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Geometry of Minkowski Space–Time

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Preface

We have written this book with the intention of providing the students (and the teachers) of the first years of university courses with a tool which is easy to be applied and allows the solution of any problem of relativistic kinematics at the same time.

The novelty of our presentation consists of the extensive use of hyperbolic numbers for a complete formalization of the kinematics in the Minkowski space–time.

In other words, in this book the mathematical relation, stated by special relativity, between space and time is formalized.

We recall from Paul Davies book [1], the different significances attributed to "time" over the centuries:

For millennia the traditional cultures have given to time an intuitive meaning. Its cyclic nature and biological rhythms predominate over its measure and time and eternity are complementary concepts.

Before Galileo and Newton, the time was subjective, not a parameter we have to measure with geometrical precision.

Newton encapsulated it in the World description just as a parameter for the mathematical description of the motion: practically the time did nothing.

Einstein has given it again its place in the heart of the Nature, as a fundamental part of the physics.

Einstein did not complete this revolution that, unfortunately, remained unfinished.

To the last sentence Einstein would have most probably replied [2] that the physical laws will never be the definitive ones. All scientists can step forwards in the advancement of the scientific knowledge, but they are sure that the results obtained cannot be the definitive ones.

The achievement of special relativity, to which P. Davies refers, can be summarized as: *space and time must be considered as equivalent quantities for the description of physical laws.*

In this book we formalize this equivalence.

Actually, even if just after the formulation of special relativity, Hermann Minkowski proposed (1907–1908) that the relation between space and time can be considered as a new geometry, but a mathematics that would allow us to operate with this geometry as we do with Euclidean geometry was not formalized yet.

This formalization has been carried out by the authors in a series of papers [3, 4], later rearranged in a book [5] where, besides the well-established aforesaid formalization, some themes of research are proposed.

The aim of the present book is supplying the tools for solving problems in space–time in the same "automatic way" as problems of analytic geometry and trigonometry are solved in secondary schools. The previous knowledge of mathematics which is required is the same required in the first year of scientific University courses.

Further we show the basic ideas of our treatment and how these ideas derive from the "scientific revolutions" of 19th and 20th centuries: in particular, the necessary link between mathematics and physics and the synergic effects that allow their development.

The papers and the books listed in the bibliography are not indispensable for understanding the contents of the book, but they can help people who want to carry on further research. Actually, even if the mathematics used can be considered elementary, the topics dealt with are the subject of research in progress. What we are saying is that an appropriate reconsideration of some points of elementary mathematics and geometry can be the starting point for obtaining original and valuable results.

Rome, January 2011

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Contents

Chapter 1
Introduction

It is largely known that the Theory of Special Relativity was born as a consequence of the demonstrated impossibility for the Maxwell's electromagnetic (e.m.) theory of obeying Galilean transformations. The non-invariance of the e.m. theory under Galilean transformations induced the theoretical physicists, at the end of the twelfth century, to invent new space–time transformations which did not allow to consider the time variable as "absolutely" independent of the space coordinates. It was thus that the transformations which, for the sake of brevity, today we call Lorentz transformations were born.[1]

A consequence of the choice of the Lorentz transformations as the ones which keep the e. m. theory invariant when passing from an inertial system to another one (e.g. the e.m. waves remain e.m. waves) was to revolutionize the kinematics, i.e. the basis from which one must start for building the mechanics. It was no longer possible, at least for high velocities, to use Galilean transformations: when passing from an inertial system to another one, also the time variable (until when considered as "the independent variable") was forced to be transformed together with the space coordinates.

After the works of Lorentz and Poincaré, it was Einstein's turn to build a relativistic mechanics starting from the kinematics based on Lorentz transformations.

However, we owe to Hermann Minkowski the creation of a four-dimensional geometry in which the time entered as the fourth coordinate: the Minkowski space–time as it is called today.

We recall that for a long period of time the introduction of "i t" (where "i" is the imaginary unit) as the fourth coordinate was into use with the aim of providing to space–time a pseudo-Euclidean structure.

We shall see in this book that it is not the introduction of an imaginary time, but of a system of numbers (the hyperbolic numbers) related in many respects with complex numbers, that can describe the relation (symmetry) between space and

[1] An exhaustive account of the subject and its historical context can be found in the book—Arthur I. Miller: Albert Einstein's Special Theory of Relativity—Springer, 1998

F. Catoni et al., *Geometry of Minkowski Space–Time*, SpringerBriefs in Physics,
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time. Moreover this system of numbers allows one a mathematical formalization that, from a logical point of view (an axiomatic–deductive method starting from axioms of empirical evidence) as well as from a practical one (the problems are solved in the same automatic way as the problems of analytical geometry and trigonometry are), is equivalent to the analytical formalization of Euclidean geometry.

