Contributions in Mathematical and Computational Sciences 2

Jakob Stix *Editor*

The Arithmetic of Fundamental Groups

PIA 2010

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Editor

The Arithmetic of Fundamental Groups

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to Lucie Ella Rose Stix

Preface to the Series

Contributions to Mathematical and Computational Sciences

Mathematical theories and methods and effective computational algorithms are crucial in coping with the challenges arising in the sciences and in many areas of their application. New concepts and approaches are necessary in order to overcome the complexity barriers particularly created by nonlinearity, high-dimensionality, multiple scales and uncertainty. Combining advanced mathematical and computational methods and computer technology is an essential key to achieving progress, often even in purely theoretical research.

The term mathematical sciences refers to mathematics and its genuine sub-fields, as well as to scientific disciplines that are based on mathematical concepts and methods, including sub-fields of the natural and life sciences, the engineering and social sciences and recently also of the humanities. It is a major aim of this series to integrate the different sub-fields within mathematics and the computational sciences, and to build bridges to all academic disciplines, to industry and other fields of society, where mathematical and computational methods are necessary tools for progress. Fundamental and application-oriented research will be covered in proper balance.

The series will further offer contributions on areas at the frontier of research, providing both detailed information on topical research, as well as surveys of the state-of-the-art in a manner not usually possible in standard journal publications. Its volumes are intended to cover themes involving more than just a single "spectral line" of the rich spectrum of mathematical and computational research.

The Mathematics Center Heidelberg (MATCH) and the Interdisciplinary Center for Scientific Computing (IWR) with its Heidelberg Graduate School of Mathematical and Computational Methods for the Sciences (HGS) are in charge of providing and preparing the material for publication. A substantial part of the material will be acquired in workshops and symposia organized by these institutions in topical areas of research. The resulting volumes should be more than just proceedings collecting papers submitted in advance. The exchange of information and the discussions during the meetings should also have a substantial influence on the contributions.

This series is a venture posing challenges to all partners involved. A unique style attracting a larger audience beyond the group of experts in the subject areas of specific volumes will have to be developed.

Springer Verlag deserves our special appreciation for its most efficient support in structuring and initiating this series.

Heidelberg University,
Germany

Hans Georg Bock
Willi Jäger
Otmar Venjakob

Preface

During the more than 100 years of its existence, the notion of the fundamental group has undergone a considerable evolution. It started by Henri Poincaré when topology as a subject was still in its infancy. The fundamental group in this setup measures the complexity of a pointed topological space by means of an algebraic invariant, a discrete group, composed of deformation classes of based closed loops *within* the space. In this way, for example, the monodromy of a holomorphic function on a Riemann surface could be captured in a systematic way.

It was through the work of Alexander Grothendieck that, raising into the focus the role played by the fundamental group in governing covering spaces, so spaces *over* the given space, a unification of the topological fundamental group with Galois theory of algebra and arithmetic could be achieved. In some sense the roles have been reversed in this *discrete Tannakian* approach of abstract Galois categories: first, we describe a suitable class of objects that captures *monodromy*, and then, by abstract properties of this class alone and moreover uniquely determined by it, we find a pro-finite group that describes this category completely as the category of discrete objects continuously acted upon by that group.

But the different incarnations of a fundamental group do not stop here. The concept of describing a fundamental group through its category of objects upon which the group naturally acts finds its pro-algebraic realisation in the theory of Tannakian categories that, when applied to vector bundles with flat connections, or to smooth ℓ-adic étale sheaves, or to iso-crystals or ..., gives rise to the corresponding fundamental group, each within its natural category as a habitat.

In more recent years, the influence of the fundamental group on the geometry of Kähler manifolds or algebraic varieties has become apparent. Moreover, the program of anabelian geometry as initiated by Alexander Grothendieck realised some spectacular achievements through the work of the Japanese school of Hiroaki Nakamura, Akio Tamagawa and Shinichi Mochizuki culminating in the proof that hyperbolic curves over p-adic fields are determined by the outer Galois action of the absolute Galois group of the base field on the étale fundamental group of the curve.

A natural next target for pieces of arithmetic captured by the fundamental group are rational points, the genuine object of study of Diophantine geometry. Here there are two related strands: Grothendieck's section conjecture in the realm of the étale arithmetic fundamental group, and second, more recently, Minhyong Kim's idea to use the full strength of the different (motivic) realisations of the fundamental group to obtain a nonabelian unipotent version of the classical Chabauty approach towards rational points. In this approach, one seeks for a nontrivial p-adic Coleman analytic function that finds all global rational points among its zeros, whereby in the one-dimensional case the number of zeros necessarily becomes finite. This has led to a spectacular new proof of Siegel's theorem on the finiteness of S-integral points in some cases and, moreover, raised hope for ultimately (effectively) reproving the Faltings–Mordell theorem. A truely motivic advance of Minhyong Kim's ideas due to Gerd Faltings and Majid Hadian is reported in the present volume.

This volume originates from a special activity at Heidelberg University under the sponsorship of the MAThematics Center Heidelberg (MATCH) that took place in January and February 2010 organised by myself. The aim of the activity was to bring together people working in the different strands and incarnations of the fundamental group all of whose work had a link to arithmetic applications. This was reflected in the working title PIA for our activity, which is the (not quite) acronym for π_1–arithmetic, short for *doing arithmetic with the fundamental group* as your main tool and object of study. PIA survived in the title of the workshop organised during the special activity: *PIA 2010 — The arithmetic of fundamental groups*, which in reversed order gives rise to the title of the present volume.

The workshop took place in Heidelberg, 8–12 February 2010, and the abstracts of all talks are listed at the end of this volume. Many of these accounts are mirrored in the contributions of the present volume. The special activity also comprised expository lecture series by Amnon Besser on Coleman integration, a technique used by the non-abelian Chabauty method, and by Tamás Szamuely on Grothendieck's fundamental group with a view towards anabelian geometry. Lecture notes of these two introductory courses are contained in this volume as a welcome addition to the existing literature of both subjects.

I wish to extend my sincere thanks to the contributors of this volume and to all participants of the special activity in Heidelberg on the arithmetic of fundamental groups, especially to the lecturers giving mini-courses, for the energy and time they have devoted to this event and the preparation of the present collection. Paul Seyfert receives the editor's thanks for sharing his marvelous TEX–expertise and help in typesetting this volume. Furthermore, I would like to take this opportunity to thank Dorothea Heukäufer for her efficient handling of the logistics of the special activity and Laura Croitoru for coding the website. I am very grateful to Sabine Stix for sharing her organisational skills both by providing a backbone for the *to do* list of the whole program and also in caring for our kids Antonia, Jaden and Lucie. Finally, I would like to express my gratitude to Willi Jäger, the former director of MATCH, for his enthusiastic support and for the financial support of MATCH that made PIA 2010 possible and in my opinion a true success.

