NONLINEAR
PHYSICAL
SCIENCE

S.N. Gurbatov
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Waves and Structures in Nonlinear Nondispersive Media

General Theory and Applications to Nonlinear Acoustics

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With 181 figures

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Preface

The book is aimed at natural science undergraduates, as well as at graduate and post-graduate students studying the theory of nonlinear waves of various physical nature. It may also be useful as a handbook for engineers and researchers who encounter the necessity of taking nonlinear wave effects into account in their work.

Evolution of sufficiently intense waves is determined by nonlinear processes, in which the progress is substantially influenced by dispersion (a dependence of the phase velocity on its frequency). Media without dispersion, where the phase velocity does not depend on the frequency, are the simplest ones with respect to their physical properties and are the most common in nature. But nonlinear interactions of the Fourier spectral components in such media are particularly complex and diverse. Here, practically all "virtual" energy-exchange processes between waves of different frequencies become resonant ones and occur with a high efficiency. An avalanche-like increase of the number of spectral components of the field takes place, which, within the space-time representation, corresponds to formation of structures with strongly pronounced nonlinear properties. Examples of such structures are discontinuities of a function describing the wave field or discontinuities of its derivative, steep shock fronts of various types and multidimensional cellular structures.

Nonlinear structures can be stable only in strong fields, under the conditions of competition with effects of absorption, dispersion, etc, which contribute to the decay of such structures. These objects have properties of quasiparticles. For instance, shock fronts undergo inelastic collisions. Thus, in nondispersive media, nonlinearity provides both a possibility of interactions between stable structures and their very existence. Solitons are other well-known objects in nonlinear physics, which are, generally speaking, stable only in idealized conservative systems. At the same time, quasi stability of shock-front structures or sawtooth waves occurs in real dissipative systems.

Structures of different physical nature are described by similar mathematical models. These models are used not only in the wave theory, but also to describe various non-wave objects, *viz.*: forest-fire fronts, density of a flow of non-interacting particles, etc. Because of the universality of such nonlinear models, it is necessary to analyze them on the basis of general principles of mathematical physics, irrespective of the nature of the described phenomena.

On the other hand, nondispersive waves and structures are widely used in science and technology. A review of these applications, from the authors' viewpoint, is what "brightens up" the theory and may be of interest to many readers.

The theory of nonlinear waves and structures is a very extensive and constant developing field of physics (especially radiophysics and mathematical physics). It has many specific applications. Among them there are both the well-known problems of acoustics, electrodynamics and plasma physics (see, e.g., [1–5]), and the less-known problems, such as surface-growth description [6, 7], dynamics of turbulence [8, 9] and development of a gravitational instability of the large-scale distribution of matter in the Universe [10–14]. A wide range of phenomena arising here have led to the development of a variety of mathematical methods, which are effective in addressing various kinds of nonlinear fields and waves (see, e.g., [15–17]). It is clear that within a single monograph, it is not possible to give an exhaustively comprehensive overview of the whole problem. For this reason, the authors limited themselves to a discussion of the "hydrodynamic" type of nonlinear waves in nondispersive medium. First of all, the properties of solutions to such standard nonlinear wave equations in nondispersive media as the simple wave equation, the Burgers equation and the Kardar-Parisi-Zhang equation have been studied in detail. Apart from the importance of these equations for the theory and applications, an analysis of these solutions allows us to trace stages of development of typical nonlinear processes and, above all, nonlinear distortion of profiles, the gradient catastrophe and emergence of shock waves. In order for the theory of nonlinear waves in nondispersive media not to look too abstract, the presentation is based on illustrative geometric interpretations of both the equations themselves and their solutions, as well as on a comprehensive discussion of the physical meaning of these solutions and the methods used to obtain them.

The monograph consists of two parts. The first part is devoted to a detailed description of the concepts and analysis methods of nonlinear waves and structures in nondispersive media. The second part focuses on an in-depth description of the nonlinear theory as applied only to one type of waves — high intensity acoustic waves. This object, on the one hand, is the most straightforward and, on the other hand, has important practical applications.

The authors have attempted to communicate all materialls at the following "two levels" of complexity. The first level is intended to introduce beginning investigators (above all undergraduate, graduate and PhD students) to the concepts and methods of the theory of nonlinear waves and structures in nondispersive media. In order to achieve a deeper understanding of the foundations, it is useful to solve the problems given in the end of the chapters in Part I. The second, higher, level is meant for researchers, who already have experience in this field of study and are interested in the state of the art or in specific results. Naturally, it is impossible to reflect the entire diversity of approaches used to study nonlinear fields and waves in a single monograph. This is why the material is presented at a simple, "physical" level of rigor, where possible. Those, who are interested in a more rigorous mathematical foundation of the problems discussed here, are advised to turn to monographs [15, 17], where mathematical foundations of many topics touched upon in this book are thoroughly discussed. An in-depth review of the methods used to solve nonlinear problems, along with profound results of the nonlinear field theory, can be found in book [16]. In monograph [18], and also in textbook [19], the theory of generalized functions necessary for construction of generalized solutions of non-linear equations is comprehensively elucidated. We recommend those who intend deeper to delve into the nonlinear field theory, without burying themselves in mathematical subtleties, the following thorough monographs and textbooks: [1, 2, 4, 5], which are written by physicists for physicists. Basic concepts of the nonlinear wave theory, along with illustrative physical examples, can be found in the remarkable textbook [14]. To those who are going professionally to engage themselves in the field of nonlinear acoustics, we recommend monograph [3] and the books of problems [20, 21], where a set of problems aiding in mastering various aspects of nonlinear acoustics is given. If one is interested in statistical properties of nonlinear random waves as applied to nonlinear acoustics, astrophysics and turbulence, he or she can pick up necessary information from monograph [10]. We also advise to turn to monograph [8], which covers the foundations of the theory of strong turbulence and its inherent phenomena, such as intermittency and multifractality.

We are grateful to the renowned scientists, fruitful interactions with whom over the years have formed our vision of the problems and methods of the nonlinear science. First of all, they are: academicians A.V. Gaponov-Grekhov, Ya.B. Zeldovich, R.V. Khokhlov, V.I. Arnold and Ya.G. Sinai; corresponding members of the Russian Academy of Sciences M.I. Rabinovich and D.I. Trubetskov; Professors A.N. Malakhov, L.A. Ostrovsky, S.A. Rybak, S.I. Soluyan, A.P. Sukhorukov, A.S. Chirkin and S.F. Shandarin. We are delighted to remember the years of collaboration with international colleagues, among whom are: D. Crighton, U. Frisch, B. Enflo, D. Blackstock, M. Hamilton, L. Cram, E. Aurell, A. Noullez, W.A. Woyczynski and many others.

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Part I
Foundations of the Theory of Waves in Nondispersive Media

Chapter 1
Nonlinear Equations of the First Order

The basic patterns of nonlinear fields and waves of the hydrodynamic type already can be discerned by the behavior of solutions to the simplest nonlinear partial differential equations of the first order. This chapter discusses solutions of such equations. Those wishing to study the theory of the first-order nonlinear equations more fully are advised to turn to the following literature: [1–4].

1.1 Simple wave equation

The simplest and, at the same time, crucial equation of the nonlinear wave theory of the hydrodynamic type is the *simple wave equation*. In what follows, we will pay tribute to the remarkable mathematician Riemann, who laid the foundations of the nonlinear wave theory, and call this equation the *Riemann equation*. In mathematical literature, this equation is often called the Hopf equation. By using the equation as an example it is most instructive to explain such typically nonlinear effects as the wave steepening and gradient catastrophe.

1.1.1 The canonical form of the equation

The *simple wave* (*Riemann*) *equation* is the following first order partial differential equation:

$$\frac{\partial u}{\partial t} + C(u)\frac{\partial u}{\partial x} = 0 \tag{1.1}$$

with respect to the function $u(x,t)$ which has different geometric, mechanical, economic, etc. meanings in different applications.

By multiplying Eq. (1.1) by $C'(u)$, it is reduced to the equivalent, but simpler in form, canonical Riemann equation:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = 0 \tag{1.2}$$

with respect to the new function $v(x,t) = C(u(x,t))$. Thus, without loss of generality, in what follows, we will limit ourselves to a detailed analysis of the Riemann equation (1.2) with an initial condition $v(x,t=0) = v_0(x)$. The following instructive mechanical interpretation of solutions to the Riemann equation helps better familiarize oneself with peculiarities of solutions to this equation.

1.1.2 Particle flow

The easiest way to comprehend properties of solutions to the Riemann equation is by using a flow of particles uniformly moving along the x-axis as an example. Let a particle at the point y at the initial moment of time $t = 0$ have the velocity $v_0(y)$. Then the particle's motion is given by the following equations:

$$X(y,t) = y + v_0(y)t, \quad V(y,t) = v_0(y). \tag{1.3}$$

By varying y, we obtain the laws of motion of other particles in the flow. Note that apart from the time t, another argument y, the initial particle position, appears here. Such coordinates, which are rigidly bound to the particles of a flow, are called *Lagrangian coordinates* (a pictorial comparative discussion of flow descriptions in the Lagrangian and Eulerian coordinate systems is given in textbook [4]).

Usually, an observer measures the velocity of a flow at some fixed position with a Cartesian coordinate x. These, more natural for an external observer, coordinates are called *Eulerian*. The mapping from the Lagrangian into Eulerian coordinates is described by the following equation:

$$x = X(y,t). \tag{1.4}$$

In the case of uniformly moving particles, this equation has the following form:

$$x = y + v_0(y)t. \tag{1.5}$$

Let the field $v(x,t)$ of particle velocities in a flow be given as a function of the Eulerian coordinate x and time t. If, in addition to that, the mapping (1.4) of the Lagrangian to Eulerian coordinates is also known, then the dependence of the velocity field on the Lagrangian coordinates is given by the following equation:

$$V(y,t) = v(X(y,t),t). \tag{1.6}$$

In what follows, the fields describing the behavior of particles in the Lagrangian coordinate system will be called the Lagrangian fields, and the fields in the Eulerian coordinate system will be referred to as the Eulerian fields. So $v(x,t)$ is the Eulerian particle-velocity field, and $X(y,t)$ is the Lagrangian field of the Eulerian coordinates of the particles.

From the uniformity of particle motion follows that the velocity $V(y,t)$ of a particle with the Lagrangian coordinate y does not depend on time, i.e. it satisfies the following simplest differential equation:

$$\frac{dV}{dt} = 0, \tag{1.7}$$

and its coordinate obeys a no less obvious equation:

$$\frac{dX}{dt} = V. \tag{1.8}$$

Equations (1.7) and (1.8) are nothing else than *characteristic equations* for the first order partial differential equation (1.2). In order to reconstruct the solution of the Riemann equation from the solutions of the characteristic equation (1.7), (1.8), it is sufficient to find the inverse of function (1.4)

$$y = y(x,t),$$

which maps the Eulerian coordinates to the Lagrangian ones. If this function is known, then, with provision for (1.3) and (1.6), the solution of the Riemann equations takes on the following form:

$$v(x,t) = V(y(x,t),t) = v_0(y(x,t)). \tag{1.9}$$

Let us emphasize that the single-valued inverse function $y(x,t)$ exists, and Eq. (1.9) gives the classical Riemann solution of Eq. (1.2), only if the mapping from the Lagrangian coordinates to the Eulerian ones (1.4), (1.5) is a monotonically increasing function y from $\mathbb{R}$ onto $\mathbb{R}$. In the following chapter we will discuss in detail what happens if this condition is violated. At the moment, let us assume that it is satisfied.

1.1.3 Discussion of the Riemann solution

Let us discuss the characteristic peculiarities of the behavior of the Riemann solution $v(x,t)$ as a function of the x-coordinate and time t. But, before doing that, let us list the main forms of notation for solutions of the Riemann equation. By substituting $y(x,t)$ for y in the equation of uniform motion of a particle (1.5)

$$y(x,t) = x - v_0(y(x,t))\,t \Rightarrow y(x,t) = x - v(x,t)\,t \tag{1.10}$$

and by inserting the right-hand side of this expression into Eq. (1.9), we obtain the implicit form of the Riemann solution:

$$v(x,t) = v_0(x - v(x,t)t). \tag{1.11}$$

After elementary computations, an even more direct form of the Riemann solution follows from Eq. (1.10):

$$v(x,t) = \frac{x - y(x,t)}{t}. \tag{1.12}$$

Its meaning is absolutely clear: the velocity v of a uniformly moving particle is equal to the distance $x - y$ travelled by the particle by the moment of time t, divided by the total time of motion. In what follows, a deeper mechanical and geometric meaning of the expression (1.12) will be uncovered.

While constructing a plot of an Eulerian field $v(x,t)$, it is convenient to employ Lagrangian fields and to construct $v(x,t)$ parametrically:

$$x = y + v_0(y)t, \quad v = v_0(y). \tag{1.13}$$

The Riemann solution in Fig. 1.1 is so constructed in the case when the initial profile of the velocity field has a Gaussian form:

$$v_0(x) = V_0 \exp\left(-\frac{x^2}{2\ell^2}\right). \tag{1.14}$$

This figure shows, in the following dimensionless variables

$$z = \frac{x}{\ell}, \quad \tau = \frac{V_0}{\ell}t, \tag{1.15}$$

the velocity field at the moment of time $\tau = 1$. It also depicts (dashed line) the initial velocity $v_0(x)$. Arrows show particle displacements travelled during the time interval τ. It is seen that the greater the velocity of a particle, the greater the displacement travelled by the particle during a time interval. This leads to the steepening of the front of the field $v(x,t)$ on the right-hand-side and to the stretching of the left front.

1.1.4 Compressions and expansions of the particle flow

The steepening of the right part of the velocity-field profile $v(x,t)$ in Fig. 1.1 is accompanied by the thickening of the particle flow. Indeed, particles within the left-hand-side of this interval have a greater velocity than the particles on the right-hand-side. As a result, the faster left particles in time catch up with the slower right particles. On the contrary, the expansion of the left part of the velocity-field profile $v(x,t)$ leads to the rarefaction of the flow. Quantitatively, the measure of rarefaction of different parts of a flow is expressed by a Jacobian, which, in the one-dimensional case, is equal to

$$J(y,t) = \frac{\partial X(y,t)}{\partial y}. \tag{1.16}$$

For uniformly moving particles, whose law of motion is given by Eq. (1.5), the Jacobian is equal to

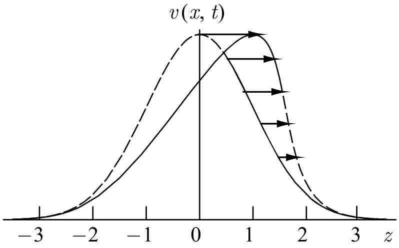


Fig. 1.1 The Riemann solution in the case of a Gaussian initial field.

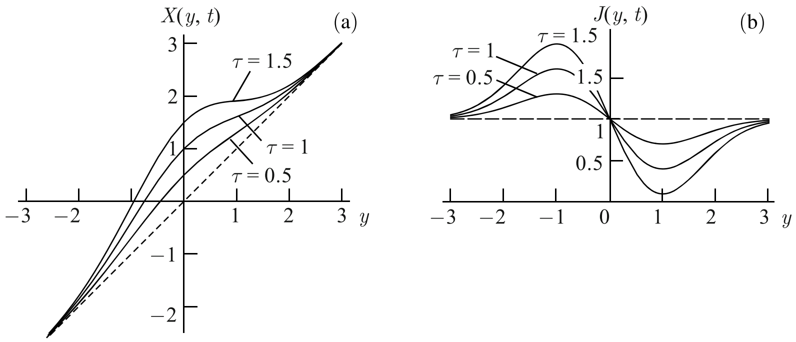


Fig. 1.2 Particle motion pattern $X(y,t)$ (*left*) and the corresponding Lagrangian divergence field $J(y,t)$ (*right*) at different moments of time. It is seen that compressed and expanded parts appear in the flow, where $J(y,t)$ is not equal to unity.

$$J(y,t) = 1 + v_0'(y)t. \tag{1.17}$$

The greater the Jacobian at a given value of y, the more rarefied the flow in a vicinity of the particle with the Lagrangian coordinate y. For this reason, let us call $J(y,t)$ the flow's *divergence*. A plot of the law of motion $X(y,t)$ and the corresponding divergence for uniformly moving particles (whose Eulerian field obeys the Riemann equation (1.2) with the initial condition (1.14)), are depicted in Fig. 1.2.

The field $J(y,t)$ (1.16) is a Lagrangian divergence field. The corresponding Eulerian field is obviously equal to

$$j(x,t) = J(y(x,t),t) \iff J(y,t) = j(X(y,t),t). \tag{1.18}$$

If the rule for transformation of Eulerian coordinates into Lagrangian ones $y(x,t)$ is known, the divergence field can be determined by a more direct method by means of the following geometrically evident expression:

$$\frac{\partial y(x,t)}{\partial x} = \frac{1}{j(x,t)}. \tag{1.19}$$

1.1.5 Continuity equation

A natural question arises in the framework of the mechanical interpretation of the solution to the Riemann equation (1.2) as the velocity field of a flow of uniformly moving particles: how does their density $\rho(x,t)$ change in time and space? It is known that the density obeys the universal *continuity equation*, which describes the law of mass conservation of the particles in a flow. Let us derive this equation by using a method, which allows one better to understand the following analysis of solutions to partial differential nonlinear equations.

Let, for definiteness, the initial particle density of a flow $\rho_0(x)$ be such that the mass of particles to the left of any point x

$$m_0(x) = \int_{-\infty}^{x} \rho_0(z)dz$$

is finite. Let the function $m(x,t)$ describe the variation of the mass of particles to the left of an arbitrary x. Like the velocity field $v(x,t)$, this is an Eulerian field. Knowing the law of particle motion $X(y,t)$, it is easy to make a transformation from the Eulerian left-mass field to the corresponding Lagrangian field:

$$M(y,t) = m(X(y,t),t).$$

The latter is easily obtained from obvious physical considerations. Indeed, if by the current moment of time t the particles have not overtaken each other (have not swapped places), the mass of particles on the left from any point with the Lagrangian coordinate y does not depend on time:

$$M(y,t) = \int_{-\infty}^{y} \rho_0(z)\,dz = m_0(y). \tag{1.20}$$

In other words, the Lagrangian mass field on the left satisfies the following equation:

$$\frac{dM}{dt} = 0.$$

The equivalent to it equation of the Eulerian field is

$$\frac{\partial m}{\partial t} + v\frac{\partial m}{\partial x} = 0. \tag{1.21}$$

Now let us determine the particle density. In the one-dimensional case, the Eulerian density field is equal to the derivative of the mass on the left:

$$\rho(x,t) = \frac{\partial m(x,t)}{\partial x}. \tag{1.22}$$

Hence, by differentiating Eq. (1.21) term by term with respect to x, we arrive at the sought-for one-dimensional variant of the continuity equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(v\rho) = 0. \tag{1.23}$$

Note. Let us note that while deriving the continuity equation, we nowhere used the fact of the uniformity of motion. Therefore, in this derivation, the universality of the continuity equation has been exhibited, which holds for any laws of particle motion.

1.1.6 Construction of the density field

In order to find a solution of the continuity equation (1.23), let us write down the Eulerian mass field on the left. From Eq. (1.20) and the link between Lagrangian and Eulerian fields follows that

$$M(y,t) = m_0(y) \iff m(x,t) = m_0(y(x,t)). \tag{1.24}$$

By differentiating the last equation with respect to x, we obtain

$$\rho(x,t) = \rho_0(y(x,t))\frac{\partial y(x,t)}{\partial x} \tag{1.25}$$

or, by taking Eq. (1.19) into account,

$$\rho(x,t) = \frac{\rho_0(y(x,t))}{j(x,t)} \iff R(y,t) = \frac{\rho_0(y)}{J(y,t)}. \tag{1.26}$$

These formulas have an apparent geometric meaning: the flow density at any point is equal to the initial density in a vicinity of the particle at this point divided by the degree of compression of particles.

Let us separately discuss the density of a flow of uniformly moving particles, whose velocity field $v(x,t)$ obeys the Riemann equation, and $y(x,t)$ is given by (1.10). Here, as it is seen from (1.25), (1.10), the flow density is expressed via the solution of the Riemann equation in the following way:

$$\rho(x,t) = \rho_0\left(x - v(x,t)t\right)\left(1 - \frac{\partial v(x,t)}{\partial x}t\right). \tag{1.27}$$

In particular, when the initial density is uniform, i.e. if $\rho_0 = const$ does not depend on x, the density is described by the following relation:

$$\rho(x,t) = \rho_0\left(1 - \frac{\partial v(x,t)}{\partial x}t\right), \tag{1.28}$$

which demonstrates a close link between the density of a flow and the steepening of its velocity-field profile.

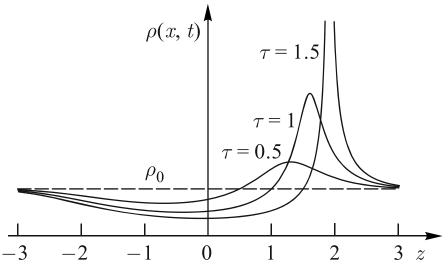


Fig. 1.3 Density of uniformly moving particles with a Gaussian initial velocity field (1.14) and a constant initial density $\rho(x,t=0)=\rho_0$.

As in the case of the velocity field, it is convenient to plot the density of a flow of uniformly moving particles parametrically, by using the fact that the Lagrangian laws of flow evolution are given explicitly:

$$x = y + v_0(y)t, \quad \rho = \frac{\rho_0(y)}{1+v_0'(y)t}. \tag{1.29}$$

The plots of $\rho(x,t)$ in Fig. 1.3 are constructed in this way.

1.1.7 Momentum-conservation law

Apart from the mass conservation law, a flow of uniformly moving particles also possesses an infinite set of invariants (see, e.g., [3, 5]). Most of them do not have any significant physical meaning, while others, for instance the law of conservation of momentum, play a paramount role in physical applications. Here, we discuss this law in more detail.

Let us remind you, that the total momentum of particles to the left of a point x is equal

$$p(x,t) = \int_{-\infty}^{x} v(x,t)\rho(x,t)\,dx.$$

By substituting here Eqs. (1.9) and (1.25) for the velocity and density of a flow, respectively, and then by changing to integration with respect to the Lagrangian coordinate, we reduce the expression for the momentum on the left to the following form:

$$p(x,t) = \int_{-\infty}^{x} v_0(y(x,t))\rho_0(x(y,t))\frac{\partial y(x,t)}{\partial x}dx = \int_{-\infty}^{y(x,t)} v_0(y)\rho_0(y)\,dy.$$

From here, it is seen that, for uniformly moving particles, the Lagrangian field of the momentum on the left does not depend on time:

$$P(y,t) = p_0(y) = \int_{-\infty}^{y} \rho_0(y)v_0(y)\,dy. \tag{1.30}$$

Hence, in analogy with the mass on the left, the corresponding Eulerian field of the momentum on the left obeys the following equation:

$$\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial x} = 0, \tag{1.31}$$

and the momentum density

$$g(x,t) = \rho(x,t)v(x,t) = \frac{\partial p(x,t)}{\partial x}$$

satisfies the continuity equation

$$\frac{\partial g}{\partial t} + \frac{\partial}{\partial x}(vg) = 0. \tag{1.32}$$

Note 1. It is easy to derive this equation as a corollary of the Riemann equation (1.2) and the continuity equation (1.23). But we have deliberately chosen the round-about "integral" method, for it will help, in the following, construct generalized solutions of the equations mentioned here.

Note 2. Since the momentum density obeys the same equation as the simple density $\rho(x,t)$, we obtain expressions for the Eulerian and Lagrangian momentum-density fields "gratis" by substituting the initial momentum for the initial density in (1.26):

$$g(x,t) = \frac{\rho_0(y(x,t))v_0(y(x,t))}{j(x,t)} \iff G(y,t) = \frac{\rho_0(y)v_0(y)}{J(y,t)}. \tag{1.33}$$

1.1.8 Fourier transforms of density and velocity

In applications, it is often important to know not the fields themselves, but their spectra. Therefore we will find expressions for the spatial Fourier transforms of the velocity and density. Let us start with the density

$$\tilde{\rho}(\kappa,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\kappa x}\rho(x,t)\,dx. \tag{1.34}$$

By substituting the solution of the continuity equation found earlier (1.25) into the integral (1.34), we obtain

$$\tilde{\rho}(\kappa,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\kappa x}\rho_0(y(x,t))\,dy(x,t).$$

By transforming to integration with respect to the Lagrangian coordinate, we finally obtain

$$\tilde{\rho}(\kappa,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\kappa X(y,t)}\rho_0(y)\,dy. \tag{1.35}$$

More unwieldy calculations, based on the same idea of transformation to integration with respect to the Lagrangian coordinates in the Fourier integral, give

$$\tilde{v}(\kappa,t) = \frac{i}{2\pi\kappa t}\int_{-\infty}^{\infty}\left[e^{-i\kappa X(y,t)} - e^{-i\kappa y}\right] dy. \tag{1.36}$$

A discussion of this solution can be found in monograph [6].

Example. Generation of harmonics. Formulas (1.35) and (1.36) are remarkable in that they express the Fourier transforms of implicitly given (e.g. by Eq. (1.11)) fields $\rho(x,t)$ and $v(x,t)$ via integrals of explicitly given functions. ∎

Let us use this opportunity to find an explicit expression for the Fourier transform of the density $\rho(x,t)$ in the case of the initial harmonic field and uniform density

$$v_0(x) = a\sin(kx), \quad \rho_0(x) = \rho_0 = const. \tag{1.37}$$

In doing so, we need the following formula from the theory of Bessel functions:

$$e^{iw\sin z} = \sum_{n=-\infty}^{\infty} J_n(w)\, e^{inz}. \tag{1.38}$$

In the case under study, the law of transformation from Lagrangian to Eulerian coordinates is given as

$$x = X(y,t) = y + at\,\sin(ky). \tag{1.39}$$

By substituting this equality into (1.35), we obtain

$$\tilde{\rho}(\kappa,t) = \frac{\rho_0}{2\pi k}\int_{-\infty}^{\infty} e^{-i\mu z - i\mu\tau\sin z}\, dz.$$

Here we introduced the dimensionless variable of integration $z = ky$, time $\tau = kat$ and spatial frequency $\mu = \kappa/k$. By taking Eq. (1.38) into account we obtain

$$\tilde{\rho}(\kappa,t) = \rho_0 \sum_{n=-\infty}^{\infty} J_n(-\mu\tau)\frac{1}{2\pi k}\int_{-\infty}^{\infty} e^{-i(\mu-n)z}\, dz.$$

According to the theory of generalized functions, the last integral (see, e.g., [7, 8]) has the form:

$$\frac{1}{2\pi k}\int_{-\infty}^{\infty} e^{-i(\mu-n)z}\, dz = \frac{1}{k}\,\delta(\mu-n) = \delta(\kappa - kn).$$

From this and the previous expression, it follows that the density field investigated here possess the following generalized Fourier transform:

$$\tilde{\rho}(\kappa,t) = \rho_0 \sum_{n=-\infty}^{\infty} J_n(-n\tau)\delta(\kappa - kn).$$

By substituting it into the inverse Fourier integral

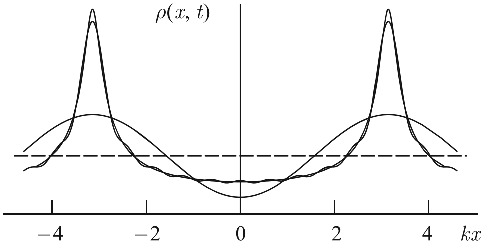


Fig. 1.4 Plot of $\rho(x,t)$ for the initial harmonic velocity field and the uniform density (1.37) at $\tau = akt = 0.7$. Profiles of the first two (one harmonic term) and eleven terms of the Fourier series are also shown. It is seen that the profile of the last sum nearly coincides with the density profile.

$$\rho(x,t) = \int_{-\infty}^{\infty} \tilde{\rho}(\kappa,t)\, e^{i\kappa x}\, d\kappa$$

and taking the symmetry properties of the Bessel function

$$J_{-n}(-w) = J_n(w) \tag{1.40}$$

into account, we arrive at the explicit expression for the density field in the form of a Fourier series:

$$\rho(x,t) = \rho_0 + 2\rho_0 \sum_{n=1}^{\infty} (-1)^n J_n(n\tau)\cos(kx). \tag{1.41}$$

Comparison between the sum of the first terms of this series and the exact density profile constructed parametrically by means of Eqs. (1.29), shows that a few first terms of Fourier series already give a good approximation to the exact solution.

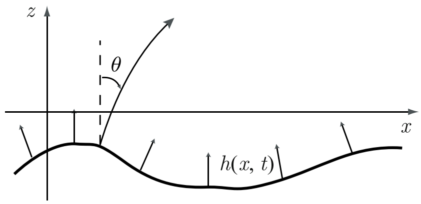
1.2 Line-growth equation

Let us discuss one more of numerous applications of the first order nonlinear partial differential equations: analysis of line and surface growth. It may be: deposited surface of an electronic chip, wave front of a light wave, shock wave of a jet plane and a fire line consuming a forest. All of these surfaces and lines are described by nonlinear partial differential equations (see, e.g., [9–12]). The simplest example of these equations is given below.

1.2.1 Forest-fire propagation

Let a fire move within a forest. In order mathematically to describe the process of fire propagation, we assume that the surface of the forest is flat and introduce within this plane the Cartesian coordinates (x,z). Let us direct the z-axis along the predominant direction of the fire. As a result, it is possible to describe the fire-front

Fig. 1.5 Fire line and its tendency to grow



line by a function

$$z = h(x,t). \tag{1.42}$$

It is natural to assume that a fire spreads perpendicularly to the line of fire $h(x,t)$ with a speed c. This means that if one selects a point $\{y, h(y, t = 0)\}$ on the line of fire at the initial moment of time $t = 0$ and traces its motion along the trajectory perpendicular to the lines of fire, as it is seen in Fig. 1.5, then the velocity of the point will be equal to c. Let the coordinates of the specified point change with time according to the laws $\{X(t), Z(t)\}$. Let us call the trajectory of the point's motion a *ray*.

From what has just been said, it is clear that the coordinates of the ray satisfy the equations:

$$\frac{dX}{dt} = c\sin\theta, \quad \frac{dZ}{dt} = c\cos\theta, \tag{1.43}$$

where θ is the angle between the ray and the z-axis. Further we note that the vertical coordinate $Z(t)$ of the ray can be expressed in terms of the fire line (1.42):

$$Z(t) = h(X(t),t). \tag{1.44}$$

By substituting this equality into the second of Eqs. (1.43), we obtain

$$\frac{\partial h}{\partial t} + \frac{dX}{dt}\frac{\partial h}{\partial x} = c\cos\theta$$

or, by taking the first of Eqs. (1.43) into account, we arrive at the partial differential equation for the sought-for line of fire $h(x,t)$:

$$\frac{\partial h}{\partial t} + c\sin\theta\,\frac{\partial h}{\partial x} = c\cos\theta. \tag{1.45}$$

