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Probability in Complex Physical Systems

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Editors

Probability in Complex Physical Systems

In Honour of Erwin Bolthausen and Jürgen Gärtner

Springer

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Preface

Probabilistic approaches have played a prominent role in the study of complex physical systems for more than 30 years. Two outstanding protagonists of this approach are Jürgen Gärtner and Erwin Bolthausen, to whom this volume is dedicated. Each of them was honored with a workshop in 2010; these took place at Technische Universität Berlin, where they both worked for decades. The conferences were devoted to the most important aspects of their interests: 'Random media' and 'Probabilistic techniques in complex physical systems'. They were organized by the DFG Research Unit FOR718 *Analysis and Stochastics in Complex Physical Systems* on the occasion of Jürgen's 60th birthday and Erwin's 65th birthday.

Jürgen and Erwin have been recognized for decades as outstanding experts in the probabilistic treatment, spiced with a dash of analysis, of problems in statistical mechanics and related fields. Their high esteem and profound impact are reflected by their great number of students and collaborators and by their large number of invitations to conferences, editorships, etc. over the years.

Erwin started his career with various distributional limit results of central limit and martingale type, but soon turned to problems coming from large-deviation analysis, like Laplace approximations and the maximum entropy principle. One of the main types of problems that accompanied his career for decades are intricate questions about the extremal behavior of the volume of the path of a random walk or a Wiener sausage and of the intersection of two independent such objects. Here he has derived a number of striking and deep results over the years. Another core area of his research, which is closely related, is the description of paths under the influence of a self-attracting or self-repellent force, partially motivated by the polaron problem. In particular, Erwin derived several fundamental properties of polymers with various kinds of interactions. His results also had a strong influence on the understanding of interface models with gradient-type interactions. Some of his favorite subjects in recent years have been random walks in random environments, and spin glasses and the little-understood phenomenon of ultrametricity.

Jürgen was educated within the Russian school in the 1970s, pioneering the application of large deviation analysis to various models in statistical mechanics.

One of the fundamental tools, the Gärtner–Ellis theorem, is a side-result of his thesis. Later he built up a theory of large deviations for projective limits. Also his contributions to the McKean–Vlasov equation remain a vital element of the theory. Over the last two decades, he has been one of the most active researchers on the parabolic Anderson model, the Cauchy problem for the heat equation with random potential.

Many of the above results were derived in close collaboration with students, colleagues, and friends, many of whom also presented talks on the occasion of the two 2010 workshops. The present volume collects 20 research and review papers by participants in the fields in which Jürgen and Erwin are best known for their contributions. Most of these papers are, in some way or another, influenced by Jürgen's and Erwin's work, and all of them present state-of-the-art results in topics that accompanied the two for decades and received significant impacts from them over the years. All papers have been peer-refereed according to highest standards.

Almost half of the contributions to this volume are devoted to the parabolic Anderson model, one of the most active research fields of Jürgen. For more than 20 years, Jürgen has formed and extended this subject like nobody else. Jürgen's co-authors and students and their students and colleagues give an impressive account on some of the latest developments for the parabolic Anderson model, among which there are results on the long-time behavior for various time-dependent and time-independent potentials, and novel aspects like several moving catalysts, acceleration/deceleration, and front propagation.

Another main topic covered by this volume is random polymers interacting with random and nonrandom environment and their critical behavior, a topic that received much attention from Erwin and his coauthors. Furthermore, special aspects of branching processes and interacting measure-valued processes are considered, topics that Jürgen studied many years ago. Finally, this volume offers a choice of results on various models that Erwin worked on or was interested in for many years, like Parisi's formulas for the generalized random energy model, metastability, hydrodynamic limits for gradient models and dimers.

In total, the collection of 20 papers in this volume presents important contributions to and surveys on research areas that are of current interest and have been strongly influenced by these two eminent mathematicians. It is not too much to say that these fields have benefited tremendously from their work.