Heidelberg *Jakob Stix*

Contents

Part I
Heidelberg Lecture Notes

Chapter 1
Heidelberg Lectures on Coleman Integration

Amnon Besser*

Abstract Coleman integration is a way of associating with a closed one-form on a p-adic space a certain locally analytic function, defined up to a constant, whose differential gives back the form. This theory, initially developed by Robert Coleman in the 1980s and later extended by various people including the author, has now found various applications in arithmetic geometry, most notably in the spectacular work of Kim on rational points. In this text we discuss two approaches to Coleman integration, the first is a semi-linear version of Coleman's original approach, which is better suited for computations. The second is the author's approach via unipotent isocrystals, with a simplified and essentially self-contained presentation. We also survey many applications of Coleman integration and describe a new theory of integration in families.

1.1 Introduction

In the first half of February 2010 I spent 2 weeks at the Mathematics Center Heidelberg (MATCH) at the university of Heidelberg, as part of the activity *PIA 2010 – The arithmetic of fundamental groups*. In the first week I gave 3 introductory lectures on Coleman integration theory and in the second week I gave a research lecture on new work, which was (and still is) in progress, concerning Coleman integration in families. I later gave a similar sequence of lectures at the Hebrew University in Jerusalem.

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This article gives an account of the 3 instructional lectures as well as the lecture I gave at the conference in Heidelberg with some (minimal) additions. I largely left things as they were presented in the lectures and I therefore apologize for the sometimes informal language used and the occasional proof which is only sketched. As in the lectures I made an effort to make things as self-contained as possible.

The main goal of these lectures is to introduce Coleman integration theory. The goal of this theory is (in very vague terms) to associate with a closed 1-form $\omega \in \Omega^1(\mathrm{X})$, where X is a "space" over a p-adic field K, by which we mean a finite extension of $\mathbb{Q}_p$, for a prime p fixed throughout this work, a locally analytic primitive F_ω, i.e., such that $d\mathrm{F}_\omega = \omega$, in such a way that it is unique up to a constant.

In Sect. 1.4 we introduce Coleman theory. The presentation roughly follows Coleman's original approach [Col82, CdS88]. One essential difference is that we emphasize the semi-linear point of view. This turns out to be very useful in numerical computations of Coleman integrals. The presentation we give here, which does not derive the semi-linear properties from Coleman's work, is new.

In Sect. 1.5 we give an account of the Tannakian approach to Coleman integration developed in [Bes02]. The main novelty is a more self contained and somewhat simplified proof from the one given in loc. cit. Rather than rely on the work of Chiarellotto [Chi98], relying ultimately on the thesis of Wildeshaus [Wil97], we unfold the argument and obtain some simplification by using the Lie algebra rather than its enveloping algebra.

At the advice of the referee we included a lengthy section on applications of Coleman integration. In the final section we explain a new approach to Coleman integration in families. We discuss two complementary formulations, one in terms of the Gauss-Manin connection and one in terms of differential Tannakian categories.

Acknowledgements. I would like to thank MATCH, and especially Jakob Stix, for inviting me to Heidelberg, and to thank Noam Solomon and Ehud de Shalit for organizing the sequence of lectures in Jerusalem. I also want to thank Lorenzo Ramero for a conversation crucial for the presentation of Kim's work. I would finally like to thank the referee for making many valuable comments that made this work far more readable than it originally was, and for a very careful reading of the manuscript catching a huge number of mistakes.

1.2 Overview of Coleman Theory

To appreciate the difficulty of integrating a closed form on a p-adic space, let us consider a simple example. We consider a form $\omega = dz/z$ on a space

$$\mathrm{X} = \{z \in \mathrm{K}\,;\ |z| = 1\}.$$

Morally, the primitive F_ω should just be the logarithm function $\log(z)$. To try to find a primitive, we could pick $\alpha \in \mathrm{X}$ and expand ω in a power series around α as follows:

$$\omega = \frac{\mathrm{d}(\alpha + x)}{\alpha + x} = \frac{\mathrm{d}x}{\alpha + x} = \frac{1}{\alpha}\frac{\mathrm{d}x}{1 + x/\alpha} = \frac{1}{\alpha}\sum\left(\frac{-x}{\alpha}\right)^n \mathrm{d}x$$

and integrating term by term we obtain

$$\mathrm{F}_\omega(\alpha + x) = -\sum \frac{1}{n+1}\left(\frac{-x}{\alpha}\right)^{n+1} + \mathrm{C}$$

where these expansions converge on the disc for which $|x| < 1$.

So far, we have done nothing that could not be done in the complex world. However, in the complex world we could continue as follows. Fix the constant of integration C on one of the discs. Then do analytic continuation: For each intersecting disc it is possible to fix the constant of integration on that disc uniquely so that the two expansions agree on the intersection. Going around a circle around 0 gives a non-trivial monodromy, so analytic continuation results in a multivalued function, which is the log function.

In the p-adic world, we immediately realize that such a strategy will not work because two open discs of radius 1 are either identical or completely disjoint. Thus, there is no obvious way of fixing simultaneously the constants of integration.

Starting with [Col82], Robert Coleman devises a strategy for coping with this difficulty using what he called **analytic continuation along Frobenius**. To explain this in our example, we take the map $\phi : \mathrm{X} \to \mathrm{X}$ given by $\phi(x) = x^p$ which is a lift of the p-power map. One notices immediately that $\phi^*\omega = p\omega$. Coleman's idea is that this relation should imply a corresponding relation on the integrals

$$\phi^*\mathrm{F}_\omega = p\mathrm{F}_\omega + \mathrm{C}$$

where C is a constant function. It is easy to see that by changing F_ω by a constant, which we are allowed to do, we can assume that $\mathrm{C} = 0$. The equation above now reads

$$\mathrm{F}_\omega(x^p) = p\mathrm{F}_\omega(x)\,.$$

Suppose now that α satisfies the relation $\alpha^{p^k} = \alpha$. Then we immediately obtain

$$\mathrm{F}_\omega(\alpha) = \mathrm{F}_\omega(\alpha^{p^k}) = p^k\mathrm{F}_\omega(\alpha) \Rightarrow \mathrm{F}_\omega(\alpha) = 0\,.$$

This condition, together with the assumption that $\mathrm{dF}_\omega = \omega$ fixes F_ω on the disc $|z - \alpha| < 1$. But it is well known that every $z \in \mathrm{X}$ resides in such a disc, hence F_ω is completely determined.

In [Col82] Coleman also introduces iterated integrals (only on appropriate subsets of $\mathbb{P}^1$) which have the form

$$\int(\omega_n \times \int(\omega_{n-1} \times \cdots \int(\omega_2 \times \int \omega_1)\cdots)\,,$$

and in particular defines p-adic polylogarithms $\mathrm{Li}_n(z)$ by the conditions

$$\begin{aligned} \mathrm{d}\,\mathrm{Li}_1(z) &= \frac{\mathrm{d}z}{1-z}\,, \\ \mathrm{d}\,\mathrm{Li}_n(z) &= \mathrm{Li}_{n-1}(z)\frac{\mathrm{d}z}{z}\,, \\ \mathrm{Li}_n(0) &= 0\,, \end{aligned}$$

so that locally one finds

$$\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}\,.$$

Then, in the paper [Col85b] he extends the theory to arbitrary dimensions but without computing iterated integrals. In [CdS88] Coleman and de Shalit extend the iterated integrals to appropriate subsets of curves with good reduction.

In [Bes02] the author gave a Tannakian point of view to Coleman integration and extended the iterated theory to arbitrary dimensions. Other approaches exist. Colmez [Col98], and independently Zarhin [Zar96], used functoriality with respect to algebraic morphisms. This approach does not need good reduction but cannot handle iterated integrals. Vologodsky has a theory for algebraic varieties, which is similar in many respects to the theory in [Bes02]. Using alterations and defining a monodromy operation on the fundamental group in a very sophisticated way he is able to define iterated Coleman integrals also in the bad reductions case. Coleman integration was later extended by Berkovich [Ber07] to his p-adic analytic spaces, again without making any reductions assumptions.

Remark 1. There are two related ways of developing Coleman integration: the linear and the semi-linear way. For a variety over a finite field κ of characteristic p the absolute Frobenius φ_a is just the p-power map and its lifts to characteristic 0 are semi-linear. A linear Frobenius is any power of the absolute Frobenius which is κ-linear.

What makes the theory work is the description of weights of a linear Frobenius on the first cohomology (crystalline, rigid, Monsky-Washnitzer) of varieties over finite fields, see Theorem 4. The theory itself can be developed by imposing an equivariance conditions with respect to a lift of the linear Frobenius, as we have done above and as done in Coleman's work, or imposing equivariance with respect to a semi-linear lift of the absolute Frobenius. Even in this approach one ultimately relies on weights for a linear Frobenius.

The two approaches are equivalent. Since a power of a semi-linear Frobenius lift is linear, equivariance for a semi-linear Frobenius implies one for a linear Frobenius. Conversely, as Coleman integration is also Galois equivariant [Col85b, Corollary 2.1e] one recovers from the linear equivariance the semi-linear one.

The linear approach is cleaner in many respects, and it is used everywhere in this text with the exception of Sects. 1.3 and 1.4. There are two main reasons for introducing the semi-linear approach:

- It appears to be computationally more efficient.
- It may be applied in some situations where the linear approach may not apply, see Remark 11.

1.3 Background

Let K be a complete discrete valuation field of characteristic 0 with ring of integers R, residue field κ of prime characteristic p, uniformizer π and algebraic closure $\bar{K}$. We also fix an automorphism σ of K which reduces to the p-power map on κ, and when needed extend it to $\bar{K}$ such that it continues to reduce to the p-power map. When the cardinality of κ is finite we denote it by $q = p^r$.

1.3.1 Rigid Analysis

Let us recall first a few basic facts about rigid analysis. An excellent survey can be found in [Sch98]. The Tate algebra T_n is by definition

$$T_n = K\langle t_1, \ldots, t_n \rangle = \left\{ \sum a_I t^I \; ; \; a_I \in K, \lim_{I \to \infty} |a_I| = 0 \right\} ,$$

which is the same as the algebra of power series with coefficients in K converging on the unit polydisc

$$B_n = \{(z_1, \ldots, z_n) \in \bar{K}^n \; ; \; |z_i| \leq 1\} .$$

An affinoid algebra A is a K-algebra for which there exists a surjective map $T_n \to A$ for some n. One associates with A its maximal spectrum

$$\begin{aligned} X = \mathrm{spm}(A) :&= \{m \subset A \text{ maximal ideal }\} \\ &= \{\psi : A \to \bar{K} \text{ a K-homomorphism}\}/\mathrm{Gal}(\bar{K}/K) \end{aligned}$$

i.e., the quotient of the set of K-algebra homomorphisms from A to $\bar{K}$ (no topology involved) by the Galois group of $\bar{K}$ over K. The latter equality is a consequence of the Noether normalization lemma for affinoid algebras from which it follows that a field which is a homomorphic image of such an algebra is a finite extension of K. Two easy examples are

$$\begin{aligned} \mathrm{spm}(T_n) &= B_n/\mathrm{Gal}(\bar{K}/K) , \\ \mathrm{spm}(T_2/(t_1t_2 - 1)) &= \{(z_1, z_2) \in B_2 \; ; \; z_1z_2 = 1\}/\mathrm{Gal}(\bar{K}/K) \\ &= \{z \in \bar{K} \; ; \; |z| = 1\}/\mathrm{Gal}(\bar{K}/K) \end{aligned}$$

In what follows we will shorthand things so that the last space will simply be written $\{|z| = 1\}$ when there is no danger of confusion.

The maximal spectrum $X = \mathrm{spm}(A)$ of an affinoid algebra with an appropriate Grothendieck topology and sheaf of functions will be called an affinoid space, and in a Grothendieckian style we associate with it its ring of functions $\mathcal{O}(X) = A$. Rigid geometry allows one to glue affinoid spaces into more complicated spaces, and obtain the ring of functions on these spaces as well. We will say nothing about this except to mention that the space $B_n^\circ = \{|z_i| < 1\} \subset B_n$ can be obtained as the union of the spaces $\{|z_i^k/\pi| \le 1\}$ for $k \in \mathbb{N}$, and its ring of functions is not surprisingly

$$\mathcal{O}(\mathrm{B}_n^\circ) = \left\{ \sum a_{\mathrm{I}} t^{\mathrm{I}} \ ; \ \lim_{\mathrm{I}\to\infty} |a_{\mathrm{I}}| r^{|\mathrm{I}|} = 0 \text{ for any } r < 1 \right\}$$

where $|(i_1, \ldots, i_n)| = i_1 + \cdots + i_n$.

1.3.2 Dagger Algebras and Monsky-Washnitzer Cohomology

The de Rham cohomology of rigid spaces is problematic in certain respects. To see an example of this, consider the first de Rham cohomology of T_1, which is the cokernel of the map

$$\mathrm{d} : \mathrm{T}_1 \to \mathrm{T}_1 dt \, .$$

This cokernel is infinite as one can write down a power series $\sum a_i t^i$ such that the a_i converge to 0 sufficiently slowly to make the coefficients of the integral

$$\sum a_i t^{i+1}/(i+1)$$

not converge to 0. On the other hand, as B_1 can be considered a lift of the affine line, one should expect its cohomology to be trivial.

To remedy this, Monsky and Washnitzer [MW68] considered so called weakly complete finitely generated algebras. An excellent reference is the paper [vdP86].

We consider the algebra

$$\mathcal{T}_n^\dagger = \left\{ \sum a_{\mathrm{I}} t^{\mathrm{I}} \ ; \ a_{\mathrm{I}} \in \mathrm{R}, \ \exists r > 1 \text{ such that } \lim_{\mathrm{I}\to\infty} |a_{\mathrm{I}}| r^{\mathrm{I}} = 0 \right\} .$$

In other words, these are the power series converging on something slightly bigger than the unit polydisc, hence the term **overconvergence**. Integration reduces the radius of convergence, but only slightly: if the original power series converges to radius r the integral will no longer converge to radius r but will converge to any smaller radius, hence still overconverges.

An R-algebra $\mathrm{A}^\dagger$ is called a **weakly complete finitely generated** (wcfg) algebra if there is a surjective homomorphism $\mathcal{T}_n^\dagger \to \mathrm{A}^\dagger$. Since $\mathcal{T}_n^\dagger$ is Noetherian, see [vdP86] just after (2.2), such an algebra may be presented as

$$\mathrm{A}^\dagger = \mathcal{T}_n^\dagger/(f_1, \ldots, f_m) \, . \tag{1.1}$$

The module of differentials $\Omega^1_{\mathrm{A}^\dagger}$ is given, in the presentation (1.1), as

$$\Omega^1_{\mathrm{A}^\dagger} = \bigoplus_{i=1}^{n} \mathrm{A}^\dagger \mathrm{d}t_i/(\mathrm{d}f_j \, , \ j = 1, \ldots m) \, ,$$

where $\mathrm{d}f = \sum_i \frac{\partial f}{\partial t_i}\mathrm{d}t_i$ as usual, see [vdP86, (2.3)]. Be warned that this is not the algebraic module of differentials. Taking wedge powers one obtains the modules of higher differential forms $\Omega^i_{\mathrm{A}^\dagger}$ and the de Rham complex $\Omega^\bullet_{\mathrm{A}^\dagger}$.

One observes that $\mathcal{T}_n^\dagger/\pi$ is isomorphic to the polynomial algebra $\kappa[t_1,\ldots,t_n]$. Thus, if $\mathrm{A}^\dagger$ is a wcfg algebra then $\bar{\mathrm{A}} := \mathrm{A}^\dagger/\pi$ is a finitely generated κ-algebra.

Assume from now throughout the rest of this work that the κ-algebras considered are finitely generated and smooth. Any such κ-algebra can be obtained as an $\bar{\mathrm{A}}$ for an appropriate $\mathrm{A}^\dagger$ by a result of Elkik [Elk73]. In addition, we have the following results on those lifts.

Proposition 2 ([vdP86, Theorem 2.4.4]). *We have:*

(1) Any two such lifts are isomorphic.
(2) Any morphism $\bar{f} : \bar{\mathrm{A}} \to \bar{\mathrm{B}}$ can be lifted to a morphism $f^\dagger : \mathrm{A}^\dagger \to \mathrm{B}^\dagger$.
(3) Any two maps $\mathrm{A}^\dagger \to \mathrm{B}^\dagger$ with the same reduction induce homotopic maps

$$\Omega^\bullet_{\mathrm{A}^\dagger} \otimes \mathrm{K} \to \Omega^\bullet_{\mathrm{B}^\dagger} \otimes \mathrm{K}\,.$$

Thus, the following definition makes sense.

Definition 3. The **Monsky-Washnitzer cohomology** of $\bar{\mathrm{A}}$ is the cohomology of the de Rham complex $\Omega^\bullet_{\mathrm{A}^\dagger} \otimes \mathrm{K}$

$$\mathrm{H}^i_{\mathrm{MW}}(\bar{\mathrm{A}}/\mathrm{K}) = \mathrm{H}^i(\Omega^\bullet_{\mathrm{A}^\dagger} \otimes \mathrm{K})\,.$$

It is a consequence of the work of Berthelot [Ber97, Corollaire 3.2] that $\mathrm{H}^i_{\mathrm{MW}}(\bar{\mathrm{A}})$ is a finite-dimensional K-vector space.

The absolute Frobenius morphism $\varphi_a(x) = x^p$ of $\bar{\mathrm{A}}$ can be lifted, by Proposition 2, to a σ-linear morphism $\phi_a : \mathrm{A}^\dagger \to \mathrm{A}^\dagger$. Indeed, $\mathrm{A}^\dagger$ with the homomorphism

$$\mathrm{R} \xrightarrow{\sigma} \mathrm{R} \to \mathrm{A}^\dagger$$

is a lift of $\bar{\mathrm{A}}$ with the map

$$\kappa \xrightarrow{x^p} \kappa \to \bar{\mathrm{A}}$$

and φ_a induces a homomorphism between $\bar{\mathrm{A}}$ and this new twisted κ-algebra. The σ-linear ϕ_a induces a well defined σ-linear endomorphism φ_a of $\mathrm{H}^i_{\mathrm{MW}}(\bar{\mathrm{A}})$. On the other hand, if κ is a finite field with $q = p^r$ elements, then φ_a^r is already κ-linear and therefore induces an endomorphism $\varphi = \varphi_a^r$ of $\mathrm{H}^i_{\mathrm{MW}}(\bar{\mathrm{A}})$. By [Chi98, Theorem I.2.2] one knows the possible eigenvalues of φ on Monsky-Washnitzer cohomology. This result, modeled on Berthelot's proof [Ber97] of the finiteness of rigid cohomology, ultimately relies on the computation of the eigenvalues of Frobenius on crystalline cohomology by Katz and Messing [KM74], and therefore on Deligne's proof of the Weil conjectures [Del74].

Theorem 4. *The eigenvalues of the κ-linear Frobenius φ on $\mathrm{H}^1_{\mathrm{MW}}(\bar{\mathrm{A}})$ are Weil numbers of weights 1 and 2. In other words, they are algebraic integers and have absolute values q or $\sqrt{q}$ under any embedding into $\mathbb{C}$.*

1.3.3 Specialization and Locally Analytic Functions

One associates with a wcfg algebra $A^\dagger$ the K-algebra A, which is the completion $\mathcal{T}_n^\dagger$ of $A^\dagger \otimes K$ by the quotient norm induced from the Gauss norm, the maximal absolute value of the coefficients of the power series. This is easily seen to be an affinoid algebra. If $A^\dagger = \mathcal{T}_n^\dagger/I$, then $A = T_n/I$. We further associate with A the affinoid space $X = \mathrm{spm}(A)$. Letting $X_\kappa = \mathrm{Spec}(\bar{A})$ we have a specialization map

$$\mathrm{Sp}: X \to X_\kappa$$

which is defined as follows. Take a homomorphism $\psi : A \to L$, with L a finite extension of K. Then one checks by continuity that $A^\dagger$ maps to $\mathcal{O}_L$ and one associates with the kernel of ψ the kernel of its reduction mod π.

For our purposes, it will be convenient to consider the space X^{geo} of geometric points of X, which means K-linear homomorphisms $\psi : A \to \bar{K}$. This has a reduction map to the set of geometric points of X_κ obtained in the same way as above.

Definition 5. The inverse image of a geometric point $x : \mathrm{Spec}\,\bar{\kappa} \to X_\kappa$ under the reduction map will be called the **residue disc** of x, denoted $U_x \subset X^{\mathrm{geo}}$.

By Hensel's Lemma and the smoothness assumption on $\bar{A}$ it is easy to see that U_x is naturally isomorphic to the space of geometric points of a unit polydisc.