It would seem that the equation is not closed, because it links two functions: the line of fire $h(x,t)$ and the angle $\theta(x,t)$ between the z-axis and the normal to the line of fire. However, it is easy to make it closed by using the obvious geometric relationship between the line $h(x,t)$ and the angle θ:

$$\frac{\partial h}{\partial x} = -\tan\theta. \tag{1.46}$$

In terms of the last result, Eq. (1.45) can be rewritten as

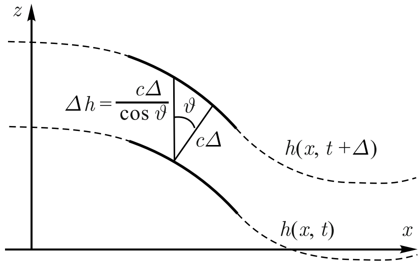


Fig. 1.6 Geometric illustration of the validity of the equation (1.47)

$$\frac{\partial h}{\partial t} = \frac{c}{\cos\theta}. \tag{1.47}$$

Finally, by noting that

$$\cos\theta = \frac{1}{\sqrt{1+\tan^2\theta}},$$

we obtain the final form of the sought-for equation:

$$\frac{\partial h}{\partial t} = c\sqrt{1+\left(\frac{\partial h}{\partial x}\right)^2}. \tag{1.48}$$

Note 1. At first, it seems that the juggling with analytical transformations has led to an absurd from the point of view of the common geometric sense Eq. (1.47). Indeed, one would think, the more the normal to the line $h(x,t)$ deviates from z-axis, i.e. the greater the angle θ between the z-axis and the direction of line growth, the slower the line $h(x,t)$ must grow along the axis z. But the Eq. (1.47) signifies that the greater θ, the faster the growth, and at $\theta = \pi/2$ the growth rate becomes infinite. However, an accurate geometric study convinces one in the validity of Eq. (1.47). Relevant geometrical constructions are given in Fig. 1.6, where fragments of the line $h(x,t)$ are shown at close moments of time t and $t+\Delta$. It is seen that the increment of the line's height at an arbitrary point x

$$\Delta h = h(x,t+\Delta) - h(x,t) \approx \frac{c\Delta}{\cos\theta}$$

is inversely proportional to $\cos\theta$. Perhaps this geometric derivation will convince someone of the correctness of Eqs. (1.47) and (1.48) sooner than the above-stated formal analysis.

Note 2. If the speed in Eq. (1.48) is negative ($c < 0$), we obtain the equation not for growth, but for decay of the line $h(x,t)$. Accordingly, Eq. (1.48) will describe, e.g., the melting of ice in a glass of water or hull corrosion of an oceanic ship.

Note 3. We deliberately called the trajectory $\{X(t), Z(t)\}$ perpendicular to the fire front a ray. This is because the wavefront of an optic wave obeys the above-stated law of propagation perpendicularly to the front with a given speed. The lines, everywhere perpendicular to the wave fronts, by definition, are optic rays. Thus Eq. (1.48) represents a one-dimensional version of the equations describing evolution of wavefronts of optic waves.

Note 4. Typically, optic waves propagate in a preferential direction — at small angles to it. If the preferential direction of the optic wave is directed along the z-axis, then tilt angles of rays to the z-axis are small, and instead of Eq. (1.48) a simpler, approximate equation is used. That is the following substitutions are used:

$$-\frac{\partial h}{\partial x} = \tan\theta \approx \theta, \quad \sqrt{1+\tan^2\theta} \approx 1 + \frac{1}{2}\left(\frac{\partial h}{\partial x}\right)^2$$

and Eq. (1.48) is rewritten as

$$\frac{\partial h}{\partial t} = c + \frac{c}{2}\left(\frac{\partial h}{\partial x}\right)^2.$$

For a plane wave propagating exactly along the z-axis, this equation has a simple solution: $h = ct$. If we are interested only in the shape of wavefront, and not in its exact position, then one can exclude the indicated trivial forward motion (translation) by introducing a new function:

$$w(x,t) = h(x,t) - ct. \tag{1.49}$$

The latter satisfies a more elegant equation:

$$\frac{\partial w}{\partial t} = \frac{c}{2}\left(\frac{\partial w}{\partial x}\right)^2. \tag{1.50}$$

1.2.2 Anisotropic surface growth

Let us introduce the following new notation

$$u(x,t) = -\frac{\partial h(x,t)}{\partial x}. \tag{1.51}$$

Recall that $u = \tan\theta$ characterizes the direction of surface growth. Therefore, $u(x,t)$ came to be called a *tilt-angle field*. Sometimes the surface is growing at different speeds in different directions [9]. For example, during the propagation of an optic wave in an anisotropic medium, or the melting of glaciers, when the melt rate depends on the angle at which a part of the glacier surface is facing the sun. Let us take an anisotropy of growth into account assuming that the speed depends on u and transform from Eq. (1.47) to a more general equation:

$$\frac{\partial h}{\partial t} = \Phi(u), \quad h(x,t=0) = h_0(x), \tag{1.52}$$