Berlin
June 2011

Jean-Dominique Deuschel
Barbara Gentz
Wolfgang König
Max von Renesse
Michael Scheutzow
Uwe Schmock

Workshop on

Probabilistic Techniques in Statistical Mechanics
Celebrating the 65th Birthday of Erwin Bolthausen

Organized by:
Jean-Dominique Deuschel, Wolfgang König, Max von Renesse, Michael Scheutzow and the DFG Research Unit FOR718 *Analysis and Stochastics in Complex Physical Systems*

Venue:
Technische Universität Berlin, Institute for Mathematics, Str. des 17. Juni 136, 10623 Berlin, Germany, Room MA043

Period:
October 14–16, 2010

Speaker/Title:

Gérard Ben Arous (Courant Institute New York)
Extreme gaps in the spectrum of random matrices

Michiel van den Berg (University of Bristol)
Minimization of Dirichlet eigenvalues with geometric constraints

Anton Bovier (University of Bonn)
Almost sure ageing

David Brydges (University of British Columbia, Vancouver)
The strong interaction limit of continuous-time weakly self-avoiding walk

Francesco Caravenna (University of Padova)
The weak coupling limit of disordered copolymer models

Amir Dembo (Stanford University)
Low temperature expansion for matrix models

Tadahisa Funaki (University of Tokyo)
Hydrodynamic limit for two- and three-dimensional Young diagrams

Giambattista Giacomin (University Paris Diderot – Paris 7)
Impurities, defects and critical phenomena

Ilya Goldsheid (Queen Mary College)
Simple random walks in one-dimensional random environment: limiting behaviour in the sub-diffusive regimes

Alice Guionnet (ENS Lyon)
Potts model on random graphs

Frank den Hollander (Leiden University and EURANDOM)
Variational approach to copolymers near linear interfaces

Dmitry Ioffe (Technion)
Stretched polymers in random environment

Nicola Kistler (University of Bonn)
Traveling waves through the spin glass

Alain-Sol Sznitman (ETH Zurich)
Random walks and random interlacements

Béatrice de Tilière (University Pierre et Marie Curie, Paris)
The critical Z-invariant Ising model via dimers: local statistics and combinatorics

Fabio Toninelli (ENS Lyon)
On the zero temperature dynamics of the three-dimensional Ising model
(joint work with P. Caputo, F. Martinelli and F. Simenhaus)

Bálint Tóth (Technical University Budapest)
Superdiffusive lower bound for self-repelling processes in the critical dimension

Yvan Velenik (University of Geneva)
A new approach to the Aizenman–Higuchi theorem

Wendelin Werner (University Paris-Sud 11 in Orsay and ENS Paris)
Self-interacting random walks with finite spatial interaction range

Ofer Zeitouni (University of Minnesota and Weizmann Institute of Science)
Fluctuations of the (discrete) Gaussian free field, and branching random walks

Workshop on

Random Media
Celebrating the 60th Birthday of Jürgen Gärtner

Organized by:
Jean-Dominique Deuschel, Wolfgang König, Max von Renesse, Michael Scheutzow and the DFG Research Unit FOR718 *Analysis and Stochastics in Complex Physical Systems*

Venue:
Technische Universität Berlin, Institute for Mathematics, Str. des 17. Juni 136, 10623 Berlin, Germany, Room MA043

Period:
April 8–10, 2010

Speaker/Title:

Gérard Ben Arous (Courant Institute New York)
Random matrices and Morse theory in many dimensions: the case of mixtures of spherical spin glasses and the (possibility of) full replica symmetry breaking

Marek Biskup (UCLA and University of South Bohemia)
Eigenvalue order statistics for the random Schrödinger operator

Erwin Bolthausen (University of Zurich)
Kac-type interactions in a one-dimensional system with a continuous symmetry

Anton Bovier (University of Bonn)
Metastability in Ginzburg–Landau type stochastic differential equations

Donald Dawson (University of Ottawa)
McKean–Vlasov mutation–selection dynamics

Klaus Fleischmann (Berlin)
Recent properties of states of super-α-stable motion with branching of index $1 + \beta$

Mark Freidlin (University of Maryland)
Perturbation theory for systems with many invariant measures

Markus Heydenreich (Free University of Amsterdam)
Random walk on high-dimensional incipient infinite cluster

Frank den Hollander (Leiden University and EURANDOM)
The mathematical work of Jürgen Gärtner

Grégory Maillard (University of Provence, Marseille)
Parabolic Anderson model with a finite number of moving catalysts

Peter Mörters (University of Bath)
Geometric approaches to intermittency in the parabolic Anderson model

Tom Mountford (EPF Lausanne)
Parabolic Anderson model with voter model noise

Alejandro Ramírez (Pontifical Catholic University of Chile)
Ballisticity conditions for random walk in random environment

Vladas Sidoravicius (CWI Amsterdam)
On random growth and random walks in dynamically evolving random environment

Rongfeng Sun (University of Singapore)
Annealed versus quenched asymptotics for the parabolic Anderson model with moving catalysts or traps

Alain-Sol Sznitman (ETH Zurich)
Disconnecting discrete cylinders

Contents

Laudatio: The Mathematical Work of Jürgen Gärtner

Frank den Hollander

Abstract Over the past 35 years, Jürgen Gärtner has made seminal contributions to probability theory and analysis. In this brief laudatio, I describe what I consider to be his five most important lines of research: (1) Gärtner-Ellis large deviation principle; (2) Kolmogorov–Petrovskii–Piskunov equation; (3) Dawson–Gärtner projective limit large deviation principle; (4) McKean–Vlasov equation; (5) Parabolic Anderson model. Each of these lines is placed in its proper context, but no attempt is made to fully trace the literature. What characterizes the papers of Jürgen is that they all deal with hard fundamental problems requiring a delicate combination of probabilistic and analytic techniques. A red thread through his work is the symbiosis of large deviation theory and potential theory, which he masterfully combines to reach powerful and elegant solutions.

1 Gärtner–Ellis Large Deviation Principle

In 1977, Jürgen proved what is nowadays considered to be the *most general form of Cramér's theorem in large deviation theory* [21, 22]. This work, which was suggested to him by Mark Freidlin, took place while the architectural foundations of large deviation theory were being laid. As such, Jürgen's theorem belongs to the very heart of the field, as developed in the 1970s by Freidlin and Wentzell [20] and Donsker and Varadhan [16]. In 1984, the assumptions under which Jürgen had proved his theorem were weakened by Richard Ellis [17].

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Theorem 1.1. *Given a sequence* $(X_n)_{n\in\mathbb{N}}$ *of random variables taking values in* $\mathbb{R}^d$*, let*

$$\phi_n(t) = E(e^{\langle t, X_n\rangle}), \qquad t \in \mathbb{R}^d,$$

denote their moment generating functions (where $\langle\cdot,\cdot\rangle$ *is the standard inner product on* $\mathbb{R}^d$*). Suppose that*

$$\lim_{n\to\infty} \frac{1}{n} \log \phi_n(nt) = \Phi(t) \quad \text{exists for all } t \in \mathbb{R}^d,$$

and is everywhere finite and differentiable. Then the family $(P_n)_{n\in\mathbb{N}}$ *with* $P_n(\cdot) = P(X_n \in \cdot)$ *satisfies the large deviation principle (LDP) on* $\mathbb{R}^d$ *with rate function* $I\colon \mathbb{R}^d \to [0,\infty]$ *given by the Legendre transform*

$$I(x) = \sup_{t\in\mathbb{R}^d} [\langle t, x\rangle - \Phi(t)], \qquad x \in \mathbb{R}^d.$$

Theorem 1.1 says that

$$\liminf_{n\to\infty} \frac{1}{n} \log P(X_n \in O) \geq -\inf_{x\in O} I(x) \qquad \forall\ O \subset \mathbb{R}^d \text{ open},$$
$$\limsup_{n\to\infty} \frac{1}{n} \log P(X_n \in C) \leq -\inf_{x\in C} I(x) \qquad \forall\ C \subset \mathbb{R}^d \text{ closed},$$

which informally reads as

$$P(X_n \approx x) \approx e^{-nI(x)} \qquad \forall\ x \in \mathbb{R}^d, \qquad n \to \infty.$$

Hence Theorem 1.1 gives full control of the deviations of the random variable X_n away from its typical values for large n.

For the special case where

$$X_n = \frac{1}{n}(Y_1 + \cdots + Y_n), \quad n \in \mathbb{N}, \quad (Y_i)_{i\in\mathbb{N}} \text{ i.i.d.},$$

we have $\phi_n(nt) = [\phi(t)]^n$ with $\phi(t) = E(e^{\langle t, Y_1\rangle})$ the moment-generating function of Y_1, and Theorem 1.1 reduces to Cramér's theorem for the empirical mean of i.i.d. random sequences. However, in its full generality, the theorem is applicable far beyond the i.i.d. setting, including Markov sequences, Gibbs random fields, and random processes in random media.

Over the years, the Gärtner–Ellis LDP has become *one of the workhorses of large deviation theory*. Due to its simplicity, generality, and flexibility, it appears in every textbook on large deviations. It has been and is being applied in a great many different contexts. For refinements as well as additional background, see the monographs by Varadhan [39] and Dembo and Zeitouni [15].

2 Kolmogorov–Petrovskii–Piskunov Equation

In 1982, Jürgen wrote a seminal paper [23] on the famous semi-linear diffusion equation introduced in 1937 by Kolmogorov, Petrovskii, and Piskunov [32]:

$$\frac{\partial u}{\partial t}(x,t) = \frac{1}{2}\Delta u(x,t) + f(u(x,t)), \quad x \in \mathbb{R}^d,\ t \geq 0.$$

Here, $f\colon [0,1] \to [0,\infty)$ is assumed to be once continuously differentiable with $f(0) = f(1) = 0$ and $0 < f(u)/u \leq f'(0)$ for $u \in (0,1)$. The initial condition is taken to be

$$u(x,0) = g(x), \qquad x \in \mathbb{R}^d,$$

for some appropriate $g\colon \mathbb{R}^d \to [0,1]$ that is strictly positive near $x = 0$ and tends to zero rapidly at infinity. The KPP equation describes a system of particles that diffuse and that split into two at a rate that depends on their local density via the function f, both in the continuum limit of many particles with small mass. This is why it is referred to as a *reaction-diffusion equation*. The KPP equation plays a key role in the understanding of wave front propagation phenomena, occurring, e.g., in combustion processes.

Theorem 2.1. *Abbreviate* $v^* = [2f'(0)]^{1/2}$*, and define*

$$h(z) = \sup_{u\in(0,z]} [f'(0) - f(u)/u], \qquad z \in (0,1].$$

Suppose that

$$\int_0^1 h(z)\, z^{-1} \log^2(1/z)\, \mathrm{d}z < \infty,$$

and that $g(x) = \bar{g}(\|x\|)$ *with*

$$\limsup_{r\to\infty} r^{-1/2} \log[e^{v^* r}\, \bar{g}(r)] < \infty.$$

Then, for every $\epsilon \in (0, \frac{1}{2})$*, there exists a* $\rho(\epsilon) \in (0,\infty)$ *such that, for all* t *sufficiently large,*

$$\begin{aligned}&\{x \in \mathbb{R}^d\colon \epsilon < u(x,t) < 1-\epsilon\}\\ &\qquad \subset \{x \in \mathbb{R}^d\colon m(t) - \rho(\epsilon) < \|x\| < m(t) + \rho(\epsilon)\},\end{aligned}$$

where

$$m(t) = v^*\, t - \frac{d+2}{2v^*}\, \log t + \frac{1}{v^*} \log \int_0^\infty r^{(d+1)/2}\, e^{v^* r}\, \bar{g}(r)\, \mathrm{d}r.$$

The first condition in Theorem 2.1 controls the behavior of f near zero, and the second condition controls the behavior of g near infinity. The result identifies the location of the *expanding wave front* around which u drops from $u \approx 1$ to $u \approx 0$: this wave front is an annulus of finite width around the surface of the ball of radius $m(t)$. The leading term in $m(t)$ says that the speed of the wave front is v^*, the correction terms in $m(t)$ are computed up to and including order 1.

Earlier work by McKean [33, 34], Aronson and Weinberger [1], Bramson [5] and Uchiyama [38] had fallen short of identifying the constant in $m(t)$ and had required more severe restrictions on g, such as compact support. Part of this work was for $d = 1$ only.

The proof of Theorem 2.1 centers around a delicate estimate of the first-exit time distribution for a Brownian motion in a time-dependent domain. In later work, Jürgen extended Theorem 2.1 to a much broader class of reaction-diffusion equations. This work was subsequently picked up and pushed further by others. See Freidlin [18] for a survey.