Definition 6. The K-algebra A_{loc} of **locally analytic functions** on X is defined as the space of all functions $f : X^{\mathrm{geo}} \to \bar{K}$ which satisfy the following two conditions:

(i) The function f is $\mathrm{Gal}(\bar{K}/K)$-equivariant in the sense that for any $\tau \in \mathrm{Gal}(\bar{K}/K)$ we have $f(\tau(x)) = \tau(f(x))$.

(ii) For each residue disc choose parameters z_1 to z_l identifying it with a unit polydisc over some finite field extension of K. Then restricted to such a residue disc f is defined by a power series in the z_i, which is therefore convergent on the open unit polydisc.

There is an obvious injection $A^\dagger \otimes K \subset A_{\mathrm{loc}}$. The algebra of our Coleman functions will lie in between these two K-algebras.

Another way of stating the equivariance condition for locally analytic functions, given the local expansion condition, is to say that given any $\tau \in \mathrm{Gal}(\bar{K}/K)$ transforming the geometric point x of X_κ to the geometric point y, we have that τ translates the local expansion of f near x to the local expansion near y by acting on the coefficients. This way one can similarly define the A_{loc}-module Ω^n_{loc} of **locally analytic** ***n*-forms** on X, the obvious differential $d : \Omega^{n-1}_{\mathrm{loc}} \to \Omega^n_{\mathrm{loc}}$, and an embedding, compatible with the differential, $\Omega^n_{A^\dagger} \otimes K \hookrightarrow \Omega^n_{\mathrm{loc}}$.

We define an action of the σ-semi-linear lift of the absolute Frobenius ϕ_a defined in the previous subsection on the spaces above. We first of all define an action on X^{geo} as follows. Suppose $\psi : A \to \bar{K} \in X^{\mathrm{geo}}$ is a K-linear homomorphism. Then

$$\phi_a(\psi) = \sigma^{-1} \circ \psi \circ \phi_a\,, \tag{1.2}$$

recall that we have extended σ to $\bar{\mathrm{K}}$. Note that this is indeed K-linear again. We can describe this action on points concretely as follows. Suppose $\mathrm{A} = \mathrm{T}_n/(f_1,\dots,f_k)$ and let $g_i = \phi_a(t_i)$ so that ϕ_a is given by the formula

$$\phi_a(\sum a_\mathrm{I} t^\mathrm{I}) = \sum \sigma(a_\mathrm{I})(g_1,\dots,g_n)^\mathrm{I} .$$

Suppose that $\mathbf{z} := (z_1,\dots,z_n) \in \mathrm{X}^{\mathrm{geo}}$, so that $f_i(\mathbf{z}) = 0$ for each i. Then we have

$$\phi_a(\mathbf{z}) = (\sigma^{-1} g_1(\mathbf{z}),\dots,\sigma^{-1} g_1(\mathbf{z})) .$$

Having defined ϕ_a on points we now define it on functions by

$$\phi_a(f)(x) = \sigma f(\phi_a(x)) \tag{1.3}$$

From (1.2) it is quite easy to see that for $f \in \mathrm{A}$ this is just the same as $\phi_a(f)$ as previously defined. We again have a compatible action on differential forms.

1.4 Coleman Theory

We define Coleman integration in a somewhat different way than the one Coleman uses, emphasizing a semi-linear condition and stressing the Frobenius equivariance.

Theorem 7. *Suppose that* K *is a finite extension of* $\mathbb{Q}_p$. *Then there exists a unique* K*-linear integration map*

$$\int : (\Omega^1_{\mathrm{A}^\dagger} \otimes \mathrm{K})^{\mathrm{d}=0} \to \mathrm{A}_{\mathrm{loc}}/\mathrm{K}$$

satisfying the following conditions:

(i) *The map* $\mathrm{d} \circ \int$ *is the canonical map* $(\Omega^1_{\mathrm{A}^\dagger} \otimes \mathrm{K})^{\mathrm{d}=0} \to \Omega^1_{\mathrm{loc}}$.
(ii) *The map* $\int \circ \mathrm{d}$ *is the canonical map* $\mathrm{A}^\dagger_\mathrm{K} \to \mathrm{A}_{\mathrm{loc}}/\mathrm{K}$.
(iii) *One has* $\phi_a \circ \int = \int \circ \phi_a$.

In addition, the map is independent of the choice of ϕ_a. *Finally, in the above Theorem, equivariance with respect to the semi-linear Frobenius lift* ϕ_a *may be replaced by equivariance with respect to a linear Frobenius lift* ϕ, *and yields the same theory.*

Proof. Since $\mathrm{H}^1_{\mathrm{MW}}(\bar{\mathrm{A}})$ is finite-dimensional, we may choose $\omega_1,\dots,\omega_n \in \Omega^1_{\mathrm{A}^\dagger} \otimes \mathrm{K}$ such that their images in $\mathrm{H}^1(\Omega^\bullet_{\mathrm{A}^\dagger} \otimes \mathrm{K})$ form a basis. If we are able to define the integrals $\mathrm{F}_{\omega_i} := \int \omega_i$ for all the ω_i's, then the second condition immediately tell us how to integrate any other form. Namely, write

$$\omega = \sum_{i=1}^n \alpha_i \omega_i + dg , \ \alpha_i \in \mathrm{K} , \ g \in \mathrm{A}^\dagger_\mathrm{K} . \tag{1.4}$$

Put all the forms above into a column vector ω. Then we have a matrix $\mathrm{M} \in \mathrm{M}_{n\times n}(\mathrm{K})$ such that

$$\phi_a \omega = \mathrm{M}\omega + \mathrm{d}\mathbf{g}\,, \tag{1.5}$$

where $\mathbf{g} \in (\mathrm{A}_{\mathrm{K}}^{\dagger})^n$. Conditions (ii) and (iii) in the theorem tell us that (1.5) implies the relation

$$\phi_a \mathrm{F}_\omega = \mathrm{M}\mathrm{F}_\omega + \mathbf{g} + \mathbf{c}\,, \tag{1.6}$$

where $\mathbf{c} \in \mathrm{K}^n$ is some vector of constants. We first would like to show that $\mathbf{c}$ may be assumed to vanish. For this we have the following key lemma.