Even though Minkowski already introduced hyperbolas in place of calibration circles and a copious literature on the subject does exist, until now a formalization, rigorous and extremely simple at the same time, was not obtained.

In this book we explain how to treat any problem of relativistic kinematics. The expounded formalization allows one to reach an exhaustive and non-ambiguous solution.

The final appendix, while contains a short outline on the evolution of the concept of geometry, helps us to reflect upon the nature of the operation made by the authors: a Euclidean way for facing a non-Euclidean geometry.

Chapter 2
Hyperbolic Numbers

Abstract Complex numbers can be considered as a two components quantity, as the plane vectors. Following Gauss complex numbers are also used for representing vectors in Euclidean plane. As a difference with vectors the multiplication of two complex numbers is yet a complex number. By means of this property complex numbers can be generalized and hyperbolic numbers that have properties corresponding to Lorentz group of two-dimensional Special Relativity are introduced.

Keywords Complex numbers · Gauss-Argand · Generalization of complex numbers · Hyperbolic numbers · Space-time geometry · Lorentz group

Complex numbers represent one of the most intriguing and emblematic discoveries in the history of science. Even if they were introduced for an important but restricted mathematical purpose, they came into prominence in many branches of mathematics and applied sciences. This association with applied sciences generated a synergistic effect: applied sciences gave relevance to complex numbers and complex numbers allowed formalizing practical problems. A similar effect can be found today in the "system of hyperbolic numbers", which has acquired the meaning and importance as the *Mathematics of Special Relativity*, as shown in this book.

Let us recall some points from the history of complex numbers and their generalization.

Complex numbers are today introduced with the purpose of extending the field of real numbers and for having always two solutions for the second degree equations and, as an important applicative example, we recall the Gauss *Fundamental theorem of algebra* stating that "all the algebraic equations of degree N has N real or imaginary roots". Further Gauss has shown that complex and real numbers are adequate for obtaining all the solutions for any degree equation.

Coming back to complex numbers we now recall how their introduction has a practical reason. Actually they were introduced in the 16th century for solving a

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mathematical paradox: to give a sense to the real solutions of cubic equations that appear as the sum of square roots of negative quantities (see Sect. 2.6.1). Really the goal of mathematical equations was to solve practical problems, in particular geometrical problems, and if the solutions were square roots of negative quantities, as can happen for the second degree equations, it simply meant that the problem does not have solutions. Therefore it was unexplainable that the real solutions of a problem were given by some "imaginary quantities" as the square roots of negative numbers.

Their introduction was thorny and the square roots of negative quantities are still called *imaginary numbers* and contain the symbol "i" which satisfies the relation $\mathrm{i}^2 = -1$. *Complex numbers* are those given by the symbolic sum of one real and one imaginary number $z = x + \mathbf{i}y$. This sum is a symbolic one because it does not represent the usual sum of "homogeneous quantities", rather a "two components quantity" written as $z = \mathbf{1}x + \mathrm{i}y$, where $\mathbf{1}$ and i identify the two components.

Today we know another two-component quantity: the plane vector, which we write $\mathbf{v} = \mathbf{i}x + \mathbf{j}y$, where $\mathbf{i}$ and $\mathbf{j}$ represent two unit vectors indicating the coordinate axes in a Cartesian representation. Despite there being no a priori indication that a complex number could represent a vector on a Cartesian plane, complex numbers were the first representation of two-component quantities on a Cartesian (or Gauss–Argand) plane (see Fig. 2.1), and they are also used for representing vectors in a Euclidean Cartesian plane.

Now we can ask: what are the reasons that allow complex numbers to represent plane vectors? The answer to this question has allowed us to formalizing the geometry and trigonometry of Special Relativity space–time.

2.1 The Geometry Associated with Complex Numbers

Let us now recall the properties that allow us to use complex numbers for representing plane vectors. The first property derives from the invariant of complex numbers the modulus, indicated with $|z|$, and given by $|z| = \sqrt{(x + \mathrm{i}y)(x - \mathrm{i}y)} \equiv \sqrt{x^2 + y^2}$.

An important property of the modulus is: given two complex numbers z_1, z_2, we have $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$.

If we represent the complex number $x + \mathrm{i}y$ as a point $P \equiv (x, y)$ of the Gauss–Argand plane (Fig. 2.1), the quantity $\sqrt{x^2 + y^2}$ represents the distance of P from the coordinates origin. This quantity is invariant with respect to translations and rotations of the coordinate axes. Now if in $z = x + \mathrm{i}y$, we give to 1 and i the same meaning of $\mathbf{i}, \mathbf{j}$ in the vectors representation, $|z|$ is the modulus of the vector.

In addition another relevant property allows complex numbers representing plane vectors and the related linear algebra. Actually let us consider the product of a complex constant, $a = a_r + \mathrm{i}a_i$ by a complex number: