 \mod 4$. Hence $(-1)^{k/2}\det A$ is always 1 or 0 modulo by 4 if k is even.*

Let h be a vector in $\mathbb{Z}^k$ such that $hA \in N\mathbb{Z}^k$ and define a function on $\mathbb{H}$ as follows

$$\theta(z; h, A, N) = \sum_{m\equiv h(N)} e\Big(\frac{zmAm^{\mathrm{T}}}{2N^2}\Big),$$

where $e(z) = \mathrm{e}^{2\pi \mathrm{i} z}$.

Proposition 1.1 *We have the following transformation formula*

$$\theta(-1/z; h, A, N) = (\det A)^{-1/2}(-\mathrm{i}z)^{k/2} \sum_{k \bmod N,\ kA\equiv 0(N)} e(hAk^{\mathrm{T}}/N^2)\theta(z; k, A, N).$$

Proof Let v be a positive real number, $x = (x_1, \cdots, x_k) \in \mathbb{R}^k$, and

$$g(x) = \sum_{m\in\mathbb{Z}^k} e(\mathrm{i}v(x+m)A(x+m)^{\mathrm{T}}/2).$$

Then $g(x)$ has Fourier expansion

$$g(x) = \sum_{m \in \mathbb{Z}^k} a_m e(x \cdot m^{\mathrm{T}}), \tag{1.3}$$

where

$$a_m = \int \cdots \int_{0 \leqslant x_j < 1} g(x) e(-x \cdot m^{\mathrm{T}}) \mathrm{d}x = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e(\mathrm{i}v x A x^{\mathrm{T}}/2 - x \cdot m^{\mathrm{T}}) \mathrm{d}x.$$

There exists a real orthogonal matrix S such that SAS^{T} is a diagonal matrix $\mathrm{diag}\{\alpha_1, \cdots, \alpha_k\}$ with $\alpha_i > 0\,(1 \leqslant i \leqslant k)$. We make a variable change $x = yS$ in the above integral and denote $Sm^{\mathrm{T}} = (u_1, \cdots, u_k)^{\mathrm{T}}$. Then

$$\begin{aligned} a_m &= \prod_{j=1}^{k} \int_{-\infty}^{\infty} \mathrm{e}^{-\pi v \alpha_j y^2 - 2\pi \mathrm{i} u_j y} \mathrm{d}y \\ &= \prod_{j=1}^{k} \int_{-\infty}^{\infty} \mathrm{e}^{-\pi v \alpha_j \left(y + \frac{\mathrm{i} u_j}{v \alpha_j}\right)^2 - \frac{\pi u_j^2}{v \alpha_j}} \mathrm{d}y \\ &= v^{-k/2} \prod_{j=1}^{k} \alpha_j^{-1/2} \mathrm{e}^{-\frac{\pi u_j^2}{v \alpha_j}} \\ &= v^{-k/2} (\det A)^{-1/2} \mathrm{e}^{-\pi m A^{-1} m^{\mathrm{T}}/v}. \end{aligned} \tag{1.4}$$

For any $m \in \mathbb{Z}^k$, let $k \equiv mNA^{-1} \pmod N$. Then $kA \equiv 0 \pmod N$ and m can be written as $(Nu + k)A/N (u \in \mathbb{Z}^k)$. Inserting (1.4) into (1.3), we get

$$\begin{aligned} g(x) = v^{-k/2} (\det A)^{-1/2} & \sum_{\substack{k \bmod N, \\ kA \equiv 0 (N)}} e(xAk^{\mathrm{T}}/N) \\ & \cdot \sum_{u} e(xAu^{\mathrm{T}} + \mathrm{i}(Nu + k)A(Nu + k)^{\mathrm{T}}/(2vN^2)). \end{aligned}$$

Since $\theta(\mathrm{i}v; h, A, N) = g(h/N)$, we get by the above equality

$$\theta(\mathrm{i}v; h, A, N) = v^{-k/2} (\det A)^{-1/2} \sum_{\substack{k \bmod N, \\ kA \equiv 0 (N)}} e(hAk^{\mathrm{T}}/N^2) \theta\left(-\frac{1}{\mathrm{i}v}; k, A, N\right),$$

which shows that Proposition 1.1 holds for $z = -1/\mathrm{i}v$. This implies that the proposition holds because $\theta(z; h, A, N)$ is holomorphic on the whole of $\mathbb{H}$. □

Now we define the full modular group of order 2 as follows

$$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

Let

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

We want to find the transformation formula of $\theta(z;\, h,\, A,\, N)$ under the transformation $z \mapsto \gamma(z) = (az+b)/(cz+d)$. We first assume that $c > 0$, then we get by Proposition 1.1 that

$$\begin{aligned}
\theta(\gamma(z); h, A, N) &= \sum_{m\equiv h(N)} e\left(mAm^{\mathrm{T}}\left(a - \frac{1}{cz+d}\right)/(2cN^2)\right) \\
&= \sum_{\substack{g \bmod (cN),\\ g\equiv h(N)}} e(agAg^{\mathrm{T}}/(2cN^2)) \\
&\quad \cdot \sum_{m\equiv g \bmod (cN)} e(-cmAm^{\mathrm{T}}/[2(cz+d)(cN)^2]) \\
&= (\det A)^{-1/2} c^{-k/2} (-\mathrm{i}(cz+d))^{k/2} \\
&\quad \cdot \sum_{\substack{k \bmod (cN),\\ kA\equiv 0(N)}} \varPhi(h,k)\theta(cz; k, cA, cN),
\end{aligned} \tag{1.5}$$

where

$$\varPhi(h,k) = \sum_{\substack{g \bmod (cN),\\ g\equiv h(N)}} e([agAg^{\mathrm{T}} + 2kAg^{\mathrm{T}} + dkAk^{\mathrm{T}}]/(2cN^2))$$

and we also used the fact that mAm^{T} is even for any $m \in \mathbb{Z}^k$. Since $ad = bc + 1$, it follows

$$\begin{aligned}
\varPhi(h,k) &= \sum_{\substack{g \bmod (cN),\\ g\equiv h(N)}} e\big(a(g+dk)A(g+dk)^{\mathrm{T}}/(2cN^2)\big) e\big(-b[2gAk^{\mathrm{T}} + dkAk^{\mathrm{T}}]/(2N^2)\big) \\
&= e\big(-b[2hAk^{\mathrm{T}} + dkAk^{\mathrm{T}}]/(2N^2)\big)\, \varPhi(h+dk, 0),
\end{aligned}$$

which implies that $\varPhi(h,k)$ is only dependent on $k \bmod N$. By equality (1.5) we get

$$\begin{aligned}
&\theta(\gamma(z); h, A, N)(\det A)^{1/2} c^{k/2} (-\mathrm{i}(cz+d))^{-k/2} \\
&= \sum_{\substack{k \bmod (N),\\ kA\equiv 0(N)}} \varPhi(h,k) \sum_{\substack{g \bmod (cN),\\ g\equiv k(N)}} \theta(cz; g, cA, cN) \\
&= \sum_{\substack{k \bmod (N),\\ kA\equiv 0(N)}} \varPhi(h,k)\theta(z; k, A, N).
\end{aligned}$$

Substituting z by $-1/z$, we get by Proposition 1.1

$$\theta\left(\frac{bz-a}{dz-c}; h, A, N\right) \det Ac^{k/2}\big(-\mathrm{i}(d-c/z)\big)^{-k/2}(-\mathrm{i}z)^{-k/2}$$
$$= \sum_{\substack{l \mod N, \\ lA\equiv 0(N)}} \left\{ \sum_{\substack{k \mod N, \\ kA\equiv 0(N)}} e(lAk^{\mathrm{T}}/N^2)\varPhi(h,k)\right\}\theta(z;l,A,N). \tag{1.6}$$

Now suppose that $d \equiv 0(N)$. Since NA^{-1} is an integral matrix with even entries on the diagonal,

$$kAk^{\mathrm{T}}/(2N) = (N^{-1}kA \cdot NA^{-1} \cdot N^{-1}Ak^{\mathrm{T}})/2$$

is an integer. Hence

$$\varPhi(h,k) = e(-bhAk^{\mathrm{T}}/N^2)\varPhi(h,0)$$

and the right hand of (1.6) becomes

$$\varPhi(h,0) \sum_{\substack{l \bmod N, \\ lA\equiv 0(N)}} \left\{ \sum_{\substack{k \bmod N, \\ kA\equiv 0(N)}} e((l-bh)Ak^{\mathrm{T}}/N^2)\right\}\theta(z;l,A,N).$$

We now compute the inner summation of the formula above. There exist modular matrices P, Q, such that $PAQ = \mathrm{diag}\{\alpha_1, \cdots, \alpha_k\}$. Since NA^{-1} is an integral matrix, then $\alpha_i | N (1 \leqslant i \leqslant k)$. Since

$$kA \equiv (l-bh)A \equiv 0(N),$$

a direct computation shows that

$$\sum_{\substack{k\mathrm{mod}(N), \\ kA\equiv 0(N)}} e((l-bh)Ak^{\mathrm{T}}/N^2) = \begin{cases} 0, & \text{if } 1 \not\equiv bh(N), \\ \det A, & \text{if } 1 \equiv bh(N). \end{cases}$$

Now substituting $\begin{pmatrix} b & -a \\ d & -c \end{pmatrix}$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we assume that $c \equiv 0(N), d < 0$. Then we have that

$$\theta((az+b)/(cz+d); h, A, N) = (-\mathrm{i}(c+d/z))^{k/2}(-\mathrm{i}z)^{k/2} W\theta(z; ah, A, N), \tag{1.7}$$

where

$$W = |d|^{-k/2} \sum_{\substack{g\mathrm{mod}(|d|N), \\ g\equiv h(N)}} e\big(-bgAg^{\mathrm{T}}/(2|d|N^2)\big).$$

Since $\mathrm{Im}(-\mathrm{i}) < 0$, $\mathrm{Im}(c + d/z) > 0$, then $(-\mathrm{i}(c + d/z))^{k/2} = (-\mathrm{i})^{k/2}(c + d/z)^{k/2}$. Similarly, since $\mathrm{Im}(-\mathrm{i}) < 0$, $\mathrm{Im}(z) > 0$, we get $(-\mathrm{i}z)^{k/2} = (-\mathrm{i})^{k/2} z^{k/2}$. Again since $\mathrm{Im}(cz + d) = c\mathrm{Im}(z)$, it follows

$$z^{k/2}(c + d/z)^{k/2} = \mathrm{sgn}(c)^k (cz + d)^{k/2},$$

where

$$\mathrm{sgn}(c) = \begin{cases} 1, & \text{if } c \geqslant 0, \\ -1, & \text{if } c < 0. \end{cases}$$

Therefore

$$(-\mathrm{i}(c + d/z))^{k/2}(-\mathrm{i}z)^{k/2} = (-\mathrm{i}\mathrm{sgn}(c))^k (cz + d)^{k/2}. \tag{1.8}$$

Since $ad \equiv 1(N)$, we can express g in W as $adh + Nu$ with $u \in (\mathbb{Z}/|d|\mathbb{Z})^k$. Then

$$W = e(abhAh^{\mathrm{T}}/(2N^2))w(b, |d|), \tag{1.9}$$

where

$$w(b, |d|) = |d|^{-k/2} \sum_{u \bmod |d|} e(-buAu^{\mathrm{T}}/(2|d|)).$$

If $c = 0$ or $b = 0$, then $d = -1$ and hence $w(b, |d|) = 1$. Now suppose that $bc \neq 0$ and d is an odd. We substitute z by $z + 8m (m \in \mathbb{Z})$ in (1.7) such that $d + 8mc < 0$. By (1.8) and (1.9) we know that

$$w(b, |d|) = w(b + 8ma, |d + 8mc|).$$

Because d and $8c$ are co-prime, we can find an integer m such that $-d - 8mc$ is an odd prime which will be denoted by p. Let $\beta = -(b + 8ma)$. Then

$$w(b, |d|) = w(-\beta, p) = p^{-k/2} \sum_{u \bmod p} e(\beta uAu^{\mathrm{T}}/(2p)).$$

Suppose that $\beta \equiv 2\beta'(p)$. Since $c \equiv 0(N)$, d and c are co-prime, then p and N are co-prime, and hence p and $\det A$ are co-prime. There exists an integral matrix S such that $\det S$ is prime to p and SAS^t is congruent to $\mathrm{diag}\{q_1, \cdots, q_k\}$ modulo p. By Gauss sum, we have that

$$w(b, |d|) = p^{-k/2} \prod_{i=1}^{k} \left(\sum_{x=1}^{k} e(\beta' q_i x^2/p) \right) = \varepsilon_p^k \left(\frac{(\beta')^k \det A}{p} \right),$$

where $\left(\dfrac{q}{p}\right)$ is the Legendre symbol

$$\left(\frac{q}{p}\right) = \begin{cases} 1, & \text{if } q \text{ is a quadratic residue modulo } p, \\ -1, & \text{otherwise.} \end{cases}$$

The symbol ε_n is defined for all odd integers:

$$\varepsilon_n = \begin{cases} 1, & \text{if } n \equiv 1(4), \\ \mathrm{i}, & \text{if } n \equiv 3(4). \end{cases}$$

It is clear that $\varepsilon_p = \varepsilon_{-d} = \mathrm{i}\varepsilon_d^{-1}$. Since all prime divisors of $\det A$ are divisors of N, $p \equiv -d(8N)$,

$$\left(\frac{\det A}{p}\right) = \left(\frac{\det A}{-d}\right).$$

Since $\begin{pmatrix} a & -\beta \\ c & -p \end{pmatrix} \in SL_2(\mathbb{Z})$, i.e., $\beta c - ap = 1$, we get $2\beta' c \equiv 1(p)$. Hence

$$\left(\frac{\beta'}{p}\right) = \left(\frac{2c}{p}\right) = \left(\frac{2c}{-d}\right).$$

Let a be an integer, $b \neq 0$ be an odd. We define a new quadratic residue symbol $\left(\frac{a}{b}\right)$ satisfying the following properties:

(1) $\left(\frac{a}{b}\right) = 0$ if $(a, b) \neq 1$;

(2) $\left(\frac{0}{\pm 1}\right) = 1$;

(3) If $b > 0$, then $\left(\frac{a}{b}\right)$ is the Jacobi symbol, i.e., if $b = \prod p^r$, then $\left(\frac{a}{b}\right) = \prod \left(\frac{a}{p}\right)^r$;

(4) If $b < 0$, then $\left(\frac{a}{b}\right) = \mathrm{sgn}(a) \left(\frac{a}{|b|}\right)$.

Hereafter, the symbol $\left(\frac{a}{b}\right)$ will be defined as above. Then we have

$$w(b, |d|) = \varepsilon_d^{-k} (\mathrm{sgn}(c)\mathrm{i})^k \left(\frac{2c \det A}{d}\right) \tag{1.10}$$

and (1.10) holds for $c = 0$ or $c \neq 0$.

Define a subgroup of the full modular group as follows

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, c \equiv 0(N) \right\}.$$

Proposition 1.2 *Let* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. *If* k *is odd, then we have*

$$\theta(\gamma(z); h, A, N) = e\big(abhAh^{\mathrm{T}}/(2N^2)\big) \left(\frac{\det A}{d}\right) \left(\frac{2c}{d}\right)^k \varepsilon_d^{-k} (cz + d)^{k/2} \theta(z; ah, A, N), \tag{1.11}$$

If k is even, then we have

$$\theta\big(\gamma(z);h,A,N\big)=e\big(abhAh^{\mathrm{T}}/(2N^2)\big)\Big(\frac{(-1)^{k/2}\det A}{d}\Big)(cz+d)^{k/2}\theta(z;ah,A,N), \tag{1.12}$$

Proof First assuming that k is odd. By Lemma 1.1, $N\equiv 0(4)$. Hence d is odd. For $d<0$, inserting (1.8), (1.9) and (1.10) into (1.7), we can get (1.11) immediately. For $d>0$, substituting γ by $-\gamma$ and noting that $(-\gamma)(z)=\gamma(z)$, we have

$$\begin{aligned}\theta(\gamma(z);h,A,N)=e(abhAh^{\mathrm{T}}/(2N^2))\left(\frac{\det A}{d}\right)\left(\frac{-2c}{-d}\right)^k\\ \times\,\varepsilon_{-d}^{-k}(-cz-d)^{k/2}\theta(z;-ah,A,N).\end{aligned}$$

It is clear that $\theta(z;-ah,A,N)=\theta(z;ah,A,N)$. If $c=0$, then $d=1$ and

$$\left(\frac{-2c}{-d}\right)^k\varepsilon_{-d}^{-k}(-cz-d)^{k/2}=\mathrm{i}^{-k}(-1)^{k/2}=1.$$

If $c\neq 0$, we have

$$\begin{aligned}\left(\frac{-2c}{-d}\right)^k\varepsilon_{-d}^{-k}(-cz-d)^{k/2}&=(-\mathrm{sgn}(c))^k\left(\frac{-2c}{d}\right)^k\mathrm{i}^{-k}\varepsilon_d^{-k}(-\mathrm{isgn}(c))^k(cz+d)^{k/2}\\ &=\varepsilon_d^{-k}\left(\frac{2c}{d}\right)^k(cz+d)^{k/2}.\end{aligned}$$

This shows that (1.12) holds also for $d>0$. Now assuming that k is even. If d is odd, we can get (1.12) by proceeding similarly as above. If d is even, then c is odd, and N is also odd. By the result for the case d odd, we have

$$\begin{aligned}&\theta\left(\frac{az+aN+b}{cz+cN+d};h,A,N\right)\\ &=e\Big(\frac{abhAh^{\mathrm{T}}}{2N^2}\Big)\left(\frac{(-1)^{k/2}\det A}{cN+d}\right)(cz+cN+d)^{k/2}\theta(z;ah,A,N),\end{aligned} \tag{1.13}$$

where we used the fact that $hAh^{\mathrm{T}}/(2N)$ is an integer. By Lemma 1.1 and Lemma 1.2 which will be proved later, we have

$$\left(\frac{(-1)^{k/2}\det A}{cN+d}\right)=\left(\frac{(-1)^{k/2}\det A}{d}\right),$$

where d is even. So the right hand side of above is equal to $\left(\dfrac{(-1)^{k/2}\det A}{\det A+d}\right)$. Substituting z by $z-N$ in (1.13) we get (1.12). □

It is clear that $\theta_f(z) = \theta(z; 0, A, N)$. Thus we obtain the main theorem of this chapter:

Theorem 1.1 *Let* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. *If* k *is odd, then*

$$\theta_f(\gamma(z)) = \left(\frac{2 \det A}{d}\right) \varepsilon_d^{-k} \left(\frac{c}{d}\right)^k (cz+d)^{k/2} \theta_f(z).$$

If k *is even, then*

$$\theta_f(\gamma(z)) = \left(\frac{(-1)^{k/2} \det A}{d}\right) (cz+d)^{k/2} \theta_f(z).$$

In particular, taking $k = 1, A = 2$, then $N = 4$. For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$, by Theorem 1.1, we have

$$\theta(\gamma(z)) = \varepsilon_d^{-1} \left(\frac{c}{d}\right) (cz+d)^{1/2} \theta(z).$$

We define the symbol

$$j(\gamma, z) = \varepsilon_d^{-1} \left(\frac{c}{d}\right) (cz+d)^{1/2}, \quad \gamma \in \Gamma_0(4).$$

If $\gamma_1, \gamma_2 \in \Gamma_0(4)$, by the above result, we have

$$\theta(\gamma_1\gamma_2(z)) = j(\gamma_1\gamma_2, z)\theta(z)$$

and

$$\theta(\gamma_1\gamma_2(z)) = j(\gamma_1, \gamma_2(z))\theta(\gamma_2(z)) = j(\gamma_1, \gamma_2(z)) j(\gamma_2, z)\theta(z).$$

Therefore

$$j(\gamma_1\gamma_2, z) = j(\gamma_1, \gamma_2(z)) j(\gamma_2, z). \tag{1.14}$$

Lemma 1.2 *Let* $a = ds^2 \neq 0$ *be an integer,* d *square-free. Let*

$$D = \begin{cases} |d|, & \text{if } d \equiv 1(4), \\ 4|d|, & \text{if } d \equiv 2, 3(4). \end{cases}$$

Then the map $b \mapsto \left(\frac{a}{b}\right)$ *(*b *is odd) defines a character modulo* $4a$ *with conductor* D.

Proof If a, b are co-prime, it is clear that

$$\left(\frac{a}{b}\right) = \left(\frac{d}{b}\right).$$

(1) Suppose $d > 0$ and d odd. If $b > 0$, then

$$\left(\frac{d}{b}\right) = \begin{cases} \left(\frac{b}{d}\right), & \text{if } d \equiv 1(4), \\ \left(\frac{-1}{b}\right)\left(\frac{b}{d}\right), & \text{if } d \equiv 3(4). \end{cases}$$

If $b < 0$, $d \equiv 1(4)$, then

$$\left(\frac{d}{b}\right) = \left(\frac{d}{|b|}\right) = \left(\frac{|b|}{d}\right) = \left(\frac{b}{d}\right).$$

If $b < 0$, $d \equiv 3(4)$, then

$$\left(\frac{d}{b}\right) = \left(\frac{d}{|b|}\right) = \left(\frac{-1}{|b|}\right)\left(\frac{|b|}{d}\right) = \left(\frac{-1}{b}\right)\left(\frac{b}{d}\right).$$

These conclusions show that the lemma holds in this case.

(2) Suppose $d < 0$, d is odd. If $b > 0$, then

$$\left(\frac{d}{b}\right) = \left(\frac{-1}{b}\right)\left(\frac{|d|}{b}\right) = \begin{cases} \left(\frac{b}{|d|}\right), & \text{if } d \equiv 1(4), \\ \left(\frac{-1}{b}\right)\left(\frac{b}{|d|}\right), & \text{if } d \equiv 3(4). \end{cases}$$

If $b < 0$, $d \equiv 1(4)$, then

$$\left(\frac{d}{b}\right) = -\left(\frac{d}{|b|}\right) = -\left(\frac{|b|}{|d|}\right) = \left(\frac{b}{|d|}\right).$$

If $b < 0$, $d \equiv 3(4)$, then

$$\left(\frac{d}{b}\right) = -\left(\frac{d}{|b|}\right) = -\left(\frac{-1}{|b|}\right)\left(\frac{|b|}{|d|}\right) = \left(\frac{-1}{b}\right)\left(\frac{b}{|d|}\right).$$

These conclusions show that the lemma holds in this case.

(3) Suppose $d = 2d'$, then

$$\left(\frac{d}{b}\right) = \left(\frac{2}{b}\right)\left(\frac{d'}{b}\right).$$

$\left(\frac{2}{b}\right)$ is a character modulo 8, gathering the results in (1) and (2), we proved the lemma. □

Remark 1.1 If $a \equiv 1(4)$, $b \mapsto \left(\frac{a}{b}\right)$ is a character modulo a. In this case, b can be an even integer.

Chapter 2

Eisenstein Series

2.1 Eisenstein Series with Half Integral Weight

In this section we always assume that k is an odd integer, N is a positive integer such that $4|N$, ω is an even character modulo N, i.e., $\omega(-1) = 1$. We shall construct a class of holomorphic functions which are named as Eisenstein series with the following property

$$f(\gamma(z)) = \omega(d_\gamma) j(\gamma, z)^k f(z), \quad \gamma = \begin{pmatrix} * & * \\ * & d_\gamma \end{pmatrix} \in \Gamma_0(N).$$

Lemma 2.1 *Let $k > 2$ be a positive integer, $z \in \mathbb{H}$. Put*

$$L = \{mz + n | m, n \in \mathbb{Z}\}.$$

Then the series

$$E_k(z) = \sum_{w \in L \setminus \{0\}} w^{-k} = {\sum_{m,n}}'(mz+n)^{-k}$$

is a holomorphic function on the upper half plane $\mathbb{H}$ where $\sum'$ indicates the summation over all $(m, n) \neq (0, 0)$.

Proof Let P_m be the parallelogram with vertices $\pm mz \pm m$. Denote

$$r = \min\{|w|, w \in P_1\},$$

for any $w \in P_m$, we have that $|w| \geqslant mr$. Since there are $8m$ points in $L \bigcap P_m$, then

$$\sum_{w \in L \setminus \{0\}} |w|^{-k} = \sum_{m=1}^{\infty} \sum_{w \in P_m} |w|^{-k} \leqslant 8 \sum_{m=1}^{\infty} m(mr)^{-k}.$$

It is clear that the right hand side of the above is convergent for $k > 2$. So $E_k(z)$ is absolutely and uniformly convergent in any bounded domain of $\mathbb{H}$. This shows that $E_k(z)$ is holomorphic on the whole of $\mathbb{H}$. □

Let

$$\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \middle| n \in \mathbb{Z} \right\},$$

which is clearly a subgroup of $\Gamma_0(N)$. Suppose $k \geqslant 5$ and define

$$E_k(\omega, N)(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \omega(d_\gamma) j(\gamma, z)^{-k}, \tag{2.1}$$

where γ runs over a complete set of representatives of right cosets of Γ_∞ in $\Gamma_0(N)$. For $\gamma' \in \Gamma_\infty$, by (1.14), we have that

$$\omega(d_{\gamma'\gamma}) j(\gamma'\gamma, z)^{-k} = \omega(d_\gamma) j(\gamma, z)^{-k},$$

which implies that $E_k(\omega, N)(z)$ is well defined. By Lemma 2.1 it is a holomorphic function on $\mathbb{H}$. For any $\gamma' \in \Gamma_0(N)$, it is easy to verify

$$E_k(\omega, N)(\gamma'(z)) = \overline{\omega}(d_{\gamma'}) j(\gamma', z)^k E_k(\omega, N)(z).$$

For $1 \leqslant k < 5$, the series defined in (2.1) is not absolutely convergent. We now introduce the following function

$$E_k(s, \omega, N)(z) = y^{s/2} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \omega(d_\gamma) j(\gamma, z)^{-k} |j(\gamma, z)|^{-2s}, \tag{2.2}$$

where $y = \mathrm{Im}(z) > 0$, s is a complex variable and we will therefore call $|j(\gamma, z)|^{-2s}$ Hecke convergence factor because it was first introduced by Hecke. It is clear that for $\mathrm{Re}(s) > 2 - k/2$ the series (2.2) is absolutely convergent and has the following transformation property

$$E_k(s, \omega, N)(\gamma(z)) = \overline{\omega}(d_\gamma) j(\gamma, z)^k E_k(s, \omega, N)(z), \quad \gamma \in \Gamma_0(N). \tag{2.3}$$

We shall study the meromorphic continuation of $E_k(s, \omega, N)$ to the whole s-plane. Then we get a holomorphic function on $\mathbb{H}$ for $s = 0$. By (2.3)

$$E_k(s, \omega, N)(z + 1) = E_k(s, \omega, N)(z),$$

i.e., $E_k(s, \omega, N)(z)$ has period 1. We shall first compute the Fourier expansion of $E_k(s, \omega, N)(z)$ with respect to $\mathrm{e}^{2\pi \mathrm{i} z}$. Then we can get the analytic continuation with respect to s. Now we assume that $k \geqslant 1$. We need some lemmas.

Lemma 2.2 *Let $\lambda, y \in \mathbb{R}, \beta \in \mathbb{C}$, and $y > 0, \mathrm{Re}(\beta) > 0$. Then*

$$\int_{y - \mathrm{i}\infty}^{y + \mathrm{i}\infty} v^{-\beta} \mathrm{e}^{\lambda v} \mathrm{d}v = \begin{cases} 2\pi \mathrm{i} \lambda^{\beta - 1} \Gamma(\beta)^{-1}, & \text{if } \lambda > 0, \\ 0, & \text{if } \lambda \leqslant 0. \end{cases}$$

Proof We only need to prove the lemma for $0 < \mathrm{Re}(\beta) < 1$. Let

$$\beta = a + \mathrm{i}b, \quad v = |v| \mathrm{e}^{\mathrm{i}\varphi} = s + \mathrm{i}t, \quad s, t \in \mathbb{R}.$$

For $\lambda \leqslant 0$, we integrate along a path shown in Figure 2.1. Since

$$|v^{-\beta}\mathrm{e}^{\lambda v}| = \mathrm{e}^{-a\lg|v|+b\varphi+\lambda s} \to 0, \quad |v| \to \infty, s \geqslant y,$$

by the Cauchy Theorem for path integrals, we know that the lemma holds. For $\lambda > 0$,

$$\int_{y-\mathrm{i}\infty}^{y+\mathrm{i}\infty} v^{-\beta}\mathrm{e}^{\lambda v}\mathrm{d}v = \lambda^{\beta-1}\int_{\lambda y-\mathrm{i}\infty}^{\lambda y+\mathrm{i}\infty} v^{-\beta}\mathrm{e}^{v}\mathrm{d}v,$$

we integrate along the path as in Figure 2.2. When v runs over the small circle with radius r, we get

$$r|v^{-\beta}\mathrm{e}^{v}| = r^{1-a}|\mathrm{e}^{v}| \to 0, \quad r \to 0,$$

since $0 < a < 1$. On the other hand,

$$|v^{-\beta}\mathrm{e}^{v}| = \mathrm{e}^{-a\lg|v|+b\varphi+s} \to 0, \quad |v| \to \infty, s \leqslant \lambda y.$$

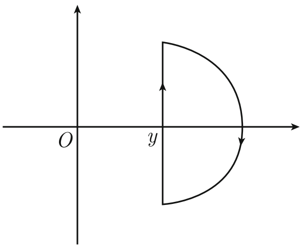

Figure 2.1

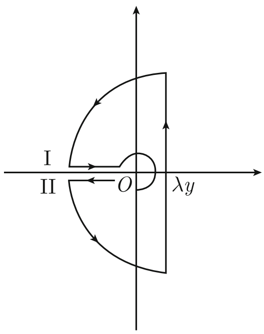

Figure 2.2

Hence by the Cauchy Theorem we have

$$\int_{\lambda y-\mathrm{i}\infty}^{\lambda y+\mathrm{i}\infty} v^{-\beta}\mathrm{e}^{v}\mathrm{d}v = -\int_{-\infty}^{0} v^{-\beta}\mathrm{e}^{v}\mathrm{d}v - \int_{0}^{-\infty} v^{-\beta}\mathrm{e}^{v}\mathrm{d}v,$$

where the variable v in the first integral runs above the negative real axis and the variable v in the second integral runs underneath the negative real axis. Therefore

$$\int_{-\infty}^{0} v^{-\beta}\mathrm{e}^{v}\mathrm{d}v = \mathrm{e}^{-\mathrm{i}\pi\beta}\int_{0}^{\infty} x^{-\beta}\mathrm{e}^{-x}\mathrm{d}x = \mathrm{e}^{-\mathrm{i}\pi\beta}\Gamma(1-\beta),$$
$$\int_{0}^{-\infty} v^{-\beta}\mathrm{e}^{v}\mathrm{d}v = -\mathrm{e}^{\mathrm{i}\pi\beta}\int_{0}^{\infty} x^{-\beta}\mathrm{e}^{-x}\mathrm{d}x = -\mathrm{e}^{\mathrm{i}\pi\beta}\Gamma(1-\beta).$$

But

$$(\mathrm{e}^{\mathrm{i}\pi\beta} - \mathrm{e}^{-\mathrm{i}\pi\beta})\Gamma(1-\beta) = 2\mathrm{i}\Gamma(1-\beta)\sin\pi\beta = 2\pi\mathrm{i}\Gamma(\beta)^{-1},$$

which completes the proof. □

Let $y > 0,\ \alpha,\ \beta \in \mathbb{C}$ and define

$$W(y,\alpha,\beta) = \Gamma(\beta)^{-1}\int_0^\infty (1+u)^{\alpha-1}u^{\beta-1}\mathrm{e}^{-yu}\mathrm{d}u,$$

which is called the Whittaker function. It is clear that the integral is convergent for $\mathrm{Re}(\beta) > 0$. Applying integration by parts we get

$$W(y,\alpha,\beta) = yW(y,\alpha,\beta+1) + (1-\alpha)W(y,\alpha-1,\beta+1). \tag{2.4}$$

Due to the above equality $W(y,\alpha,\beta)$ can be continued analytically to $\mathbb{C}^2$ for (α,β). We will also denote the continued function by $W(y,\alpha,\beta)$.

Lemma 2.3 $W(y,\alpha,0) = 1, W(y,\alpha,-1/2) = y^{1/2}$.

Proof Taking $\beta = 0$ in equality (2.4), we have

$$\begin{aligned}
W(y,\alpha,0) &= yW(y,\alpha,1) + (1-\alpha)W(y,\alpha-1,1)\\
&= y\int_0^\infty (1+u)^{\alpha-1}\mathrm{e}^{-yu}\mathrm{d}u + (1-\alpha)\int_0^\infty (1+u)^{\alpha-2}\mathrm{e}^{-yu}\mathrm{d}u\\
&= y\int_0^\infty (1+u)^{\alpha-1}\mathrm{e}^{-yu}\mathrm{d}u - \int_0^\infty \mathrm{e}^{-yu}\mathrm{d}(1+u)^{\alpha-1}\\
&= -\mathrm{e}^{-yu}(1+u)^{\alpha-1}\ |_0^\infty = 1.
\end{aligned}$$

Similarly taking $\beta = -1/2$ in (2.4), we have

$$W(y,1,-1/2) = yW(y,1,1/2) = y\Gamma(1/2)^{-1}\int_0^\infty u^{-1/2}\mathrm{e}^{-yu}\mathrm{d}u = y^{1/2},$$

which completes the proof. □

Lemma 2.4 *Let* $y > 0, \alpha, \beta \in \mathbb{C}$. *Then*

$$y^\beta W(y,\alpha,\beta) = y^{1-\alpha}W(y,1-\beta,1-\alpha).$$

Proof Taking the Mellin transformation of $\Gamma(\beta)W(y,\alpha,\beta)$ (assume $\mathrm{Re}(s) > 0$), we see

$$\begin{aligned}
\Gamma(\beta)\int_0^\infty W(y,\alpha,\beta)y^{s-1}\mathrm{d}y &= \int_0^\infty (u+1)^{\alpha-1}u^{\beta-1}\int_0^\infty y^{s-1}\mathrm{e}^{-yu}\mathrm{d}y\mathrm{d}u\\
&= \Gamma(s)\int_0^\infty (u+1)^{\alpha-1}u^{\beta-s-1}\mathrm{d}u.
\end{aligned}$$

Suppose $\mathrm{Re}(1-\alpha) > 0$ and insert the following equality into the formula above

$$(u+1)^{\alpha-1} = \Gamma(1-\alpha)^{-1}\int_0^\infty \mathrm{e}^{-x(u+1)}x^{-\alpha}\mathrm{d}x,$$

we get

$$\Gamma(1-\alpha)\Gamma(\beta)\int_0^{\infty} W(y,\alpha,\beta)y^{s-1}\mathrm{d}y = \Gamma(s)\Gamma(\beta-s)\Gamma(1-\alpha-\beta+s).$$

By the inverse Mellin transformation, we see

$$W(y,\alpha,\beta) = \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\Gamma(s)\Gamma(\beta-s)\Gamma(1-\alpha-\beta+s)}{\Gamma(1-\alpha)\Gamma(\beta)} y^{-s}\mathrm{d}s,$$

where c satisfies the inequalities $c>0$, $\mathrm{Re}(\beta)>c>\mathrm{Re}(\alpha+\beta-1)$. There exists such a c if $\mathrm{Re}(\beta)>0$, $\mathrm{Re}(1-\alpha)>0$. Let $S=s-\beta$, we have

$$y^{\beta}W(y,\alpha,\beta) = \frac{1}{2\pi\mathrm{i}}\int_{-p-\mathrm{i}\infty}^{-p+\mathrm{i}\infty} \frac{\Gamma(-S)\Gamma(\beta+S)\Gamma(1-\alpha+S)}{\Gamma(1-\alpha)\Gamma(\beta)} y^{-S}\mathrm{d}S,$$

where p satisfies $0<p<\min\{\mathrm{Re}(1-\alpha),\mathrm{Re}(\beta)\}$. The right hand side of the above equality is stable under the transformation $\alpha\to 1-\beta, \beta\to 1-\alpha$. This shows that the lemma holds for $\mathrm{Re}(1-\alpha)>0, \mathrm{Re}(\beta)>0$. But $W(y,\alpha,\beta)$ is analytic on $\mathbb{C}^2$. So the lemma holds for any $(\alpha,\beta)\in\mathbb{C}^2$, which completes the proof. □

Lemma 2.5 *Suppose that* $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$, $\mathrm{Re}(\alpha+\beta)>1, z=x+\mathrm{i}y\in\mathbb{H}$, *then*

$$\sum_{m=-\infty}^{+\infty}(z+m)^{-\alpha}(\overline{z}+m)^{-\beta} = \sum_{n=-\infty}^{+\infty} t_n(y,\alpha,\beta)\mathrm{e}^{2\pi\mathrm{i}nx},$$

where

$$\mathrm{i}^{\alpha-\beta}(2\pi)^{-\alpha-\beta}t_n(y,\alpha,\beta) = \begin{cases} n^{\alpha+\beta-1}\mathrm{e}^{-2\pi ny}\Gamma(\alpha)^{-1}W(4\pi ny,\alpha,\beta), & \text{if } n>0,\\ |n|^{\alpha+\beta-1}\mathrm{e}^{-2\pi|n|y}\Gamma(\beta)^{-1}W(4\pi|n|y,\beta,\alpha), & \text{if } n<0,\\ \Gamma(\alpha)^{-1}\Gamma(\beta)^{-1}\Gamma(\alpha+\beta-1)(4\pi y)^{1-\alpha-\beta}, & \text{if } n=0. \end{cases}$$

Proof Let

$$f(x) = \sum_{m=-\infty}^{+\infty}(x+\mathrm{i}y+m)^{-\alpha}(x-\mathrm{i}y+m)^{-\beta}.$$

This series is absolutely convergent for $\mathrm{Re}(\alpha+\beta)>1$. Since $f(x+1)=f(x)$, we have

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n \mathrm{e}^{2\pi\mathrm{i}nx},$$

where

$$\begin{aligned} c_n &= \int_0^1 f(x)\mathrm{e}^{-2\pi\mathrm{i}nx}\mathrm{d}x \\ &= \int_{-\infty}^{+\infty} (x+\mathrm{i}y)^{-\alpha}(x-\mathrm{i}y)^{-\beta}\mathrm{e}^{-2\pi\mathrm{i}nx}\mathrm{d}x \\ &= \mathrm{i}^{\beta-\alpha}\int_{-\infty}^{+\infty} (y-\mathrm{i}x)^{-\alpha}(y+\mathrm{i}x)^{-\beta}\mathrm{e}^{-2\pi\mathrm{i}nx}\mathrm{d}x \\ &= \mathrm{i}^{\beta-\alpha-1}\mathrm{e}^{2\pi ny}\int_{y-\mathrm{i}\infty}^{y+\mathrm{i}\infty} v^{-\beta}(2y-v)^{-\alpha}\mathrm{e}^{-2\pi nv}\mathrm{d}v \\ &= \mathrm{i}^{\beta-\alpha-1}\mathrm{e}^{2\pi ny}\Gamma(\alpha)^{-1}\int_{y-\mathrm{i}\infty}^{y+\mathrm{i}\infty} v^{-\beta}\mathrm{e}^{-2\pi nv}\int_0^\infty \mathrm{e}^{-\xi(2y-v)}\xi^{\alpha-1}\mathrm{d}\xi\mathrm{d}v \\ &= \mathrm{i}^{\beta-\alpha-1}\mathrm{e}^{2\pi ny}\Gamma(\alpha)^{-1}\int_0^\infty \xi^{\alpha-1}\mathrm{e}^{-2y\xi}\left\{\int_{y-\mathrm{i}\infty}^{y+\mathrm{i}\infty} v^{-\beta}\mathrm{e}^{(\xi-2\pi n)v}\mathrm{d}v\right\}\mathrm{d}\xi, \end{aligned}$$

where we used the fact that

$$(2y-v)^{-\alpha} = \Gamma(\alpha)^{-1}\int_0^\infty \mathrm{e}^{-\xi(2y-v)}\xi^{\alpha-1}\mathrm{d}\xi$$

for $\mathrm{Re}(\alpha) > 0$.

Now let $\xi = 2\pi p, u = \max\{0, n\}$. Since $\mathrm{Re}(\beta) > 0$, by Lemma2.2 we have

$$\begin{aligned} c_n &= 2\pi\mathrm{i}^{\beta-\alpha}\mathrm{e}^{2\pi ny}\Gamma(\alpha)^{-1}\Gamma(\beta)^{-1}\int_{2\pi u}^\infty \xi^{\alpha-1}(\xi-2\pi n)^{\beta-1}\mathrm{e}^{-2y\xi}\mathrm{d}\xi \\ &= (2\pi)^{\alpha+\beta}i^{\beta-\alpha}\mathrm{e}^{2\pi ny}\Gamma(\alpha)^{-1}\Gamma(\beta)^{-1}\int_u^\infty p^{\alpha-1}(p-n)^{\beta-1}\mathrm{e}^{-4\pi py}\mathrm{d}p. \end{aligned}$$

If $n > 0$, then $u = n$, let $p - n = nq$. If $n < 0$, then $u = 0$, let $p = -nq$. Hence we have

$$\begin{aligned} &\int_u^\infty p^{\alpha-1}(p-n)^{\beta-1}\mathrm{e}^{-4\pi py}\mathrm{d}p \\ &= \begin{cases} n^{\alpha+\beta-1}\displaystyle\int_0^\infty (q+1)^{\alpha-1}q^{\beta-1}\mathrm{e}^{-4\pi n(1+q)y}\mathrm{d}q, & \text{if } n > 0, \\ |n|^{\alpha+\beta-1}\displaystyle\int_0^\infty (q+1)^{\beta-1}q^{\alpha-1}\mathrm{e}^{-4\pi|n|qy}\mathrm{d}q, & \text{if } n < 0, \\ \displaystyle\int_0^\infty p^{\alpha+\beta-2}\mathrm{e}^{-4\pi py}\mathrm{d}p, & \text{if } n = 0 \end{cases} \\ &= \begin{cases} n^{\alpha+\beta-1}\mathrm{e}^{-4\pi ny}W(4\pi ny, \alpha, \beta), & \text{if } n > 0, \\ |n|^{\alpha+\beta-1}W(4\pi|n|y, \beta, \alpha), & \text{if } n < 0, \\ (4\pi y)^{1-\alpha-\beta}\Gamma(\alpha+\beta-1), & \text{if } n = 0, \end{cases} \end{aligned}$$

which completes the proof. □

Now we can compute the Fourier expansion of $E_k(s,\omega,N)(z)$. Let

$$W=\{(c,d)|c,d\in\mathbb{Z},\gcd(c,d)=1,N|c,c\geqslant 0,d=1 \text{ if } c=0\}.$$

Then we can prove that there exists a one-to-one correspondence between W and the set of representatives of right cosets of $\varGamma_\infty$ in $\varGamma_0(N)$. Suppose $\mathrm{Re}(s)>2-k/2$, by Lemma 2.5 we have (substituting c by cN)

$$\begin{aligned}
E_k(s,\omega,N)(z)&=y^{s/2}\left\{1+\sum_{d=-\infty}^{+\infty}\sum_{c=1}^{+\infty}\omega(d)\varepsilon_d^k\left(\frac{cN}{d}\right)(cNz+d)^{-k/2}|cNz+d|^{-s}\right\}\\
&=y^{s/2}\left\{1+\sum_{c=1}^{\infty}(cN)^{-k/2-s}\sum_{d=1}^{cN}\omega(d)\varepsilon_d^k\left(\frac{cN}{d}\right)\right.\\
&\quad\left.\times\sum_{n=-\infty}^{\infty}\left(z+\frac{d}{cN}+n\right)^{-k/2-s/2}\left(\overline{z}+\frac{d}{cN}+n\right)^{-s/2}\right\}\\
&=y^{s/2}\left\{1+\sum_{n=-\infty}^{\infty}a_k(n,s,\omega,N)t_n(y,(k+s)/2,s/2)e(nx)\right\},
\end{aligned}\tag{2.5}$$

where

$$a_k(n,s,\omega,N)=\sum_{c=1}^{\infty}(cN)^{-k/2-s}\sum_{d=1}^{cN}\omega(d)\varepsilon_d^k\left(\frac{cN}{d}\right)e\left(\frac{nd}{cN}\right).\tag{2.6}$$

For $\mathrm{Re}(s)>2-k/2$, define

$$E_k'(s,\omega,N)(z)=z^{-k/2}E_k(s,\omega,N)(-1/(Nz)).\tag{2.7}$$

Now assume that $\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\varGamma_0(N)$. Then by (2.3) we can verify easily that

$$E_k'(s,\omega,N)(\gamma(z))=\omega(d)\left(\frac{N}{d}\right)j(\gamma,z)^kE_k'(s,\omega,N)(z).\tag{2.8}$$

Now let $W'=\{(c,d)|c,d\in\mathbb{Z},\gcd(c,d)=1,N|c,d>0\}$. Then there exists a one-to-one correspondence between W' and the set of representatives of cosets of $\varGamma_\infty$ in $\varGamma_0(N)$. Then we can similarly get that

$$E_k'(s,\omega,N)(z)=y^{s/2}N^{-s/2}\sum_{n=-\infty}^{\infty}b_k(n,s,\omega,N)t_n(y,(k+s)/2,s/2)e(nx),\tag{2.9}$$

where

$$b_k(n,s,\omega,N)=\sum_{d=1}^{\infty}\left(\frac{-N}{d}\right)\omega(d)\varepsilon_d^kd^{-s-k/2}\sum_{m=1}^{d}\left(\frac{m}{d}\right)e\left(\frac{nm}{d}\right).\tag{2.10}$$

Lemma 2.6 *Let ω_0 be a primitive character modulo r, ω be a character modulo rs, and $\omega(n) = \omega_0(n)$ for $\gcd(n,s) = 1$. Then for any integer q we have*

$$\sum_{n=1}^{rs} \omega(n) e\left(\frac{nq}{rs}\right) = \sum_{m=1}^{r} \omega_0(m) e(m/r) \sum_{c|(s,q)} c\mu(s/c)\omega_0(s/c)\overline{\omega}_0(q/c).$$

Proof We have that

$$\begin{aligned}\sum_{n=1}^{rs} \omega(n) e\left(\frac{nq}{rs}\right) &= \sum_{n=1}^{rs} \omega_0(n) \sum_{d|(s,n)} \mu(d) e\left(\frac{nq}{rs}\right) \\ &= \sum_{d|s} \mu(d) \sum_{n=1}^{rs/d} \omega_0(nd) e\left(\frac{ndq}{rs}\right) \\ &= \sum_{d|s} \mu(d)\omega_0(d) \sum_{n=1}^{r} \omega_0(n) e\left(\frac{ndq}{rs}\right) \sum_{u=1}^{s/d} e\left(\frac{uq}{s/d}\right).\end{aligned}$$

Denote $c = s/d$, then the inner summation in the above formula is zero for all $c \nmid q$ and is c for $c|q$ which shows the lemma. □

Now let $d = ru^2$ be an odd positive integer with r square free. Taking $\omega = \left(\frac{\cdot}{d}\right)$, $\omega_0 = \left(\frac{\cdot}{r}\right)$, $q = n, s = u^2$ in Lemma 2.6, we have

$$\sum_{m=1}^{d} \left(\frac{m}{d}\right) e\left(\frac{nm}{d}\right) = \varepsilon_r r^{1/2} \sum_{c|(u^2,n)} c\mu(u^2/c) \left(\frac{u^2/c}{r}\right) \left(\frac{n/c}{r}\right), \tag{2.11}$$

where we used the fact

$$\sum_{m=1}^{r} \left(\frac{m}{r}\right) e\left(\frac{m}{r}\right) = \varepsilon_r r^{1/2}.$$

Let $\lambda = (k-1)/2$ and n be an integer. We define a primitive character $\omega_k^{(n)}$ satisfying

$$\omega_k^{(n)}(d) = \left(\frac{(-1)^\lambda nN}{d}\right) \omega(d), \quad \text{if } (d, nN) = 1.$$

We also define a primitive character ω' satisfying

$$\omega'(d) = \omega^2(d), \quad \text{if } (d, N) = 1.$$

Suppose that χ is a character modulo a factor of N. Define

$$L_N(s,\chi) = \sum_{(n,N)=1}^{\infty} \chi(n) n^{-s} = \prod_{p\nmid N} (1 - \chi(p)p^{-s})^{-1},$$

where p runs over all primes co-prime to N.

Proposition 2.1 *We have*

$$L_N(2s+2\lambda,\omega')b_k(0,s,\omega,N) = L_N(2s+2\lambda-1,\omega').$$

For $n \neq 0$, we have

$$L_N(2s+2\lambda,\omega')b_k(n,s,\omega,N) = L_N(s+\lambda,\omega_k^{(n)})\beta_k(n,s,\omega,N),$$

where

$$\beta_k(n,s,\omega,N) = \sum_{a,b}\mu(a)\omega_k^{(n)}(a)\omega'(b)a^{-s-\lambda}b^{-2s-2\lambda+1}, \tag{2.12}$$

where a, b run over all positive integers such that $(ab,N)=1$ and $(ab)^2|n$.

Proof For $n=0$, the inner summation of (2.10) is nonzero only for d a square. Therefore

$$\begin{aligned}
b_k(0,s,\omega,N) &= \sum_{u=1}^{\infty}\omega(u^2)u^{-2s-k}\varphi(u^2)\\
&= \prod_{p\nmid N}\left\{\sum_{i=0}^{\infty}\omega(p^{2i})p^{-(2s+k)i}\varphi(p^{2i})\right\}\\
&= \prod_{p\nmid N}\left\{1+\sum_{i=1}^{\infty}\left(1-\frac{1}{p}\right)(\omega(p^2)p^{-(2s+k-2)})^i\right\}\\
&= \prod_{p\nmid N}\frac{1-\omega(p^2)p^{-2s-k-1}}{1-\omega(p^2)p^{-2s-k+2}}\\
&= L_N(2s+2\lambda-1,\omega')L_N(2s+2\lambda,\omega')^{-1}.
\end{aligned}$$

Now assume that $n=tm^2\neq 0$, t square free. Since N is even, the summation in (2.10) is nonzero only for odd integer d. By (2.11) we get

$$\begin{aligned}
b_k(n,s,\omega,N) = &\sum_{r,u}\left(\frac{-N}{ru^2}\right)\varepsilon_r^{k+1}\omega(ru^2)(ru^2)^{-s-k/2}r^{1/2}\\
&\times\sum_{c|(u^2,n)}c\mu(u^2/c)\left(\frac{u^2/c}{r}\right)\left(\frac{n/c}{r}\right),
\end{aligned}$$

where r,u run over all positive integers with r square free. Denote $u^2=ac$, then $\mu(a)\neq 0$ only for a square free. So we can suppose $u=ab$. Then

$$c=ab^2,\quad u^2n/c^2=n/b^2,$$

hence

$$b_k(n,s,\omega,N) = \sum_{r,a,b}\mu(a)r^{-s-\lambda}a^{-2s-2\lambda}b^{-2s-2\lambda+1}\omega(ra^2b^2)\left(\frac{(-1)^\lambda nN/b^2}{r}\right),$$

where we used the fact

$$\varepsilon_r^{k+1} = \left(\frac{(-1)^{(k+1)/2}}{r} \right)$$

and r, a, b run over all positive integers such that $(rab, N) = 1$, $ab^2 \mid n$ and r square free. Since $ab^2 \mid n = tm^2$, we see that $b \mid m$. Let $m = bh$, then $a \mid th$, $n/b^2 = th^2$. Since

$$\omega(r) \left(\frac{(-1)^\lambda N th^2}{r} \right) = \begin{cases} 0, & \text{if } (r, thN) > 1, \\ \omega_k^{(n)}(r), & \text{if } (r, thN) = 1, \end{cases}$$

we have

$$\begin{aligned} b_k(n, s, \omega, N) = & \sum_{b|m} \omega^2(b) b^{-2s-2\lambda+1} \sum_{a|th} \mu(a) \omega^2(a) a^{-2s-2\lambda} \\ & \times \sum_{(r,thN)=1} \mu^2(r) \omega_k^{(n)}(r) r^{-s-\lambda}. \end{aligned} \tag{2.13}$$

It is clear that

$$\sum_{a|th} \mu(a) \omega^2(a) a^{-2s-2\lambda} = \prod_{p|th, p\nmid N} \left(1 - \omega'(p) p^{-2s-2\lambda}\right) \tag{2.14}$$

and

$$\begin{aligned} \sum_{(r,thN)=1} \mu^2(r) \omega_k^{(n)}(r) r^{-s-\lambda} &= \prod_{p\nmid thN} \left(1 + \omega_k^{(n)}(p) p^{-s-\lambda}\right) \\ &= \prod_{p\nmid thN} \frac{1 - \omega'(p) p^{-2s-2\lambda}}{1 - \omega_k^{(n)}(p) p^{-s-\lambda}} \\ &= \frac{L_N(s+\lambda, \omega_k^{(n)})}{L_N(2s+2\lambda, \omega')} \prod_{p|th, p\nmid N} \frac{1 - \omega_k^{(n)}(p) p^{-s-\lambda}}{1 - \omega'(p) p^{-2s-2\lambda}}, \end{aligned} \tag{2.15}$$

For primes p such that $p \mid t, p \nmid N$, we have $\omega_k^{(n)}(p) = 0$. Inserting (2.14) and (2.15) into (2.13), we get

$$\begin{aligned} b_k(n, s, \omega, N) &= \frac{L_N(s+\lambda, \omega_k^{(n)})}{L_N(2s+2\lambda, \omega')} \sum_{b|m} \omega^2(b) b^{-2s-2\lambda+1} \prod_{p|h, p\nmid N} \left(1 - \omega_k^{(n)}(p) p^{-s-\lambda}\right) \\ &= \frac{L_N(s+\lambda, \omega_k^{(n)})}{L_N(2s+2\lambda, \omega')} \sum_{a,b} \mu(a) \omega_k^{(n)}(a) \omega'(b) a^{-s-\lambda} b^{-2s-2\lambda+1}, \end{aligned}$$

which completes the proof. □

Let n be any integer and χ_n a primitive character satisfying

$$\chi_n(d) = \left(\frac{n}{d} \right) \qquad \text{for all } (d, 4n) = 1.$$