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137

Nico Sneeuw
Pavel Novák
Mattia Crespi
Fernando Sansò *Editors*

VII Hotine-Marussi Symposium on Mathematical Geodesy

Proceedings of the Symposium in Rome, 6-10 June, 2009

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Preface

This volume contains the proceedings of the VII Hotine-Marussi Symposium on Mathematical Geodesy, which was held from 6 to 10 July 2009. The symposium took place at the Faculty of Engineering of the Sapienza University of Rome, Italy, in the ancient *chiostro* of the Basilica of S. Pietro in Vincoli, famously known for its statue of Moses by Michelangelo.

The traditional name *mathematical geodesy* for the series of Hotine-Marussi Symposia may not fully do justice to the symposium's broad scope of theoretical geodesy in general. However, the name for the series has been used since 1965, i.e., the days of Antonio Marussi, which is a good reason to adhere to it. The venue of the Hotine-Marussi Symposia has traditionally been in Italy. The choice for Rome, if a reason is needed at all, was partially made because 2009 was the International Year of Astronomy. Two important astronomical events were commemorated: the publication of Kepler's *Astronomia Nova* in 1609, in which he published his first two laws of planetary motion, as well as the very first astronomical use of a telescope by Galileo and his discovery of Jupiter's moons. Besides one of the founding fathers of geodesy, the unit of Gal being named after him, he was one of the cofounders and an early member of the Accademia Nazionale dei Lincei in Rome. It was a pleasure, therefore, that a special session was organized by Fernando Sansò at the Villa Farnesina, located at the Academy. The special session was dedicated to the memory of Antonio Marussi (1908–1984), who was the driving force behind the series of Hotine (later Hotine-Marussi) Symposia.

Since 2006 the series is under the responsibility of the InterCommission Committee on Theory (ICCT), a cross-commission entity within the International Association of Geodesy (IAG). The overall goal of the Hotine-Marussi Symposia has always been the advancement of theoretical geodesy. This goal is aligned with the objectives of the ICCT, which has the developments in geodetic modeling and data processing in the light of recent advances of geodetic observing systems as well as the exchange between geodesy and neighboring Earth sciences as its central themes. Indeed, the current proceedings are testimony to the width and vibrancy of theoretical geodesy.

The symposium attracted 132 participants who contributed 75 papers (51 oral and 24 poster), organized in eight regular sessions plus the session at the Accademia Nazionale dei Lincei. To a large extent, the sessions' topics were modeled on the study group structure of the ICCT. The chairs of the ICCT study groups, who constituted the Symposium's Scientific Committee, were at the same time responsible for organizing the sessions:

1. *Geodetic sensor systems and sensor networks*
   S. Verhagen
2. *Estimation and filtering theory, inverse problems*
   H. Kutterer, J. Kusche
3. *Time series analysis and prediction of multi-dimensional signals in geodesy*
   W. Kosek, M. Schmidt
4. *Geodetic boundary value problems and cm-geoid computational methods*
   Y.M. Wang, P. Novák
5. *Satellite gravity theory*
   T. Mayer-Gürr, N. Sneeuw
6. *Earth oriented space techniques and their benefit for Earth system studies*
   F. Seitz, R. Gross
7. *Theory, implementation and quality assessment of geodetic reference frames*
   Dermanis, Z. Altamimi
8. *Temporal variations of deformation and gravity*
   G. Spada, M. Crespi, D. Wolf

We want to express our gratitude to all those who have contributed to the success of the VII Hotine-Marussi Symposium. The aforementioned study group chairs (Scientific Committee) put much effort in organizing attractive sessions and convening them. They also organized the peer review process. We equally owe thanks to all reviewers. Although much of the review process itself remains anonymous, the complete list of the reviewers is printed in this volume as a token of our appreciation of their dedication.

Financial and promotional support was given by a number of agencies and institutions. Special thanks go to Federazione delle Associazioni Scientifiche per le Informazioni Territoriali e Ambientali (ASITA), Agenzia Spaziale Italiana (ASI), the European Space Agency (ESA), and the Faculty of Engineering of the Sapienza University of Rome.

But most of all we like to thank Mattia Crespi and his team (Gabriele Colosimo, Augusto Mazzoni, Francesca Fratarcangeli, and Francesca Pieralice) who hosted the symposium. It is well known that the quality of a Local Organizing Committee (LOC) is decisive to a successful scientific meeting. Beyond responsibility for website, registration, technical support, and all kinds of other arrangements, the LOC organized a great social event to the St. Nilus' Abbey, the archeological area of Monte Tuscolo and the Villa Grazioli in Frascati. Through their able organization and improvisation skills, Mattia Crespi and his team have done more than their share in bringing the VII Hotine-Marussi Symposium to success.

Stuttgart

Nico Sneeuw
Pavel Novák
Mattia Crespi
Fernando Sansò

Fifty Years of Hotine-Marussi Symposia

In 1959, Antonio Marussi, in cooperation with the Italian Geodetic Commission, started a series of symposia in Venice. The first three of these covered the entire theoretical definition of 3D Geodesy, as delineated in discussions with renowned contemporary scientists:

- 1959, Venice, 16–18 July, 1st Symposium on Three Dimensional Geodesy, published in Bollettino di Geodesia e Scienze Affini, XVIII, N ° 3, 1959
- 1962, Cortina d'Ampezzo, 29 May-1 June, 2nd Symposium on Three Dimensional Geodesy, published in Bollettino di Geodesia e Scienze Affini, XXI, N ° 3,1962
- 1965, Turin, 21–22 April, 3rd Symposium on Mathematical Geodesy, published by Commissione Geodetica Italiana, 1966

From the very beginning, Martin Hotine provided essential inspiration to these symposia. After his death in 1968, the following symposia bear his name:

- 1969, Trieste, 28–30 May, 1st Hotine Symposium (4th Symposium on Mathematical Geodesy), published by Commissione Geodetica Italiana, 1970
- 1972, Florence, 25–26 October, 2nd Hotine Symposium (5th Symposium on Mathematical Geodesy), published by Commissione Geodetica Italiana, 1973
- 1975, Siena, 2–5 April, 3rd Hotine Symposium (6th Symposium on Mathematical Geodesy), published by Commissione Geodetica Italiana, 1975
- 1978, Assisi, 8–10 June, 4th Hotine Symposium (7th Symposium on Mathematical Geodesy), published by Commissione Geodetica Italiana, 1978
- 1981, Como, 7–9 September, 5th Hotine Symposium (8th Symposium on Mathematical Geodesy), published by Commissione Geodetica Italiana, 1981

After Marussi's death, in 1984, the symposia were finally named the Hotine-Marussi Symposia:

- 1985, Rome, 3–6 June, I Hotine-Marussi Symposium (Mathematical Geodesy)
- 1989, Pisa, June, II Hotine-Marussi Symposium (Mathematical Geodesy)
- 1994, L'Aquila, 29 May-3 June, III Hotine-Marussi Symposium (Mathematical Geodesy, Geodetic Theory Today), published by Springer, IAG 114
- 1998, Trento, 14–17 September, IV Hotine-Marussi Symposium (Mathematical Geodesy), published by Springer, IAG 122
- 2003, Matera, 17–21 June, V Hotine-Marussi Symposium (Mathematical Geodesy), published by Springer, IAG 127
- 2006, Wuhan, 29 May-2 June, VI Hotine-Marussi Symposium (Theoretical and Computational Geodesy, 1st time under ICCT), published by Springer, IAG 132
- 2009, Rome, 6–10 June, VII Hotine-Marussi Symposium (Mathematical Geodesy), published by Springer, IAG

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Part I

Session at the Accademia Nazionale dei Lincei

1 Welcome to the Participants to the VII° Hotine-Marussi Symposium

M. Caputo

Good morning. I am Michele Caputo. The president of our Accademia prof. G. Conso could not come to the meeting. He asked me to present his greetings, to welcome you on his behalf and wish a good visit of the Accademia.

Our Academia was founded in by the young Federico Cesi, 18 years old, in the year 1603. The name of the Accademia comes from the lynx, the elegant feline, which was supposed to have excellent eyes and see well at incommensurable distances. Galilei observed the planets from the highest portion of the garden outside this building. He had joined the Accademia in 1625.

Few of you may know of the Pizzetti–Somigliana theory, but all know of the International Gravity Formula. It was all born and developed within the walls of this building.

In fact following the path indicate by Pizzetti in a series of papers published between 1894 and 1913, Somigliana (1929) developed the general theory of the gravity field of a rotating ellipsoid of revolution. At the same time Silva (1928, 1930) estimated the values to adopt for the parameters appearing in the formula from the average values obtained using the observed gravity on the surface of the Earth.

Finally Cassinis (1930) presented the series expansion of the original closed form formula at the 1930 IUGG Assembly in Stockholm which adopted the formula to be used for the normal values of gravity on the surface of the international ellipsoid of revolution. This ellipsoid had been adopted by the International Association of Geodesy in the 1923 assembly. 57 years later the closed form formula of the Pizzetti–Somigliana theory was extended to space, for whatever it may be useful, introducing the then available satellite data (Caputo and Benavidez 1987).

It was almost all discussed within the walls of the Accademia dei Lincei and published in its proceedings Now all theoretical geodesists who are familiar with the gravity field of the Earth know that Somigliana, Pizzetti and Cassinis were members of the Accademia where they often met and discussed of theoretical Geodesy. One more notable member of the Accademia was Antonio Marussi who was one of the most complete professionals of geodesy I knew in my life; he knew the use of the data resulting from the observations made with the Stark Kammerer theodolite and how to make sophisticated maps, at the time when the Brunswiga Addiermachine desk mechanical computer was the most advanced instrument to make multiplications and divisions; Marussi had the expertise of making accurate measurements as well as that to use differential geometry to model what is called intrinsic geodesy. And finally he made the extraordinary pendulums. We are here to honour him, as well as his colleague Hotine.

Thank you for coming to Accademia dei Lincei. I wish a good day of work.

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The Marussi Legacy: The Anholonomity Problem, Geodetic Examples

2

E.W. Grafarend

Antonio Marussi *died 16th April 1984 in Trieste, nearly exactly 25 years ago. He is the founder of the* Geodetic Anholonomity Problem *or the problem of integrability of geodetic observational functionals. My talk will try to* open your eyes by simple examples.

Top 1: Gravitostatics

Geodetic heights, better height differences are not integrable. For instance, every geodesist knows "dH", the infinitesimal change of geodetic heights. In my courses in Physics I learned the notion $đH$. In terms of *Planck notation* $đH$ *is not integrable*. But the *Gauss potential* – C.F. Gauss introduced the notion of *potential – is integrable*. We all know the transformation

$$dW = -\Gamma đH,$$

where we use the input "geometric height differential" (anholonomic) versus the physical height difference in terms of output of the potential differential (holonomic). The factor of integrability is the modulus of gravity, also called an element of the *Frobenius matrix*. *A. Marussi* recognized first this key problem and generalized it into *three- and four-dimensional Geodesy*, into space-time geodesy. Notable, the potential W consists of two quantities: the gravitational potential U and the centrifugal potential V. In contrast the *Euler rotational force* and the *Coriolis rotational force* are not integrable.

Geodetic height systems referring to the *Gauss-Listing Geoid* are founded on "potential heights". To my strong belief, the anholonomity problem established *Geodesy as a Science!*

Integrability or anholonomity problems are treated nowadays by *Cartan calculus*, also called *exterior calculus* or the calculus of *differential forms*, a calculus introduced in the twentieth century by *Elie Cartan*, a famous French scientist. *F.R. Helmert* knew already about the subject, *Heinz Draheim* of Karlsruhe University wrote an early paper about *Cartan calculus* and surface geometry. I learned it in *Thermodynamics* from the *Carnot circle* or *Carnot loop* in my *Physics Courses*.

Top 2: Gravitodynamics

I only mention the papers by *E. Doukakis*, his *Ph.D. Thesis*, to include space-time concepts on anholonomity problems, namely integrability, both in the space and in the time domain. There is no time to discuss this in more detail.

Top 3: Space-Time Geodesy

A. Marussi is the real founder of space-time Geodesy: He influenced Hotine (1969) to write his famous textbook with more than 5,000 formulae. He influenced also J. Zund (1988-1994) to write many beautiful papers on *Differential Geodesy* and the leg calculus. In addition he published *A. Marussi's works* in a remarkable Springer Edition. *J. Zund's book* on Differential Geodesy is another masterpiece of depth and wide range. (*J. Zund*, Differential Geodesy, Springer Verlag, Berlin 1994). At this point, another mathematician has to be mentioned who also applied *Cartan calculus* to the holonomity problem: Nathaniel (Nick) Grossmann from UCLA. He wrote remarkable papers on the geodetic *anholonomity problem*. He is a trained mathematician on *Cartan or exterior calculus*. See our reference list at the end.

Top 4: Refraction and Diffraction

There are excellent papers in Physics on this subject written in exterior calculus. For instance, I recall a paper by *P. Defrise et al* from Belgium.

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Top 5: Continuum Mechanics
Traditionally, plasticity problems and nonlinear stress-strain relations are treated by *Cartan calculus* and exterior differential forms.

Top 6: Deformation Analysis
There is a special geometric property within *Cartan calculus*. When we transform within Gauss surface geometry a Riemann metric to orthogonal axes, we arrive at a picture of a circle: the orthonormal axes produce a *Cartan reference system* which is *anholonomic*. For *deformation analysis*, it is possible to transform a *left metric* into a *right metric*, namely from a *left circle* into a *right ellipse* or vice versa. This is the extended Cartan system when transforming two Riemann manifolds.

Top 7: Map Projections
The *Tissot ellipse* is the proper tool when we transform a left Riemann metric to a right Riemann metric. It is the extended *Cartan reference system* from a circle (left) to an ellipse (right). Reference has to be made to *C. Boucher, A. Dermanis, E. Livieratos* and many others. For more details, we refer to our book "Map Projections" (Springer Verlag, 750 pages, Berlin-Heidelberg 2006).

Top 8: Rotational motion by Cartan calculus and Omega quantities
E. Cartan introduced his new concept by referring to the *Euler kinematical equation*. You have to introduce the transformation from rotational velocities "Omega" to *Euler angles*: $\omega = \mathbf{M}(d\alpha, d\beta, d\gamma)$. "Omega" is the rotational vector which is mapped to *Euler angles*. ω is *not integrable*, $(d\alpha, d\beta, d\gamma)$ are *integrable*.

Top 9: Relativity
Hehl (1996) referred to more than 100 authors to establish *Einstein-Cartan geometry* with *spin degrees-of-freedom*. One part of the connection symbols are anti-symmetric characterizing *Cartan torsion* related to my *M.Sc. Thesis* in *Theoretical Physics*. We refer also to the correspondence between *Elie Cartan and Albert Einstein*, published by Springer Verlag.

What has happened meanwhile?

First, Cartan geometry was generalized to *Clifford algebra* and *Clifford analysis* in order to account for *symmetric differential forms or symmetric matrices and antisymmetric differential forms or antisymmetric matrices.* Nowadays we summarize to *multilinear algebra* and *multilinear analysis*. There are special conferences every year devoted to *Clifford algebra* and *Clifford analysis*. As a reference see my review "Tensor Algebra, Linear Algebra, Multilinear Algebra" (344 References), Stuttgart 2004. The famous papers by *W.K. Clifford* were published in 1878 and 1882.

Second, Henry Cartan, son of Elie Cartan, also professor at the Sorbonne, established with 50 French mathematicians the topic of *Structure Mathematics*. In a collective series they wrote more than 20 books, first in French, then in many other languages under the pseudonym "Nicholas Bourbuki". Basically they found out that there are only three basic structures based on advanced set theory and being in interference with each other:

- Order structure
- Topological structure
- Algebraic structure

Now it is time for my examples.

Example 1. Misclosure within a local triangular network and a threedimensional Euclidean space

By Figs. 2.1–2.4 and Tables 2.1–2.6 we present a triangular network within a *threedimensional Euclidean space*. Our target is the computation of the misclosures caused by three local vertical/horizontal directions at the points $\{P_\alpha, P_\beta, P_\gamma\}$ which differ from the geometric vertical/horizontal directions. These

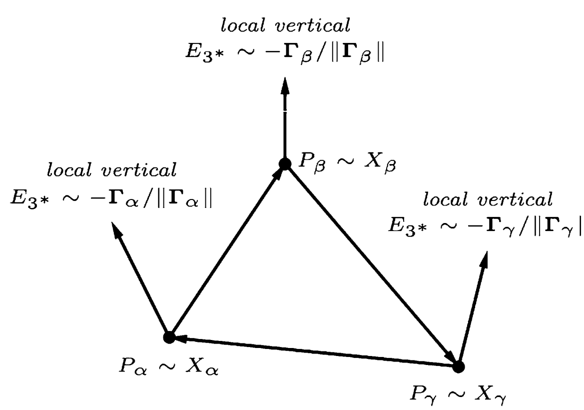

Fig. 2.1 Triangular network $\{P_\alpha, P_\beta, P_\gamma|\mathbf{O}\}$, placement vectors at the origin $\mathbf{O}$, local verticals $E_3(P_\alpha)$, $E_3(P_\beta)$, $E_3(P_\gamma)$, $\mathbf{\Gamma}_\alpha, \mathbf{\Gamma}_\beta, \mathbf{\Gamma}_\gamma$ local gravity vectors

$$\begin{tikzcd} & F^\circ \arrow[dl, "\mathbf{R}_E(\Lambda_T,\Phi_T,0)(P_\alpha)"'] \arrow[dr, "\mathbf{R}_E(\Lambda_T,\Phi_T,0)(P_\gamma)"] & \\ E_*(P_\alpha) \arrow[rr] & & E_*(P_\gamma) \end{tikzcd}$$

$$\begin{tikzcd} & F^\circ \arrow[dl, "\mathbf{R}_E(\Lambda_T,\Phi_T,0)(P_\alpha)"'] \arrow[dr, "\mathbf{R}_E(\Lambda_T,\Phi_T,0)(P_\beta)"] & \\ E_*(P_\alpha) \arrow[rr] & & E_*(P_\beta) \end{tikzcd}$$

Fig. 2.2 Commutative diagrams: *moving horizon reference systems* E_* versus *fixed equatorial reference systems* F°

$$\begin{gathered} \overset{\alpha}{X}_{\alpha\beta} := \overset{\alpha}{X}_{\beta} - \overset{\alpha}{X}_{\alpha}, \ldots, \overset{\alpha}{Z}_{\gamma\alpha} := \overset{\alpha}{Z}_{\alpha} - \overset{\alpha}{Z}_{\gamma} \\ \overset{\alpha}{X}_{\alpha\beta} + \overset{\alpha}{X}_{\beta\gamma} + \overset{\alpha}{X}_{\gamma\alpha} = 0 \\ \overset{\alpha}{Y}_{\alpha\beta} + \overset{\alpha}{Y}_{\beta\gamma} + \overset{\alpha}{Y}_{\gamma\alpha} = 0 \\ \overset{\alpha}{Z}_{\alpha\beta} + \overset{\alpha}{Z}_{\beta\gamma} + \overset{\alpha}{Z}_{\gamma\alpha} = 0 \end{gathered}$$

Fig. 2.3 Holonomity condition in terms of relative coordinates in a fixed reference system, fixed to the reference point P_α

verticals/horizontals are *not parallel to each other* causing the *anholonomity problem* or the *misclosures*. Of course, we assume *parallelism in the Euclidean sense (Euclid's axiom number five).*

Our two computations are based *first* on a holonomic reference system at the point P_α which is not operational and *second* on a realistic anholonomic reference system attached to the points $\{P_\alpha, P_\beta, P_\gamma\}$, separately. We use a local network of an extension of *25* m versus *500* m.