Lemma 8. *The map* $\sigma - \mathrm{M} : \mathrm{K}^n \to \mathrm{K}^n$ *is bijective.*

Proof. We need to show that for any $\mathbf{d} \in \mathrm{K}^n$ there is a unique solution to the system of equations $\sigma(\mathbf{x}) = \mathrm{M}\mathbf{x} + \mathbf{d}$. By repeatedly applying σ to this equation we can obtain an equation for $\sigma^i(\mathbf{x})$

$$\sigma^i(\mathbf{x}) = \mathrm{M}_i \mathbf{x} + \mathbf{d}_i$$

where

$$\mathrm{M}_i = \sigma^{i-1}(\mathrm{M})\sigma^{i-2}(\mathrm{M}) \cdots \sigma(\mathrm{M})\mathrm{M}\,.$$

As $[\mathrm{K} : \mathbb{Q}_p] < \infty$ there exists some l such that σ^l is the identity on K and so we obtain the equation $\mathbf{x} = \mathrm{M}_l \mathbf{x} + \mathbf{d}_l$. Recalling that the cardinality of the residue field κ is p^r, we see that r divides l and that the matrix M_l is exactly the matrix of the l/r power of the linear Frobenius φ_a^r on $\mathrm{H}^1_{\mathrm{MW}}(\bar{\mathrm{A}}/\mathrm{K})$. It follows from Theorem 4 that the matrix $\mathrm{I} - \mathrm{M}_l$ is invertible. This shows that

$$\mathbf{x} = (\mathrm{I} - \mathrm{M}_l)^{-1} \mathbf{d}_l$$

is the unique possible solution to the equation. This shows that the map is injective, and since it is $\mathbb{Q}_p$-linear on a finite-dimensional $\mathbb{Q}_p$-vector space it is also bijective (one can also show directly that $\mathbf{x}$ above is indeed a solution). □

Remark 9. In computational applications, it is important that the modified equation $\mathbf{x} = \mathrm{M}_l \mathbf{x} + \mathbf{d}_l$ can be computed efficiently in $\mathrm{O}(\log(l))$ steps, see [LL03, LL06].

Since ϕ_a acts as σ on constant functions we immediately get from the lemma that by changing the constants in F_ω we may assume that $\mathbf{c} = 0$ in (1.6).

We claim that now the vector of functions F_ω is completely determined. Indeed, since $\mathrm{d}\mathrm{F}_\omega = \omega$ by condition (i), we may determine F_ω on any residue disc up to a vector of constants by term by term integration of a local expansion of ω. It is therefore sufficient to determine it on a single point on each residue disc. So let x be such a point. Substituting x in (1.6) and recalling the action of ϕ_a on functions (1.3) we find

$$\sigma(\mathrm{F}_\omega(\phi_a(x))) = \mathrm{M}\mathrm{F}_\omega(x) + \mathbf{g}(x)\,.$$

Since $\phi_a(x)$ is in the same residue disc as x the difference

$$\mathbf{e} := \mathrm{F}_\omega(\phi_a(x)) - \mathrm{F}_\omega(x) = \int_x^{\phi_a(x)} \omega$$

is computable from ω alone. Substituting in the previous equation we find

$$\sigma(\mathrm{F}_\omega(x)) + \sigma(\mathbf{e}) = \mathrm{MF}_\omega(x) + \mathbf{g}(x)\,,$$

and rearranging we find an equation for $\mathrm{F}_\omega(x)$ that may be solved using Lemma 8 for some finite extension of K where x is defined. This shows uniqueness and gives a method for computing the integration map.

It is fairly easy to see that the method above indeed gives an integration map satisfying all the required properties. Note that by uniqueness the integration map is independent of the choice of basis ω.

When using a linear Frobenius lift ϕ, one relies instead on the fact that any point defined over a finite field will be fixed by an appropriate power of φ. Considering equivariance with respect to that power, mapping the residue disc of the point x back to itself, we can determine the integral at x by a similar method. Thus, if ϕ is a power of ϕ_a and is linear, equivariance for ϕ_a implies one with respect for ϕ, and since the theory is determined uniquely by equivariance the converse is also true.

It remains to show that it is independent of the choice of ϕ_a. By the above, it is easy to see that it suffices to do this with respect to the equivariance property with respect to a linear Frobenius. So suppose we are given two linear Frobenii ϕ and ϕ' and that we have set up the theory for ϕ. We want to show that it also satisfies equivariance with respect to ϕ'. Let ω be a closed form and suppose we have chosen the constant in Coleman integration so that $\mathrm{F}_{\phi(\omega)} = \phi\mathrm{F}_\omega$. By Proposition 2 we have $h \in \mathrm{A}_\mathrm{K}^\dagger$ such that $\phi'(\omega) - \phi(\omega) = \mathrm{d}h$. We now compute

$$\begin{aligned}\int \phi'(\omega) - \phi' \int \omega &= \int \phi'(\omega) - \phi' \int \omega - (\int \phi(\omega) - \phi \int \omega)\\ &= \int (\phi'(\omega) - \phi(\omega)) - (\phi' \int \omega - \phi \int \omega)\\ &= h - (\phi' \int \omega - \phi \int \omega)\end{aligned}$$

and substituting at a point x we get

$$h(x) - \int_{\phi(x)}^{\phi'(x)} \omega\,.$$

We need to show that this is a constant independent of x. To do this, consider the subspace of $\mathrm{X}\times\mathrm{X}$

$$\mathrm{D} := \{(x, y) \in \mathrm{X}\times\mathrm{X}\ ;\ \mathrm{Sp}(x) = \mathrm{Sp}(y)\}\,,$$

in Berthelot's language this is the tube of the points reducing to the diagonal. This is a rigid analytic space and Coleman shows [Col85b, Proposition 1.2] that there exists a rigid analytic function H on D such that $\mathrm{dH} = \pi_y^*\omega - \pi_x^*\omega$, where π_x and π_y are the projections on the two coordinates. The pullback to the diagonal of H is thus constant, and may be assumed 0. It follows that $\mathrm{H}(x, y) = \int_x^y \omega$. The two lifts ϕ and

ϕ', having the same reduction, define a map $\Phi = (\phi, \phi') : \mathrm{X} \to \mathrm{T}$ and $\phi = \pi_x \circ \Phi$, $\phi' = \pi_y \circ \Phi$ on X. Therefore, we may take $h(x) = \mathrm{H}(\Phi(x)) = \int_{\phi(x)}^{\phi'(x)} \omega$. □

Example 10. Let us demonstrate the above on the example from the introduction. Our dagger algebra is

$$\mathrm{A}^\dagger = \mathcal{T}_2^\dagger / (t_1 t_2 - 1)$$

over $\mathbb{Z}_p$. Setting $t = t_1$ and $t^{-1} = t_2$ we have

$$\mathrm{A}^\dagger \cong \left\{ \sum_{i \in \mathbb{Z}} a_i t^i \ ; \ a_i \in \mathbb{Z}_p \ , \ \lim_{|i| \to \infty} |a_i| r^{|i|} = 0 \text{ for some } r > 1 \right\}$$

and the module of 1-forms is $\Omega^1_{\mathrm{A}^\dagger} = \mathrm{A}^\dagger \mathrm{d}t$. The associated Monsky-Washnitzer cohomology $\mathrm{H}^1(\Omega^\bullet_{\mathrm{A}^\dagger} \otimes \mathrm{K})$ is clearly one-dimensional, generated by the form $\omega = \frac{\mathrm{d}t}{t}$. Since it is clear how to integrate exact forms, it suffices to integrate ω. The integral is to be a function on the space associated with the algebra $\mathrm{A} = \mathrm{T}_2/(t_1 t_2 - 1)$ which is just $\{z \in \bar{\mathbb{Q}}_p \ ; \ |z| = 1\}$, see Sect. 1.3.1. Finally, we may take the lift of Frobenius ϕ_a such that $\phi_a(t) = t^p$.

