NONLINEAR
PHYSICAL
SCIENCE

Albert C.J. Luo
Jian-Qiao Sun
Editors

Complex Systems

Fractionality, Time-delay and Synchronization

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NONLINEAR PHYSICAL SCIENCE

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Nonlinear Physical Science focuses on recent advances of fundamental theories and principles, analytical and symbolic approaches, as well as computational techniques in nonlinear physical science and nonlinear mathematics with engineering applications.

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Preface

This edited book covers recent developments on fractional dynamics, time-delay systems, system synchronization, and neuron dynamics.

Fractional calculus is extensively used as a powerful tool to investigate complex phenomena in engineering and science, and has received renewed attention recently. Chapter 1 of the book investigates fractional dynamics of complex systems. Some recent results and applications in fractional dynamics are presented. In Chapter 2, the synthesis and application of fractional-order controllers are presented. This is an active area of research. The fractional-order *PID* controllers are designed for the velocity control of an experimental modular servo system. The system consists of a digital servomechanism and open-architecture software environment for real-time implementation. Experimental results of fractional-order controllers are presented and analyzed. The effectiveness and superior performance of the fractional-order controls are compared with classical integer-order *PID* controllers.

Time delay is a common phenomenon in engineering, economical and biological systems, and has become a popular research topic in recent years. In Chapter 3, equilibrium stability, Lindsedt's method and Hopf bifurcation, and transient behaviors in differential-delay equations are presented. Multiple-scale and the center manifold analysis are addressed. These methods are applied to investigate dynamical behaviors of a differential-delay system modeling a section of the DNA molecule. Chapter 4 focuses on the methodologies for time-domain solutions and control design of time-delayed systems. Method of semi-discretization and continuous time approximation are discussed. The spectral properties of the methods will be investigated. A comparative study of stability of time-delayed linear time invariant systems is carried out by the Lyapunov method, Pad approximation and semi-discretization. The methods of solution for stochastic dynamical systems with time delay are also discussed, and a number of control examples and an experimental validation are presented.

Chapter 5 develops a theory for synchronization of multiple dynamical systems under constraints. The metric functionals based on the constraints are introduced to describe the synchronicity of two or more dynamical systems. The chapter provides a theoretic framework for designing controllers of slave systems which can be synchronized with master systems.

Finally in Chapter 6, complex dynamics of neurons with time-delay, stochasticity and impulsive discontinuity are presented. Complex dynamical behaviors include periodic spiking, chaotic spiking, periodic and chaotic bursting, and synchronization. In this chapter, a comprehensive review on recent developments and new results in nonlinear neural dynamics are presented.

It is our hope that the book presents a reasonably broad view of the state-of-the-art of complex systems, and provides a useful reference volume to scientists, engineers and students. Furthermore, we hope that the book will stimulate more researches in the rapidly evolving and interesting field of complex systems.

Edwardsville, Illinois — Albert C.J. Luo
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Chapter 1
New Treatise in Fractional Dynamics

Dumitru Baleanu

Abstract Fractional calculus becomes a powerful tool used to investigate complex phenomena from various fields of science and engineering. In this context, the researchers paid a lot of attention for the fractional dynamics. However, the fractional modeling is still at the beginning of its developing. The aim of this chapter is to present some new results in the area of fractional dynamics and its applications.

1.1 Introduction

Fractional calculus deals with the generalization of differentiation and integration to non-integer orders. Fractional calculus is as old as the classical one and it has gained importance during the last few decades in various fields of science and engineering (Oldham and Spanier, 1974; Miller and Ross, 1993; Samko et al., 1993; Podlubny, 1999; Hilfer, 2000; Zaslavsky, 2005; Magin, 2000; Kilbas et al., 2006; West et al., 2003; Uchaikin, 2008; Lakshmikantham et al., 2009).

The fractional derivatives are the infinitesimal generators of a class of translation invariant convolution semigroups which appear universally as attractors. The fractional derivative at a point x is a local property only when α is an integer. Since the fractional derivatives represent the generalization of the classical ones, some of the classical properties are lost, e.g. the fractional Leibniz rule and the chain rule become more complicated than the classical counterparts (Oldham and Spanier, 1974; Miller and Ross, 1993; Samko et al., 1993; Podlubny, 1999; Kilbas et al., 2006).

Several applications of fractional calculus were simply based on replacing the time derivative in an evolution equation with a given derivative of fractional order.

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Various recent results confirm that fractional derivatives seem to arise for important mathematical reasons (Podlubny, 1999; Hilfer, 2000; Zaslavsky, 2005; Magin, 2000; Kilbas et al., 2006; West et al., 2003; Uchaikin, 2008; Lakshmikantham et al., 2009; Gorenflo and Mainardi, 1997; Heymans and Podlubny, 2006; Mainardi et al., 2001; Scalas et al., 2004; Jesus and Machado, 2008; Chen et al., 2004; Barbosa et al., 2004; Carpinteri and Mainardi, 1997; Solomon et al., 1993; Fogleman et al., 2001; Nigmatullin and Mehaute, 2005; Momani, 2006; Tarasov, 2006, 2005; Zaslavsky, 2002; Lorenzo and Hartley, 2004; Baleanu et al., 2009a,b; Magin et al., 2008; Magin, 2009; Silva et al., 2008; Trujillo, 1999; Maraaba et al., 2008a,b; Baleanu et al., 2008a; Baleanu and Muslih, 2005a; Caputo, 2001; Lim and Muniandy, 2004; Mainardi et al., 2001; Mainardi, 1996; Tenreiro Machado, 2003, 2001; Metzler et al., 1995).

Based on the fact that the diffusion can be described by fractional differential equations, we ask the following questions:

Are mathematical models with fractional space and/or time derivatives consistent with the fundamental laws of nature? How can the fractional order of differentiation be observed experimentally?

Recently the fractional order differential equations started to play an important role in modeling the anomalous dynamics of various processes related to complex systems in the most diverse areas of science and engineering. However, only a few steps have been taken toward what may be called a coherent theory of these equations in the applied sciences (Oldham and S'panier, 1974; Miller and Ross, 1993; Samko et al., 1993; Podlubny, 1999; Hilfer, 2000; Kilbas, 2006; Uchaikin, 2008; Lakshmikantham, 2009).

The fractional Lagrangian and Hamiltonian are typical examples of non-local theories which were investigated in several physical problems (Pais and Uhlenbeck, 1950; Gomis et al., 2004, 2001; Gomis and Mehen, 2000; Llosa and Vives, 1994; Bering, online). Besides, a Hamilton formalism for nonlocal Lagrangian was proposed in Llosa and Vives (1994) and Bering (online), an equivalent singular first order Lagrangian was obtained and the corresponding Hamiltonian was pulled back on the phase space by making use of the corresponding constraints (Llosa and Vives, 1994). It was shown the space-time non-commutative field theories are acausal and the unitarity is lost (Seiberg et al., 2000; Alvarez-Gaume and Barbon, 2001).

The fractional variational principles represent an important part of fractional calculus and it is connected to the fractional quantization procedure (Riewe, 1996, 1997; Klimek, 2001; Klimek, 2002; Agrawal, 2002; Tarasov and Zaslavsky, 2006; Agrawal and Baleanu, 2007; Agrawal, 2006, 2007; Baleanu and Agrawal, 2006; Baleanu and Avkar, 2004; Baleanu, 2009; Rabei et al., 2009; Baleanu et al., 2008b; Baleanu, 2006, 2008; Baleanu and Trujillo, 2008). There are several proposed methods to obtain the fractional Euler-Lagrange equations and the corresponding Hamiltonian (Baleanu et al., 2008a,b; Rabei et al., 2007; Baleanu and Muslih, 2005b; Muslih and Baleanu, 2005a; Baleanu et al., 2006). However, this issue has not yet completely clarified and it requires more further analysis.

Quantization of systems with fractional derivatives is a novel area in the application of fractional differential and integral calculus. The interest in fractional quantization appears simply because it describes both conservative systems and non-conservative systems (Muslih and Baleanu, 2005b; Lim and Teo, 2009).