"anholonomity condition"

$$\overset{\alpha}{X}_{\alpha\beta} := \overset{\alpha}{X}_{\beta} - \overset{\alpha}{X}_{\alpha}, \ldots, \overset{\alpha}{Z}_{\gamma\alpha} := \overset{\alpha}{Z}_{\alpha} - \overset{\alpha}{Z}_{\gamma}$$

$$\overset{\alpha}{X}_{\alpha\beta} + \overset{\beta}{X}_{\beta\gamma} + \overset{\gamma}{X}_{\gamma\alpha} \neq 0 \quad \text{(misclosure)}$$

$$\overset{\alpha}{Y}_{\alpha\beta} + \overset{\beta}{Y}_{\beta\gamma} + \overset{\gamma}{Y}_{\gamma\alpha} \neq 0 \quad \text{(misclosure)}$$

$$\overset{\alpha}{Z}_{\alpha\beta} + \overset{\beta}{Z}_{\beta\gamma} + \overset{\gamma}{Z}_{\gamma\alpha} \neq 0 \quad \text{(misclosure)}$$

"representation of the base vectors in the horizon reference frame E^*"

$$\mathbf{E}_{1*} X_{\alpha\beta} + \mathbf{E}_{2*} Y_{\alpha\beta} + \mathbf{E}_{3*} Z_{\alpha\beta}$$

"Direct and inverse transformation of Cartesian coordinates into spherical coordinates"

(horizontal coordinate $H_{\alpha\beta}$, vertical coordinate $V_{\alpha\beta}$, distance S, azimuth $A_{\alpha\beta}$, vertical angle $B_{\alpha\beta}$, horizontal orientiation unknown)

$$\begin{gathered} X_{\alpha\beta} = S_{\alpha\beta} \cos A_{\alpha\beta} \cos B_{\alpha\beta}, \\ Y_{\alpha\beta} = S_{\alpha\beta} \sin A_{\alpha\beta} \cos B_{\alpha\beta}, \\ Z_{\alpha\beta} = S_{\alpha\beta} \sin B_{\alpha\beta} \\ A_{\alpha\beta} := H_{\alpha\beta} + \Sigma_\alpha = \arctan\left(Y_{\alpha\beta}/X_{\alpha\beta}\right), \\ B_{\alpha\beta} := V_{\alpha\beta} = \arctan\left(Z_{\alpha\beta}/\sqrt{X_{\alpha\beta} + Y_{\alpha\beta}}\right) \\ S_{\alpha\beta} := \sqrt{X_{\alpha\beta}^2 + Y_{\alpha\beta}^2 + Z_{\alpha\beta}^2} \end{gathered}$$

$$\boxed{\begin{gathered} \textit{Point transformation} \\ \mathbf{E}_*^\alpha \to \mathbf{E}_*^\beta \;\; \textit{and} \;\; \mathbf{E}_*^\beta \to \mathbf{E}_*^\gamma \end{gathered}}$$

"Euler angles"

$$\mathbf{E}^\alpha(P_\alpha) \to \mathbf{E}^\beta(P_\beta):$$

$$\mathbf{E}^\alpha(P_\alpha)\mathbf{R}_E(\Lambda_\alpha, \Phi_\alpha, 0)\mathbf{R}_E^T(\Lambda_\beta, \Phi_\beta, 0)$$

$$\Delta\Lambda := \Lambda_\beta - \Lambda_\alpha, \quad \Delta\Phi := \Phi_\beta - \Phi_\alpha$$

$$\mathbf{R}_E(\Lambda_\alpha, \Phi_\alpha, 0)\mathbf{R}_E^T(\Lambda_\beta, \Phi_\beta, 0) \doteq$$

$$\doteq \begin{bmatrix} 1 & -\Delta\Lambda \sin\Phi_\alpha & \Delta\Phi \\ \Delta\Lambda \sin\Phi_\alpha & 1 & \Delta\Lambda\cos\Phi_\alpha \\ \Delta\Phi & -\Delta\Lambda\cos\Phi_\alpha & 1 \end{bmatrix}$$

"antisymmetric matrix $\mathbf{A}$*"*

$$-\begin{bmatrix} \overset{\beta}{X}_{\beta\alpha} \\ \overset{\beta}{Y}_{\beta\alpha} \\ \overset{\beta}{Z}_{\beta\alpha} \end{bmatrix} = (\mathbf{I} + \mathbf{A}^T) \begin{bmatrix} \overset{\alpha}{X}_{\alpha\beta} \\ \overset{\alpha}{Y}_{\alpha\beta} \\ \overset{\alpha}{Z}_{\alpha\beta} \end{bmatrix}$$

analogue formulae for

$$X_{\gamma\alpha}, \ldots, Z_{\gamma\alpha}$$

analogue formulae for

$$A_{\beta\alpha},\ B_{\beta\alpha},\ S_{\beta\alpha},\ \ldots,\ A_{\gamma\alpha},\ B_{\gamma\alpha},\ S_{\gamma\alpha}$$

Fig. 2.4 Anholonomity in a moving frame at points $\{P_\alpha, P_\beta, P_\gamma\}$

Table 2.1 25 meter local network

$$\overset{\alpha}{X}_{\alpha\beta} = +30\ m,\ \overset{\alpha}{X}_{\beta\gamma} = +50\ m,\ \overset{\alpha}{X}_{\gamma\alpha} = -80\ m$$
$$\overset{\alpha}{Y}_{\alpha\beta} = +30\ m,\ \overset{\alpha}{Y}_{\beta\gamma} = -50\ m,\ \overset{\alpha}{Y}_{\gamma\alpha} = +20\ m$$
$$\overset{\alpha}{Z}_{\alpha\beta} = +5\ m,\ \overset{\alpha}{Z}_{\beta\gamma} = +15\ m,\ \overset{\alpha}{Z}_{\gamma\alpha} = -20\ m$$

$$\Lambda_{\alpha\beta} = 1'' \sim 4.85 \cdot 10^{-6}\ \text{RAD},$$
$$\Phi_{\alpha\beta} = -0.5'' \sim -2.42 \cdot 10^{-6}\ \text{RAD}$$
$$\Lambda_{\alpha\gamma} = -1'' \sim -4.85 \cdot 10^{-6}\ \text{RAD},$$
$$\Phi_{\alpha\gamma} = -2.5'' \sim -12.12 \cdot 10^{-6}\ \text{RAD}$$
$$\Phi_{\alpha} = 48.783^{\circ}$$

Relative to the origin **O** attached to the mass centre of our planet we calculate *relative Cartesian coordinates* in a "fixed equatorial reference system" transformed to a "moving horizontal reference system" as illustrated by Figs. 2.1 and 2.2. The basic *holonomity condition* is presented in Fig. 2.3, the detailed computation in Fig. 2.4 related to *realistic anholonomity*. Our results are given in Tables 2.1–2.3 for the *25* m triangular network and in *Tables 2.4–2.6* for the *500* m triangular network: They document a *misclosure* in the millimeter range for our *25* m network and in the 30 cm range for our *500* m network.

For more details let us refer to the contribution by *E. Grafarend* (1987): The influence of local verticals in local geodetic networks, *Zeitschrift für Vermessungswesenv 112 (1987) 413–424*.

Table 2.2 25 meter local network, detailed computation

$$\begin{bmatrix} \overset{\beta}{X}_{\beta\gamma} \\ \overset{\beta}{Y}_{\beta\gamma} \\ \overset{\beta}{Z}_{\beta\gamma} \end{bmatrix} = \begin{bmatrix} 1 & +\Lambda_{\alpha\beta}\sin\Phi_\alpha & +\Phi_{\alpha\beta} \\ -\Lambda_{\alpha\beta} & 1 & -\Lambda_{\alpha\beta}\cos\Phi_\alpha \\ -\Phi_{\alpha\beta} & +\Lambda_{\alpha\beta}\cos\Phi\alpha & 1 \end{bmatrix} \begin{bmatrix} \overset{\alpha}{X}_{\beta\gamma} \\ \overset{\alpha}{Y}_{\beta\gamma} \\ \overset{\alpha}{Z}_{\beta\gamma} \end{bmatrix} =$$
$$= \begin{bmatrix} 1 & +3.65\cdot10^{-6} & -2.42\cdot10^{-6} \\ -3.65\cdot10^{-6} & 1 & -3.19\cdot10^{-6} \\ +2.42\cdot10^{-6} & +3.19\cdot10^{-6} & 1 \end{bmatrix} \begin{bmatrix} +50\ m \\ -50\ m \\ +15\ m \end{bmatrix}$$
$$\begin{bmatrix} \overset{\gamma}{X}_{\gamma\alpha} \\ \overset{\gamma}{Y}_{\gamma\alpha} \\ \overset{\gamma}{Z}_{\gamma\alpha} \end{bmatrix} = \begin{bmatrix} 1 & +\Lambda_{\alpha\gamma}\sin\Phi_\alpha & +\Phi_{\alpha\gamma} \\ -\Lambda_{\alpha\gamma} & 1 & -\Lambda_{\alpha\gamma}\cos\Phi_\alpha \\ -\Phi_{\alpha\gamma} & +\Lambda_{\alpha\gamma}\cos\Phi\alpha & 1 \end{bmatrix} \begin{bmatrix} \overset{\alpha}{X}_{\gamma\alpha} \\ \overset{\alpha}{Y}_{\gamma\alpha} \\ \overset{\alpha}{Z}_{\gamma\alpha} \end{bmatrix} =$$
$$= \begin{bmatrix} 1 & -3.65\cdot10^{-6} & -12.12\cdot10^{-6} \\ +3.65\cdot10^{-6} & 1 & +3.19\cdot10^{-6} \\ +12.12\cdot10^{-6} & -3.19\cdot10^{-6} & 1 \end{bmatrix} \begin{bmatrix} -80\ m \\ +20\ m \\ -20\ m \end{bmatrix}$$

Table 2.3 25 meter local network, misclosures

$$\overset{\beta}{X}_{\beta\gamma} = +50\ m - 0.22\ mm,\quad \overset{\gamma}{X}_{\gamma\alpha} = -80\ m + 0.17\ mm$$
$$\overset{\beta}{Y}_{\beta\gamma} = -50\ m - 0.23\ mm,\quad \overset{\gamma}{Y}_{\gamma\alpha} = +20\ m - 0.36\ mm$$
$$\overset{\beta}{Z}_{\beta\gamma} = +15\ m - 0.04\ mm,\quad \overset{\gamma}{Z}_{\gamma\alpha} = -20\ m - 1.03\ mm$$

$$\overset{\alpha}{X}_{\alpha\beta} + \overset{\beta}{X}_{\beta\gamma} + \overset{\gamma}{X}_{\gamma\alpha} \neq 0: \quad -0.22\ mm + 0.17\ mm = -0.05\ mm$$
$$\overset{\alpha}{Y}_{\alpha\beta} + \overset{\beta}{Y}_{\beta\gamma} + \overset{\gamma}{Y}_{\gamma\alpha} \neq 0: \quad -0.23\ mm - 0.35\ mm = -0.59\ mm$$
$$\overset{\alpha}{Z}_{\alpha\beta} + \overset{\beta}{Z}_{\beta\gamma} + \overset{\gamma}{Z}_{\gamma\alpha} \neq 0: \quad -0.04\ mm - 1.03\ mm = -1.07\ mm$$

Table 2.4 500 meter local network

$$\overset{\alpha}{X}_{\alpha\beta}=+500\,m,\ \overset{\alpha}{X}_{\beta\gamma}=+800\,m,\ \overset{\alpha}{X}_{\gamma\alpha}=-1300\,m$$
$$\overset{\alpha}{Y}_{\alpha\beta}=+500\,m,\ \overset{\alpha}{Y}_{\beta\gamma}=-800\,m,\ \overset{\alpha}{Y}_{\gamma\alpha}=+300\,m$$
$$\overset{\alpha}{Z}_{\alpha\beta}=+50\,m,\ \overset{\alpha}{Z}_{\beta\gamma}=+150\,m,\ \overset{\alpha}{Z}_{\gamma\alpha}=-200\,m$$

$$\Lambda_{\alpha\beta} = 25'' \sim 12.12\cdot 10^{-5}\ \text{RAD},$$
$$\Phi_{\alpha\beta} = -15'' \sim -7.27\cdot 10^{-5}\ \text{RAD}$$
$$\Lambda_{\alpha\gamma} = -15'' \sim -7.27\cdot 10^{-5}\ \text{RAD},$$
$$\Phi_{\alpha\gamma} = -45'' \sim -2.18\cdot 10^{-4}\ \text{RAD}$$
$$\Phi_{\alpha} = 48.783^\circ$$

Table 2.5 500 meter local network, detailed computation

$$\begin{bmatrix}\overset{\beta}{X}_{\beta\gamma}\\ \overset{\beta}{Y}_{\beta\gamma}\\ \overset{\beta}{Z}_{\beta\gamma}\end{bmatrix} = \begin{bmatrix}1 & +\Lambda_{\alpha\beta}\sin\Phi_\alpha & +\Phi_{\alpha\beta}\\ -\Lambda_{\alpha\beta} & 1 & -\Lambda_{\alpha\beta}\cos\Phi_\alpha\\ -\Phi_{\alpha\beta} & +\Lambda_{\alpha\beta}\cos\Phi\alpha & 1\end{bmatrix}\begin{bmatrix}\overset{\alpha}{X}_{\beta\gamma}\\ \overset{\alpha}{Y}_{\beta\gamma}\\ \overset{\alpha}{Z}_{\beta\gamma}\end{bmatrix} =$$
$$= \begin{bmatrix}1 & +9.12\cdot 10^{-5} & -7.27\cdot 10^{-5}\\ -9.12\cdot 10^{-5} & 1 & -7.99\cdot 10^{-5}\\ +7.27\cdot 10^{-5} & +7.99\cdot 10^{-5} & 1\end{bmatrix}\begin{bmatrix}+800\,m\\ -800\,m\\ +150\,m\end{bmatrix}$$
$$\begin{bmatrix}\overset{\gamma}{X}_{\gamma\alpha}\\ \overset{\gamma}{Y}_{\gamma\alpha}\\ \overset{\gamma}{Z}_{\gamma\alpha}\end{bmatrix} = \begin{bmatrix}1 & +\Lambda_{\alpha\gamma}\sin\Phi_\alpha & +\Phi_{\alpha\gamma}\\ -\Lambda_{\alpha\gamma} & 1 & -\Lambda_{\alpha\gamma}\cos\Phi_\alpha\\ -\Phi_{\alpha\gamma} & +\Lambda_{\alpha\gamma}\cos\Phi\alpha & 1\end{bmatrix}\begin{bmatrix}\overset{\alpha}{X}_{\gamma\alpha}\\ \overset{\alpha}{Y}_{\gamma\alpha}\\ \overset{\alpha}{Z}_{\gamma\alpha}\end{bmatrix} =$$
$$= \begin{bmatrix}1 & -5.47\cdot 10^{-5} & -2.18\cdot 10^{-4}\\ +5.47\cdot 10^{-5} & 1 & +4.79\cdot 10^{-5}\\ +2.18\cdot 10^{-4} & -4.79\cdot 10^{-5} & 1\end{bmatrix}\begin{bmatrix}-1300\,m\\ +300\,m\\ -200\,m\end{bmatrix}$$

Example 2. How to establish an orthonormal frame in Gauss surface geometry? Is the orthonormal frame anholonomic?

Table 2.7 Gauss surface geometry, Cartan surface geometry, orthonormal frame of reference, example of the sphere

$$\mathbf{x}(u,v) = r\cos u\cos v\mathbf{e}_1 + r\sin u\cos v\mathbf{e}_2 + r\sin v\mathbf{e}_3$$
$$\mathbf{g}_1 = \frac{\partial\mathbf{x}}{\partial u} = r\cos v(-\mathbf{e}_1\sin u + \mathbf{e}_2\cos u)$$
$$\mathbf{g}_2 = \frac{\partial\mathbf{x}}{\partial v} = -r\sin v\cos u\mathbf{e}_1 - r\sin v\sin u\mathbf{e}_2 + r\cos v\mathbf{e}_3$$
$$\mathbf{g}_3 = \cos v(\mathbf{e}_1\cos u + \mathbf{e}_2\sin u) + \mathbf{e}_3\sin v$$
$$\|\mathbf{g}_1\| = r\cos v,\ \|\mathbf{g}_2\| = r,\ \|\mathbf{g}_3\| = 1,\quad \langle\mathbf{g}_1|\mathbf{g}_2\rangle = 0$$
$$\mathbf{c}_1 := \frac{\mathbf{g}_1}{\|\mathbf{g}_1\|},\quad \mathbf{c}_2 := \frac{\mathbf{g}_2}{\|\mathbf{g}_2\|},\quad \mathbf{c}_3 := \mathbf{g}_3$$

Here we concentrate to the question of *how to establish* an *orthonormal frame* $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$, for instance for the *sphere* if we refer to *Gauss surface geometry*. Is the attached *orthonormal frame a coordinate base or not*? Is the orthonormal frame *anholonomic*?

Based on an *orthogonal reference frame* $\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$ with references on spherical longitude and spherical latitude called $\{u, v\}$ we compute an *orthonormal reference frame* $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$, called *Cartan frame of reference* in Table 2.7. In Table 2.8 we introduce the *displacement* $\mathbf{dx}$ on the surface of the sphere, both in an *Gaussean frame of reference* and in the *Cartan frame of reference*. We ask the key question: Are the matrix components $\{\sigma^1, \sigma^2\}$ integrable? Table 2.8 is a very short introduction to "exterior calculus" or the *Cartan derivative*. The 3-index symbol is introduced and calculated for our example of the sphere. Naturally, the Cartan derivative is *not* integrable (Table 2.9)!

Table 2.6 500 meter local network, misclosures

$$\overset{\beta}{X}_{\beta\gamma} = +800\,m - 83.9\,mm,\quad \overset{\gamma}{X}_{\gamma\alpha} = -1300\,m + 27.2\,mm$$
$$\overset{\beta}{Y}_{\beta\gamma} = -800\,m - 84.9\,mm,\quad \overset{\gamma}{Y}_{\gamma\alpha} = +300\,m - 80.7\,mm$$
$$\overset{\beta}{Z}_{\beta\gamma} = +150\,m - 5.8\,mm,\quad \overset{\gamma}{Z}_{\gamma\alpha} = -200\,m - 297.8\,mm$$
$$\overset{\alpha}{X}_{\alpha\beta} + \overset{\beta}{X}_{\beta\gamma} + \overset{\gamma}{X}_{\gamma\alpha} \neq 0:\quad -83.9\,mm + 27.2\,mm = -56.7\,mm$$
$$\overset{\alpha}{Y}_{\alpha\beta} + \overset{\beta}{Y}_{\beta\gamma} + \overset{\gamma}{Y}_{\gamma\alpha} \neq 0:\quad -84.9\,mm - 80.7\,mm = -165.6\,mm$$
$$\overset{\alpha}{Z}_{\alpha\beta} + \overset{\beta}{Z}_{\beta\gamma} + \overset{\gamma}{Z}_{\gamma\alpha} \neq 0:\quad -5.8\,mm - 297.8\,mm = -303.6\,mm$$

Table 2.8 Displacement vector of the surface of the sphere, Gaussean frame of reference versus Cartan frame of reference, integrability

"derivational equations of the first kind"

$$\mathbf{dx} = \frac{\partial \mathbf{x}}{\partial u} du + \frac{\partial \mathbf{x}}{\partial v} dv = $$
$$= \mathbf{c}_1 r \cos v \; du + \mathbf{c}_2 r \; dv = \sigma^1 \mathbf{c}_1 + \sigma^2 \mathbf{c}_2$$

$$\begin{bmatrix} \sigma^1 \\ \sigma^2 \end{bmatrix} = \begin{bmatrix} r\cos v & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} du \\ dv \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} du \\ dv \end{bmatrix}$$

$$\begin{bmatrix} du \\ dv \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} \sigma^1 \\ \sigma^2 \end{bmatrix} = \begin{bmatrix} (r\cos v)^{-1} & 0 \\ 0 & r^{-1} \end{bmatrix} \begin{bmatrix} \sigma^1 \\ \sigma^2 \end{bmatrix}$$

$$du = (r\cos v)^{-1} \sigma^1, \quad dv = r^{-1} \sigma^2 \longleftrightarrow$$
$$\sigma^1 = r\cos v \; du, \quad \sigma^2 = r \; dv$$

? integrability ?

$$\frac{\partial \sigma^1}{\partial v} - \frac{\partial \sigma^2}{\partial u} = -r \sin v \neq 0$$

$$\begin{bmatrix} \sigma^1 \\ \sigma^2 \end{bmatrix} =: \underline{\sigma}, \quad d\underline{u} := \begin{bmatrix} du \\ dv \end{bmatrix}$$

"if $\underline{\sigma} = A \, d\underline{u}$, then $d_C \sigma = dA \wedge d\underline{u}$"

Example 3. Discussion between *A. Marussi* and *C. Mineo* and the development of Differential Geodesy

Let us refer to the discussion of *A. Marussi* (1952): Intrinsic geodesy, The Ohio State Research Foundation, Project No. 485, Columbus/Ohio/USA 1952, *C. Mineo* (1955): Intrinsic geodesy and general properties of cartographic representations, Rend. Acc. Naz. Lincei, Cl. di Sc. Fis., Mat. e Nat., Serie 18, fasc. 6 and *A. Marussi* (1955): A reply to a note by *C. Mineo*, see C. Mineo *19*, fasc. 5 in order to document these discussions in the past to accept "Differential Geodesy" as a subject of science.

The subject of *Marussian Geodesy* was established in my paper *E. Grafarend* (1978): *Marussian Geodesy*, pages 209–247, Boll. di Geodesia e Scienze Affini, No. 23, April-Septembre 1978. Refer, in addition, to our contribution "Elie Cartan and Geodesy" by *F. Bocchio, E. Grafarend, N. Grossmann, J.G. Leclerc and A. Marussi* (1978): *Elie Cartan and Geodesy*. Boll. di Geodesia e Scienze Affini, No. 4, August-October 1978, presenting five papers given at *sixth symposium of mathematical geodesy* (third Hotine Symposium) held at *Siena/Italy*, April 2–5, 1975.

Example 4. Projective heights in geometry and gravity space, the work of *Antoni Marussi*

Satellite positioning in terms of Cartesian coordinates $(X, Y, Z) \in \mathbb{T}^2 \subset \mathbb{E}^3$ establishing a triplet of

Table 2.9 1-differential forms, exterior calculus, Cartan derivative

"1-differential form"

$$\sigma^1 = a \; du + b \; dv = a_\alpha du^\alpha$$
$$\sigma^2 = c \; du + d \; dv = b_\alpha du^\alpha$$

"exterior" or E. Cartan-derivative

$$d_C \sigma^\alpha = \sum_{\beta,\gamma=1}^{2} \frac{1}{2!} \Omega^\alpha_{\beta\gamma} \sigma^\beta \wedge \sigma^\gamma \sum_{\beta<\gamma} \Omega^\alpha_{\beta\gamma} \sigma^\beta \wedge \sigma^\gamma$$

! anti-symmetry !

$$du^\alpha \wedge du^\beta = -du^\beta \wedge du^\alpha$$
$$\sigma^\alpha = a^\alpha_\beta \; du^\beta \sim \underline{\sigma} = A \, d\underline{u}$$

three-index-symbol

$$d_C \sigma^\alpha = \sum_{\beta\gamma}^{\alpha} \sigma^\beta \wedge \sigma^\gamma$$

$$d\sigma^1_C = \frac{1}{2} \Omega^1_{12} \sigma^1 \wedge \sigma^2 + \frac{1}{2} \Omega^1_{21} \sigma^2 \wedge \sigma^1$$
$$= \frac{1}{2} (\Omega^1_{12} - \Omega^1_{21}) \sigma^1 \wedge \sigma^2 = \Omega^1_{12} \sigma^1 \wedge \sigma^2$$
$$d\sigma^2_C = \frac{1}{2} \Omega^2_{12} \sigma^1 \wedge \sigma^2 + \frac{1}{2} \Omega^2_{21} \sigma^2 \wedge \sigma^1$$
$$= \frac{1}{2} (\Omega^2_{12} - \Omega^2_{21}) \sigma^1 \wedge \sigma^2 = \Omega^2_{12} \sigma^1 \wedge \sigma^2$$

$$\boxed{\sigma^1 \wedge \sigma^2 = -\sigma^2 \wedge \sigma^1}$$

comparison of coefficients

$$\Omega^1_{12} = -\Omega^1_{21} = \tfrac{1}{r} \tan v$$
$$\Omega^2_{12} = -\Omega^2_{21} = 0$$

assumption: $d_C d\underline{u} = 0$

$$\text{proof: } d_C d\underline{u} = \left(\frac{\partial du}{\partial v} - \frac{\partial dv}{\partial v} \right) dv \wedge du = 0$$

$$d_C \underline{\sigma} = dA \wedge du = \begin{bmatrix} -r \sin v \; dv \wedge du \\ 0 \end{bmatrix}$$

$$d_C \sigma = \begin{bmatrix} -\tfrac{1}{r} \tan v \; \sigma^2 \wedge \sigma^1 \\ 0 \end{bmatrix} \neq 0$$

"summary"

$$\boxed{\begin{aligned} d\sigma^\alpha_C &= \sum_{\beta,\gamma=1}^{2} \Omega^\alpha_{\beta\gamma} \sigma^\beta \wedge \sigma^\gamma \\ &= \sum_{\beta<\gamma} \Omega^\alpha_{\beta\gamma} \sigma^\beta \wedge \sigma^\gamma \end{aligned}}$$

Table 2.10 Projective heights in gravity space, geodesics

"stationary functional"

$$\delta \int_{s_1}^{s_2} ds = \delta \int_{\tau_1}^{\tau_2} \sqrt{g_{k\ell}(x^m)\frac{dx^k}{d\tau}\frac{dx^\ell}{d\tau}}\, d\tau = 0, \quad k, \ell, m \in \{1, 2, 3\}$$

subject to a conformally flat metric

$$g_{k\ell}(x^m) = \gamma^2(x^m)\delta_{k\ell}$$

"Lagrange equations"

$$\mathcal{L}\left(x^m, \frac{dx^m}{d\tau}\right) := \sqrt{\delta_{k_1\ell_1}\gamma^{k_1}\gamma^{\ell_1}}\sqrt{\delta_{k_2\ell_2}\frac{dx^{k_2}}{d\tau}\frac{dx^{\ell_2}}{d\tau}} =$$

$$= \sqrt{(\gamma^1)^2 + (\gamma^2)^2 + (\gamma^3)^2}\sqrt{\left(\frac{dx^1}{d\tau}\right)^2 + \left(\frac{dx^2}{d\tau}\right)^2 + \left(\frac{dx^2}{d\tau}\right)^2}$$

$$\delta \int_{\tau_1}^{\tau_2} \mathcal{L}\left(x^m, \frac{dx^m}{d\tau}\right) d\tau = 0 \Longleftrightarrow \frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{dx^k}{d\tau}\right)}\right) - \frac{\partial \mathcal{L}}{\partial x^k} = 0$$

$$\frac{\partial \mathcal{L}}{\partial\left(\frac{dx^k}{d\tau}\right)} = \frac{1}{2}\frac{\partial \mathcal{L}^2}{\partial\left(\frac{dx^k}{d\tau}\right)} = \frac{\sqrt{\delta_{k_1\ell_1}\gamma^{k_1}\gamma^{\ell_1}}}{\sqrt{\delta_{k_2\ell_2}\frac{dx^{k_2}}{d\tau}\frac{dx^{\ell_2}}{d\tau}}}\frac{dx^k}{d\tau} =$$

$$= \frac{\sqrt{(\gamma^1)^2 + (\gamma^2)^2 + (\gamma^3)^2}}{\sqrt{\left(\frac{dx^1}{d\tau}\right)^2 + \left(\frac{dx^2}{d\tau}\right)^2 + \left(\frac{dx^2}{d\tau}\right)^2}}\frac{dx^k}{d\tau}$$

$$\frac{\partial \mathcal{L}}{\partial x^k} = \frac{1}{2\mathcal{L}}\frac{\partial \mathcal{L}^2}{\partial x^k} = \frac{\sqrt{\delta_{k_2\ell_2}\frac{dx^{k_2}}{d\tau}\frac{dx^{\ell_2}}{d\tau}}}{\sqrt{\delta_{k_1\ell_1}\gamma^{k_1}\gamma^{\ell_1}}}\frac{1}{2}\partial_k(\delta_{k_3\ell_3}\gamma^{k_3}\gamma^{\ell_3}) =$$

$$= \frac{\sqrt{\left(\frac{dx^1}{d\tau}\right)^2 + \left(\frac{dx^2}{d\tau}\right)^2 + \left(\frac{dx^2}{d\tau}\right)^2}}{\sqrt{(\gamma^1)^2 + (\gamma^2)^2 + (\gamma^3)^2}}\frac{1}{2}\partial_k\gamma^2(x^m)$$

"transformation from τ to s"

(*affine parameter*)

$$\left.\begin{aligned} \frac{dx^k}{d\tau} &= \frac{dx^k}{ds}\frac{ds}{d\tau} = x^{\ell k}\frac{ds}{d\tau} \\ \frac{ds}{d\tau} &= \sqrt{g_{k\ell}(x^m)\frac{dx^k}{d\tau}\frac{dx^\ell}{d\tau}} \end{aligned}\right] \Rightarrow$$

$$\boxed{\begin{aligned} g_{k\ell}(x'')^\ell + [k\ell, m](x')^\ell(x')^m = 0 \\ \gamma^2(x'')^k + (\partial_\ell\gamma^2)(x')^k(x')^\ell - \tfrac{1}{2\gamma^2}\partial_k\gamma^2 = 0 \end{aligned}}$$

"Marussi gauge"

$$\delta_{k_1\ell_1}\gamma^{k_1}\gamma^{\ell_1} := \delta_{k_2\ell_2}\dot{x}^{k_2}\dot{x}^{\ell_2} \text{ or } (\gamma^1)^2 + (\gamma^2)^2 + (\gamma^3)^2 = (\dot{x}^1)^2 + (\dot{x}^2)^2 + (\dot{x}^3)^2$$

$$ds^2 = g_{k\ell}(x^m)dx^k dx^\ell = \gamma^2(x^m)\delta_{k\ell}dx^k dx^\ell = \gamma^2(x^m)((\dot{x}^1)^2 + (\dot{x}^2)^2 + (\dot{x}^3)^2)dt^2$$

$$\boxed{ds = \gamma^2(x^m)dt}$$

"transformation from s to t"

$$\boxed{\ddot{x}^k - \tfrac{1}{2}\partial_k\gamma^2(x^m) = 0}$$

"representation of the gradient of the factor of conformality in terms of gravity gradients"

$$\boxed{\begin{aligned} \tfrac{1}{2}\partial_k\gamma^2(x^m) &= \delta_{k_1\ell_1}\gamma^{k_1}\partial_k\gamma^{\ell_1} = \gamma^1\partial_k\gamma^1 + \gamma^2\partial_k\gamma^2 + \gamma^3\partial_k\gamma^3 = \\ &= \partial_1 w\partial_k\partial_1 w + \partial_2 w\partial_k\partial_2 w + \partial_3 w\partial_k\partial_3 w \end{aligned}}$$

Cartesian coordinates for *quantifying the position* of a topographic point requires a *complete redefinition* of geodetic projective heights in geometry and gravity space, namely with respect to a *deformable Earth body*. Such a redefinition has been presented *in two steps*:

(*i*) Projective heights are based upon projective lines which are

- (*i1*) *geodesics* (straight lines) in a Euclidean geometric space, *or*
- (*i2*) *geodesics* (plumblines/orthogonal trajectories with respect to a family of equipotential surfaces) in gravity space in a *conformally flat manifold*, the *Marussi manifold* with the modulus of gravity as the *factor of conformality*.

(*ii*) Projective heights are based upon a *minimal distance mapping* along those geodesics between a topographic point $(X, Y, Z) \in \mathbb{T}^2 \subset \mathbb{E}^3$ and a *reference surface*:

- (*ii1*) For projective heights in geometry space such as *standard reference surfaces* (two-dimensional Riemann manifolds) are the plane $\mathbb{P}^2$, the sphere $\mathbb{S}^2$ or the ellipsoid of revolution $\mathbb{E}^2_{a,b}$,
- (*ii2*) for projective heights in gravity space the standard reference surface is identifies by the reference equipotential surface, the *Geoid at some reference epoch* $t_0 \in \mathbb{R}$.

Here we review by Table 2.10 the variational calculus or the standard optimization routine to generate a *minimal distance mapping* between points on the topography and the reference surface, in particular the corresponding algorithm. We have referred to the *problem of holonomity of orthometric heights, normal orthometric heights* ("slightly anholonomic") for a "star-shaped gravity space" and of steric levelling heights ("pressure heights") in our contribution by *E. Grafarend, R. Syffus and R.J. You* (dedicated to the memory of *Antonio Marussi*) in "Allgemeine Vermessungsnachrichten (1995) 382–403".

Last, not least, I thank *Joseph Zund* for all previous discussions on *anholonomity*. We recommend to the reader to study his masterly written book *J. Zund* (1994): Foundations of Differential Geodesy, Springer Verlag, Berlin-Heidelberg-New York 1994 in which *Local Differential Geodesy* and *Global Geodesy in the Large* are elegantly described. We advice the reader also to study his *The work of Antonio Marussi*, Academia Nazionale dei Lincei, Atti dei Convegni Lincei, Report 91, Roma 1991, pages 9–20. Here the mathematical background as well as the geodetic background of A. Marussi based on interviews with *Mrs. Dolores Marussi de Finetti, Ian Reilly and his own research* are presented.

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