3 Dawson–Gärtner Projective Limit Large Deviation Principle

In 1987, Jürgen and Don Dawson proved a theorem that considers a *nested sequence* of LDPs and obtains from this a new LDP via a *projective limit* [8]. This theorem is a powerful tool, because it allows to first derive an LDP in a simple setting (e.g., on a finite or a compact space) and then draw from that an LDP in a more difficult setting (e.g., on an infinite or a noncompact space). Over the years, also the Dawson–Gärtner projective limit LDP has become *one of the workhorses of large deviation theory*.

Theorem 3.1. *Let $(P_n)_{n\in\mathbb{N}}$ be a family of probability measures on a Hausdorff topological space χ. Let $(\pi^N)_{N\in\mathbb{N}}$ be a nested family of projections acting on χ, and let*

$$\chi^N = \pi^N \chi, \qquad P_n^N = P_n \circ (\pi^N)^{-1}, \qquad N \in \mathbb{N}.$$

If, for each $N \in \mathbb{N}$, the family $(P_n^N)_{n\in\mathbb{N}}$ satisfies the LDP on χ^N with rate function $I^N\colon \chi^N \to [0,\infty]$, then the family $(P_n)_{n\in\mathbb{N}}$ satisfies the LDP on χ with rate function $I\colon \chi \to [0,\infty]$ given by

$$I(x) = \sup_{N\in\mathbb{N}} I^N(\pi^N x), \qquad x \in \chi.$$

The π_N's can for instance be discretizations or truncations,

$$\chi = \mathbb{R},\ \chi_N = 2^{-N}\mathbb{Z} \quad \text{or} \quad \chi = \mathbb{Z},\ \chi_N = \mathbb{Z} \cap [-N, N],$$

and the P_N's can for instance be probability distributions of the empirical means of a sequence of random variables. The supremum defining I is monotone in N and can often be computed explicitly. Apart from the nestling condition, the result in Theorem 3.1 is again simple, general, and flexible. For more background, see, e.g., the monograph by Dembo and Zeitouni [15].

4 McKean–Vlasov Equation

In the period 1987–1989, Jürgen and Don Dawson wrote a series of papers on the McKean–Vlasov equation [8–10, 24]. Their main result reads as follows. Let $H_N\colon \mathbb{R}^N \to \mathbb{R}$ be the N-particle *mean-field* Hamiltonian

$$H_N(x) = \frac{1}{2N} \sum_{i,j=1}^{N} f(x_j - x_i) + \sum_{i=1}^{N} g(x_i), \quad x = (x_1, \ldots, x_N),$$

with f even and f, g both twice continuously differentiable (f is a pair interaction, g is an external field). For $T > 0$, which plays the role of a time horizon, let $(X(t))_{[0,T]} = ((X_1(t), \ldots, X_N(t)))_{[0,T]}$ evolve according to the system of N *coupled diffusion equations*

$$\mathrm{d}X_i(t) = \frac{\partial H_N}{\partial x_i}(X(t))\,\mathrm{d}t + \mathrm{d}B_i(t), \quad i = 1, \ldots, N,$$

where $(B(t))_{[0,T]} = ((B_1(t), \ldots, B_N(t)))_{t\in[0,T]}$ are i.i.d. standard Brownian motions. This system defines a stochastic dynamics that is reversible w.r.t. the Gibbs measure with Hamiltonian H_N. A typical initial condition is where $X(0)$ has distribution λ^N for some probability measure on $\mathbb{R}$.

Define the *empirical path measure*

$$L_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{(X_i(t))_{t\in[0,T]}},$$

which is an element of $M_1(C[0,T])$, the space of probability measures on the set of continuous functions from $[0,T]$ to $\mathbb{R}$.

Theorem 4.1. *The family $(P_N)_{N\in\mathbb{N}}$ with $P_N(\cdot) = P(L_N \in \cdot)$ satisfies the LDP on $M_1(C[0,T])$ with rate function $I\colon M_1(C[0,T]) \to [0,\infty]$ given by*

$$I(Q) = \begin{cases} \displaystyle\int \log\left(\frac{\mathrm{d}Q}{\mathrm{d}P^Q}\right) \mathrm{d}Q, & \text{if } Q \ll P^Q, \\ \infty, & \text{otherwise}, \end{cases}$$

where P^Q is the law of a single diffusion with self-interaction.

Formally, P^Q is the law of the unique strong solution of the one-dimensional Itô stochastic differential equation

$$\mathrm{d}x(t) = \beta^{\pi_t Q}(x(t))\,\mathrm{d}t + \mathrm{d}b(t), \qquad t \in [0,T],$$

where $x(0)$ has probability distribution λ, $(b(t))_{[0,T]}$ is a standard Brownian motion on $\mathbb{R}$, $\pi_t Q$ is the evaluation of Q at time t, and

$$\beta^q(x) = -\int_{\mathbb{R}} f'(y-x)\,q(\mathrm{d}y) - g'(x), \qquad x \in \mathbb{R},\ q \in M_1(\mathbb{R}).$$

Theorem 4.1 describes the large deviation properties of the paths of the interacting diffusions. The rate function I has a unique zero solving the equation

$$Q = P^Q.$$

The solution of this equation determines the law of $(x(t))_{[0,T]}$ via the successive time evaluations of Q. The resulting process, which is called the *McKean–Vlasov process*, is a diffusion with a time-inhomogenous drift that is to be determined from *self-consistency*. This self-consistency is typical for mean-field models. In terms of the McKean–Vlasov process, I can be written as an action functional, in the spirit of Freidlin–Wentzell theory.

Related work was done by Sznitman [35, 36] and by Ben Arous and Brunaud [2]. The results were later extended to random mean-field interactions by Dai Pra and den Hollander [7] and to spin-glass mean-field interactions by Ben Arous and Guionnet [3, 4] and by Jürgen's student Malte Grunwald [30].

In the period 1991–1997, while extending their work on the McKean–Vlasov equation, Jürgen and Don Dawson introduced the notion of *multi-level large deviations*, describing the large deviation behavior of multi-array families of dependent random variables [11, 12]. This work in turn gave rise to the Dawson–Greven *renormalization program for hierarchically interacting diffusions* [13, 14], introduced in 1993 and since then pursued by various groups. For an overview on the latter, see den Hollander [31].

5 Parabolic Anderson Model

In 1990, Jürgen wrote a seminal paper with Stas Molchanov on *intermittency* in the Parabolic Anderson Model [26]. A lot of earlier work had been done in the physics and in the chemistry literature, but this was the first paper that put the model on a firm mathematical basis and provided a new way of looking at intermittency via the study of *Lyapunov exponents*. A follow-up paper in 1998 [27] pushed the subject further. Since then Jürgen has been working intensively on the PAM with several colleagues, both senior and junior. There are two versions of the model: *static* and *dynamic*.

The PAM is the partial differential equation:

$$\frac{\partial u}{\partial t}(x,t) = \kappa \Delta u(x,t) + \xi(x,t)\, u(x,t), \qquad x \in \mathbb{Z}^d,\ t \geq 0,$$

where Δ is the discrete Laplacian, $\kappa \in (0,\infty)$ is the diffusion constant, and $\xi(x,t)$ is a space–time random medium that drives the equation. Typical initial conditions are:

$$u(x,0) = 1 \quad \text{or} \quad u(x,0) = \delta_0(x).$$

The solution of the PAM describes the behavior of a *reactant* u under the influence of a *catalyst* ξ.

The key objects of interest are the Lyapunov exponents

$$\lambda_p = \lim_{t\to\infty} \frac{1}{pt} \log E([u(0,t)]^p), \qquad p > 0,$$

$$\lambda_0 = \lim_{t\to\infty} \frac{1}{t} \log u(0,t), \qquad \xi\text{-a.s.},$$

where E denotes expectation over the ξ-field. The λ_p's are referred to as the *annealed* Lyapunov exponents, λ_0 as the *quenched* Lyapunov exponent. The PAM is said to be *intermittent* when

$$p \mapsto \lambda_p \text{ is strictly increasing.}$$

The geometric interpretation behind this property is that the u-field develops *sparse high peaks*, with λ_p being dominated by different classes of peaks for different p (see Gärtner, König, and Molchanov [28]). This is the reason why λ_p and λ_0 provide insight into the behavior of u in space and time.

A key tool to study the PAM is the *Feynman–Kac formula*

$$u(x,t) = E_x\left(\exp\left[\int_0^t \xi(X^\kappa(s), t-s)\,\mathrm{d}s\right] u(X^\kappa(t),0)\right),$$

where $(X^\kappa(t))_{t\geq 0}$ is simple random walk jumping at rate $2d\kappa$, and E_x denotes expectation given that $X^\kappa(0) = x$. This shows that understanding the PAM is equivalent to understanding the large deviation properties of *a random walk in a random scenery*.

- *Static version:* For the case where ξ is time-independent, i.e., $\xi(x,t) = \xi(x,0) = \xi(x)$, the PAM is by now fairly well understood. The typical case is where $\xi(x)$, $x \in \mathbb{Z}^d$, are i.i.d., in which case there are *four subclasses* of distributions of $\xi(0)$ leading to qualitatively different behavior. A detailed description has been obtained for the location and the height of the peaks in the u-field, which tend to concentrate around the peaks in the ξ-field. The peaks in the u-field tend to live on *sparse islands*, whose locations and sizes change over time. For an overview, see Gärtner and König [25]. The development of the static PAM took place parallel to the work by Alain-Sol Sznitman on *Brownian motion among Poissonian obstacles* [37]. Both have substantially enriched our understanding of random processes in random media.

- *Dynamic version:* For the case where ξ is time-dependent, work is still in progress. Early work was done by Carmona and Molchanov [6] when ξ consists of i.i.d. Brownian noises. Since then the focus has been on a number of choices where ξ evolves like an interacting particle system:

  1. Independent random walks
  2. Exclusion process
  3. Voter model

  It turns out that the behavior of λ_p as a function of d and κ is extremely rich. For instance, there is a critical dimension d_c such that λ_p is constant in κ for $d < d_c$ and nonconstant in κ for $d \geq d_c$, with a delicate asymptotics for $\kappa \to \infty$ at $d = d_c$. For an overview, see Gärtner, den Hollander, and Maillard [29].

The main collaborators of Jürgen on the PAM have been S. Molchanov, F. den Hollander, W. König, and G. Maillard. Many others have made important contributions, including:

- *Jürgen's colleagues:* M. Biskup, F. Castell, M. Cranston, O. Gün, R. van der Hofstad, H. Kesten, H.-Y. Kim, L. Koralov, H. Lacoin, P. Mörters, T. Mountford, M. Ortgiese, A. Ramirez, T. Shiga, V. Sidoravicius, N. Sidorova, R. Sun, F. Viens, A. Vizcarra.
- *Jürgen's students:* A. Drewitz, J. Hähnel, M. Heydenreich, A. Schnitzler, A. Vosz, T. Wolff.

The present Festschrift contains several papers on the PAM, which include many references to the literature.

The PAM has been the main focus of Jürgen's work in the past decade. He has been the leader in the field and has shown to his colleagues what challenges the PAM is offering. The above list of names shows that he has made school.

6 Personal Remarks

Over the years, three collaborators of Jürgen have been a major inspiration to him:

- Mark Freidlin
- Don Dawson
- Stas Molchanov

Each of them has played an important role in his career: Mark as his thesis advisor, later co-authoring Jürgen's most cited paper (on wave propagation in random media [19]) and following him ever since, Don as a long-term collaborator pursuing a variety of different themes over more than a decade, and Stas as the person who brought him to the PAM, which became Jürgen's main focus in later years. Each of them has drawn Jürgen into exciting new areas of research, which he has subsequently pursued with all his force. Without them, Jürgen's mathematical itinerary would no doubt have been quite different.

For me, personally, it has been a wonderful experience to work with Jürgen. Our discussions over the past 20 years have covered a vast area. Most of what we spoke about was never written up, but part did make it to the literature: we wrote 7 papers together, and number 8 appears in the present Festschrift. What I value most in Jürgen, apart from his mastery of probability theory and analysis, is his ability to look far ahead, his constant search for elegance, his unwavering computational skills, his humor and scepticism, as well as his friendship and loyalty.

Jürgen holds the record as the most frequent visitor at EURANDOM. I trust that he will continue to push up this record in the years to come!

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