For the computation of the Coleman integral F_ω of ω we notice that, as in the introduction, $\phi_a \omega = p\omega$. Thus, we may pick our integral so that when evaluating at a point x we have $\sigma(\mathrm{F}_\omega(\phi_a(x))) = p\mathrm{F}_\omega(x)$, where here $\phi_a(x) = \sigma^{-1}(x^p)$. We can now either proceed with a general x as in the proof of the theorem or, as in the introduction, consider an x which is a root of unity of order prime to p. In this case it is easy to see that $\phi_a(x) = x$ and so one finds the relation $\sigma(c) = pc$ for $c = \mathrm{F}_\omega(x)$. Now, if $\sigma^l(c) = c$ we find $c = p^l c$ so $c = 0$. Thus, we again discover that our integral vanishes at all these roots of unity.

Remark 11. (1) Note that we have used the semi-linear approach here, whereas in the introduction we used the linear approach.

(2) It is interesting to note that the equation $\sigma(c) = pc$ yields $c = 0$ even without assuming a finite residue field, because it implies that $\sigma(c)$, hence c, are divisible by p and iterating we find that c is divisible by any power of p hence is 0. This suggests an interesting alternative to Coleman integration, applicable when all slopes are positive, using slopes rather than weights, that works without assuming finite residue fields. It also works for example for polylogarithms. We plan to come back to this method in future work.

To end this section, let us sketch how one may define iterated integrals using an extension of the method above. Note that this differs from the method of [Col82] and [CdS88] and is again geared towards computational applications. A similar method to the one sketched above is worked out (in progress) by Balakrishnan.

As explained in the introduction, prior to the introduction of isocrystals into Coleman integration, iterated integrals were only defined on one-dimensional spaces. This restriction means that any form is closed and can therefore be integrated. Let us explain then how one can define integrals $\int(\omega \times \int \eta)$ for ω and η in $\Omega^1_{\mathrm{A}^\dagger} \otimes \mathrm{K}$, where the space X^{geo} is one-dimensional. More complicated iterated integrals are derived in exactly the same manner.

We begin by observing that when $\eta = \mathrm{d}f$ is exact, then the above integral is just $\int f\omega$ which has already been defined. To proceed, we will impose an additional condition, which is the integration by parts formula

$$\int\left(\omega \times \int \eta\right) + \int\left(\eta \times \int \omega\right) = \left(\int \omega\right) \times \left(\int \eta\right) + \mathrm{C}\,, \tag{1.7}$$

see Remark 33 for a justification of this formula. Using this formula and our knowledge of $\int \omega$ and $\int \eta$ we can also compute $\int(\omega \times \int \eta)$ when ω is exact.

Consider again a basis $\omega_1, \ldots, \omega_n \in \Omega^1_{\mathrm{A}^\dagger} \otimes \mathrm{K}$. Decomposing both ω and η as in (1.4) and using the above it is sufficient to compute the integrals $\int(\omega_i \times \int \omega_j)$ for all pairs (i, j). If M is the matrix satisfying (1.5), then $\mathrm{M}\otimes\mathrm{M}$ is the matrix describing the action of ϕ_a on the basis $\{\omega_i \otimes \omega_j\}$ of $\mathrm{H}^1(\Omega^\bullet_{\mathrm{A}^\dagger} \otimes \mathrm{K}) \otimes \mathrm{H}^1(\Omega^\bullet_{\mathrm{A}^\dagger} \otimes \mathrm{K})$. Eigenvalues of (appropriately linearized) $\mathrm{M} \otimes \mathrm{M}$ are just products of eigenvalues of M (again linearized), and they are again Weil numbers of positive weight. Thus, the same arguments used for proving Theorem 7 may be used to obtain iterated integrals.

1.5 Coleman Integration via Isocrystals

In this section we explain the approach to Coleman integration using isocrystals introduced in [Bes02]. We comment that the approach there works globally as well, but we only explain it in the affine, more precisely, the affinoid situation, in which we described Coleman's work.

The main idea is that the iterated integral

$$\int(\omega_n \int(\omega_{n-1} \int(\cdots \int \omega_1)\cdots))$$

is the y_n coordinate of a solution of the system of differential equations

$$\mathrm{d}y_0 = 0,\ \mathrm{d}y_1 = \omega_1 y_0, \ldots \mathrm{d}y_n = \omega_n y_{n-1} \tag{1.8}$$

or, in vector notation

$$\mathbf{dy} = \Omega\mathbf{y}\ ,\ \Omega = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \omega_1 & 0 & 0 & \cdots & 0 \\ 0 & \omega_2 & 0 & \cdots & 0 \\ 0 & 0 & \omega_3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \omega_n & 0 \end{pmatrix},$$

with $y_0 = 1$. This is just a unipotent differential equation. The Frobenius equivariance condition can now be interpreted as saying that we have a system $\mathbf{y}$ of *good* local solutions for this equation, in such a way that $\phi\mathbf{y}$ is a *good* system of solutions for the equation $d\mathbf{y} = \phi(\Omega)\mathbf{y}$. This, as well as the independence of the choice of the lift of Frobenius, turns out to be very nicely explained by the Tannakian formalism of unipotent isocrystals.

1.5.1 The Tannakian Theory of Unipotent Isocrystals

We assume familiarity with the basic theory of neutral Tannakian categories. The standard reference is [DM82].

Definition 12. A unipotent isocrystal on $\bar{A}$ is an $A_K^\dagger$-module M together with an integrable connection

$$\nabla : M \to M \otimes_{A_K^\dagger} \Omega^1_{A^\dagger} \otimes K$$

which is an iterated extension of trivial connections, where trivial means the object

$$\mathbb{1} := (A_K^\dagger, d)\,.$$

We first observe that the module M is in fact free, because it is an iterated extension of $A_K^\dagger$, which is obviously split.

A morphism of unipotent isocrystals is just a map of $A_K^\dagger$-modules which is horizontal, meaning that it commutes with the connection.

We denote the category of unipotent isocrystals on $\bar{A}$ by $\mathcal{U}n(\bar{A})$. It is a basic fact of the theory [Ber96, (2.3.6) and following paragraph] that, as the notation suggests, the category depends only on $\bar{A}$ and not on the particular choice of lift $A^\dagger$.

Example 13. Let $M \in \mathcal{U}n(\bar{A})$ have rank 2. Then it sits in a short exact sequence

$$0 \to \mathbb{1} \to M \to \mathbb{1} \to 0$$

which is non-canonically split. It is thus isomorphic to the object having underlying module ${A_K^\dagger}^2$ and connection

$$\nabla = d - \begin{pmatrix} 0 & 0 \\ \omega & 0 \end{pmatrix}.$$

By associating with M the class of ω in $H^1(\Omega^\bullet_{A^\dagger} \otimes K) = H^1_{MW}(\bar{A}/K)$ it is easy to check that one obtains a bijection

$$\mathrm{Ext}^1_{\mathcal{U}n(\bar{A})}(\mathbb{1}, \mathbb{1}) \cong H^1_{MW}(\bar{A}/K)\,.$$

Theorem 14. *The category $\mathcal{U}n(\bar{A})$ is a rigid abelian tensor category.*

To see this, assuming the corresponding result [Cre92, p. 438] for the category of all overconvergent isocrystals, one follows the proof of [CLS99, 2.3.2] which discusses

F-isocrystals but the proof is word for word the same, to show that $\mathcal{U}n(\bar{A})$ is closed under sub and quotient objects, tensor products and duals in the category of all overconvergent isocrystals.

To make $\mathcal{U}n(\bar{A})$ into a neutral Tannakian category what is missing is a fiber functor, i.e., an exact faithful functor into K-vector spaces preserving the tensor structure. We can associate such a functor with each κ-rational point as follows.

Definition 15. Let $x \in X_\kappa(\kappa)$ be a rational point. We associate with it the functor

$$\omega_x : \mathcal{U}n(\bar{A}) \to \mathrm{Vec}_K \,,\ \omega_x(M, \nabla) = \{v \in M(U_x), \nabla(v) = 0\}$$

where U_x is the residue disc of x and $M(U_x)$ consists of the sections of M on the rigid analytic space U_x.

The fact that ω_x is indeed a fiber functor is quite standard. The key point to observe is the following: a precondition for a functor such as ω_x to be a fiber functor is that the dimension of $\omega_x(M, \nabla)$ equals the rank of M. For a general differential equation there is no reason why this should be the case and one introduces a condition of overconvergence, which among other things guarantees this. A unipotent isocrystal is always overconvergent. It is, however, easy to see without knowing this that indeed $\omega_x(M, \nabla)$ has the right dimension for a unipotent ∇ simply because finding horizontal sections amounts to iterated integration and one can integrate power series converging on the unit open polydisc to power series with the same property as the algebra of power series converging on the open polydisc of radius 1 has trivial de Rham cohomology.

In the general theory of overconvergent isocrystals one can realize the functor ω_x as simply the pullback x^* to an isocrystal on $\mathrm{Spec}(\kappa)$, see the remark just before Lemma 1.8 in [Cre92].

The general theory of Tannakian categories [DM82] tells us that the category $\mathcal{U}n(\bar{A})$ together with the fiber functor ω_x determine a fundamental group

$$G = G_x = \pi_1(\mathcal{U}n(\bar{A}), \omega_x)$$

which is an affine proalgebraic group, and an equivalence of categories between $\mathcal{U}n(\bar{A})$ and the category of finite dimensional K-algebraic representations of G. We recall that G represents the functor that sends a K-algebra F to the group

$$\begin{aligned} \mathrm{Aut}^{\otimes}(\omega_x \otimes F) := \{M \in \mathcal{U}n(\bar{A}) \to (\alpha_M &: \omega_x(M) \otimes F \to \omega_x(M) \otimes F)\,, \\ &\alpha_M \text{ natural isomorphism and} \\ &\alpha_{M\otimes N} = \alpha_M \otimes \alpha_N\,,\ \alpha_{\mathbb{1}} = \mathrm{id}\}\,. \end{aligned} \tag{1.9}$$

The description of the Lie algebra $\mathfrak{g}$ of G is well known. Consider the algebra $K[\varepsilon]$ of dual numbers where $\varepsilon^2 = 0$. Then $\mathfrak{g}$ is just the tangent space to G at the origin and is thus given by

$$\mathfrak{g} = \mathrm{Ker}(G(K[\varepsilon]) \to G(K))\,.$$

In terms of the description (1.9) to G an element $\alpha \in \mathfrak{g}$ sends $\mathrm{M} \in \mathcal{U}n(\bar{\mathrm{A}})$ to

$$\alpha_{\mathrm{M}} = \mathrm{id} + \epsilon\beta_{\mathrm{M}}\ , \ \beta_{\mathrm{M}} \in \mathrm{End}(\omega_x(\mathrm{M}))\ .$$

Such an element is automatically invertible. The conditions on the α_{M} easily translate to conditions on the β_{M} and we obtain

$$\begin{aligned}\mathfrak{g} = \{(\mathrm{M} \to &\beta_{\mathrm{M}} \in \mathrm{End}(\omega_x(\mathrm{M})))\ ,\\ &\beta_{\mathrm{M}} \text{natural}, \beta_{\mathbb{1}} = 0\ ,\\ &\beta_{(\mathrm{M}\otimes\mathrm{N})} = \beta_{\mathrm{M}} \otimes \mathrm{id}_{\omega_x(\mathrm{N})} + \mathrm{id}_{\omega_x(\mathrm{M})} \otimes \beta_{\mathrm{N}}\}\ .\end{aligned}$$

The Lie bracket is given in this representation by the commutator.

Lemma 16. *The elements of* G *are unipotent and the elements of* $\mathfrak{g}$ *are nilpotent in the sense that for every* $\mathrm{M} \in \mathcal{U}n(\bar{\mathrm{A}})$ *the corresponding* α_{M} *is unipotent and the corresponding* β_{M} *is nilpotent.*

Proof. Choose a flag $\mathrm{M} = \mathrm{M}_0 \supset \mathrm{M}_1 \supset \cdots$ with trivial consecutive quotients. Then the naturality of α and β implies that with respect to a basis compatible with the associated flag on $\omega_x(\mathrm{M})$ the matrices of α_{M} and β_{M} are upper triangular, with 1 respectively 0 on the diagonal. □

It follows that there is well-defined algebraic exponential map $\exp : \mathfrak{g} \to \mathrm{G(K)}$ sending β_{M} to $\exp(\beta_{\mathrm{M}})$ given by the usual power series. Tensoring with an arbitrary K-algebra we can easily see, using the fact that K has characteristic 0, that exp induces an isomorphism of affine schemes from the affine space associated with $\mathfrak{g}$ to G. The product structure on G translates in $\mathfrak{g}$ to the product given by the Baker-Campbell-Hausdorff formula. It is further clear that the following holds.

Proposition 17. *The reverse operations of differentiation and exponentiation give an equivalence between the categories of algebraic representations of* G *and continuous Lie algebra representations of* $\mathfrak{g}$.

Here, continuous representation means with respect to the discrete topology on the representation space and with respect to the inverse limit topology on $\mathfrak{g}$.

1.5.2 The Frobenius Invariant Path

Consider now two κ-rational points $x, z \in \mathrm{X}_\kappa$. Then we have a similarly defined space of paths $\mathrm{P}_{x,z} := \mathrm{Iso}^{\otimes}(\omega_x, \omega_z)$ (same functoriality and tensor conditions) which is clearly a right principal homogeneous space for G_x (and a left one for G_z, note that in [Bes02] the directions are wrong). In concrete terms, the path space $\mathrm{P}_{x,z}$ consists of rules for "analytic continuation" for each unipotent differential equation (M, ∇), of a solution, i.e., horizontal section, $\mathbf{y}_x \in \mathrm{M}(\mathrm{U}_x)^{\nabla=0}$ to $\mathbf{y}_z \in \mathrm{M}(\mathrm{U}_z)^{\nabla=0}$ compatible with morphisms and tensor products. Composition of paths