Schrödinger equation was considered with the first-order time derivative modified to Caputo fractional ones in Naber (2004), Dong and Xu (2008) and Jumarie (2009). However, in this case the obtained Hamiltonian was found to be non-Hermitian and non-local in time and the obtained wave functions are not invariant under the time reversal. The quantization of fractional Klein-Gordon field and fractional electromagnetic potential in the Coulomb gauge and the temporal gauge were subjected of intense debate (Lim and Teo, 2009).

The necessary conditions for the optimality in optimal control problems with dynamics described by differential equations of fractional order were obtained (Agrawal and Baleanu, 2007; Agrawal, 2004; Baleanu et al., 2009). By making use of an expansion formula for fractional derivative, optimality conditions and a new solution scheme is proposed.

It was proved that the fractional calculus models with differential equations can describe more complex biological systems by extending the scales (time and space) over which the models are effective and thus expand the range of phenomena under study.

The fractional wavelet transform (Unser and Blu, 2000b, 2002, 1999, 2000a) represents a new and important mathematical tool for signal and image analysis. The fractional wavelet analysis (Dinç and Baleanu, 2006, 2010) and the combination of this method with some other standard ones (Walczak, 2000; Dinç and Baleanu, 2007, 2004b,a; Dinç et al., 2003) were proposed very recently in order to investigate the composite signals of the components in complex drug mixtures.

This chapter is based mainly on the results obtained by the author and his collaborators in various fields of fractional calculus and its application. The chapter is organized as follows:

In Section 1.2 the main definitions and the properties of the fractional calculus are presented.

Section 1.3 is dedicated to the fractional variational principles and their applications.

Section 1.4 contains a brief review of the fractional optimal control formulation.

Section 1.5 is devoted to the application of the fractional calculus in nuclear magnetic resonance.

Fractional wavelet method and its applications in drug analysis are illustrated briefly in Section 1.6.

1.2 Basic definitions and properties of fractional derivatives and integrals

In this section we present the basic definitions of the fractional derivatives and integrals (Oldham and Spanier, 1974; Miller and Ross, 1993; Samko et al., 1993; Podlubny, 1999; Kilbas et al., 2006).

Definition 1. *Riemann-Liouville left-sided fractional integral of order* α is given by

$$ {}_a\mathbf{D}_x^{-\alpha}\phi(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\phi(t)dt, \;\; x > a. \tag{1.1}$$

Definition 2. *Riemann-Liouville right-sided fractional integral of order* α has the form

$$ {}_x\mathbf{D}_b^{-\alpha}\phi(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}\phi(t)dt, \;\; x < b. \tag{1.2}$$

Here $\alpha > 0$ and $\Gamma(\alpha) = \int_0^\infty s^{\alpha-1}e^{-s}ds$ denotes the Gamma function.

Having defined the fractional integral, define the fractional derivative as the inverse operation, namely

$$ {}_a\mathbf{D}_x^{\alpha}\,{}_aD_x^{-\alpha}\phi = \phi, \tag{1.3}$$

$$ {}_a\mathbf{D}_x^{\mu}\phi = \tfrac{d^N}{dx^N}[{}_aD_x^{-\alpha}], \tag{1.4}$$

where $\alpha = N - \mu$, N is the smallest integer bigger than μ and $\mu > 0$.

Definition 3. *Left Riemann-Liouville fractional derivative of order* α is defined as

$$ {}_a\mathbf{D}_x^{\alpha}\phi(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x \frac{\phi(t)}{(x-t)^{\alpha+1-n}}dt, x > a. \tag{1.5}$$

Definition 4. *Right Riemann-Liouville fractional derivative of order* α becomes

$$ {}_x\mathbf{D}_b^{\alpha}\phi(x) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^n \int_x^b \frac{\phi(t)}{(t-x)^{\alpha+1-n}}dt, \;\; x < b, \tag{1.6}$$

$n - 1 \leq \alpha < n$, and $\alpha > 0$.

Note that, for $\alpha = 1, 2, ...$ we have

$$ {}_a\mathbf{D}_x^{\alpha} = \left(\frac{d}{dx}\right)^\alpha, \tag{1.7}$$

$$ {}_x\mathbf{D}_b^{\alpha} = \left(-\frac{d}{dx}\right)^\alpha. \tag{1.8}$$

Definition 5. *The left Caputo fractional derivative* is defined as

$$ {}_a^CD_x^{\alpha}\phi(x) = {}_aD_x^{\alpha}\left(\phi(x) - \sum_{k=0}^{n-1}\frac{\phi^{(k)}(a)}{k!}(x-a)^k\right), \tag{1.9}$$

or

$$ {}_a^CD_x^{\alpha}\phi(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{\phi^n(t)}{(x-t)^{\alpha+1-n}}dt, \tag{1.10}$$

Definition 6. *The right Caputo fractional derivative* is defined as

$$ {}_x^C D_b^\alpha \phi(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \frac{\phi^n(t)}{(t-x)^{\alpha+1-n}} dt, \tag{1.11} $$

where $0 \leq n-1 < \alpha < n$ and $\phi(x)$ has n+1 continuous and bounded derivatives in $[a, b]$.

We notice that

$$ {}_a^C D_x^\alpha A = 0, \quad (A = constant) \tag{1.12} $$

and

$$ \lim_{x \to a} {}_a^C D_x^\alpha \phi = 0. \tag{1.13} $$

In an infinite domain we have the following results

$$ {}_{-\infty}^{\;\;C} D_x^\alpha \phi = {}_{-\infty}\mathbf{D}_x^\alpha \phi, {}_x^C D_\infty^\alpha \phi = {}_x\mathbf{D}_\infty^\alpha \phi. \tag{1.14} $$

Let us consider a function $f(x_1, x_2)$.

Definition 7. *A partial left Riemann-Liouville fractional derivative* of order α_2, $0 < \alpha_2 < 1$, in the second variable is defined as

$$ (\mathbf{D}_{a_2+}^{\alpha_2} f)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial x_2} \int_{a_2}^{x_2} \frac{f(x_1, u)}{(x_2-u)^{\alpha_2}} du. \tag{1.15} $$

Definition 8. *A partial right Riemann-Liouville fractional derivative* of order α_k has the form

$$ (\mathbf{D}_{a_2-}^{\alpha_2} f)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial x_2} \int_{x_2}^{a_2} \frac{f(x_1, x_2)}{(-x_2+u)^{\alpha_2}} du. \tag{1.16} $$

If the function f is differentiable we obtain

$$ \begin{aligned} (\mathbf{D}_{a_k+}^{\alpha_k} f)(x) = \frac{1}{\Gamma(1-\alpha_k)} & \left[\frac{f(x_1, \ldots, x_{k-1}, a_k, x_{k+1}, \ldots, x_n)}{(x_k-a_k)^{\alpha_k}} \right] \\ & + \int_{a_k}^{x_k} \frac{\frac{\partial f}{\partial u}(x_1, \ldots, x_{k-1}, u, x_{k+1}, \ldots, x_n)}{(x_k-u)^{\alpha_k}} du. \end{aligned} \tag{1.17} $$

Definition 9. *Reflection operator Θ* is defined as follows

$$ \Theta\phi(x) = \phi(a+b-x) \tag{1.18} $$

and it has the following properties

$$ \Theta[{}_a\mathbf{D}_x^{-\alpha}\phi] = {}_x\mathbf{D}_b^{-\alpha}[\Theta\phi], \tag{1.19} $$

$$ \Theta[{}_x\mathbf{D}_b^{-\alpha}\phi] = {}_a\mathbf{D}_x^{-\alpha}[\Theta\phi]. \tag{1.20} $$

In addition, we have other properties of fractional derivatives and integrals, namely:

Semi-group property

$$ {}_a\mathbf{D}_x^{-\alpha}\,{}_a\mathbf{D}_x^{-\beta}\phi = {}_a\mathbf{D}_x^{-\alpha-\beta}\phi, \tag{1.21}$$

$$ {}_x\mathbf{D}_b^{-\alpha}\,{}_x\mathbf{D}_b^{-\beta}\phi = {}_x\mathbf{D}_b^{-\alpha-\beta}\phi, \tag{1.22}$$

where $\alpha > 0, \beta > 0$.

Reciprocity

$$ {}_a\mathbf{D}_x^{\alpha}\,{}_a\mathbf{D}_x^{-\alpha}\phi = \phi, \quad \alpha > 0, t > a, \tag{1.23}$$

In the more general case we have

$$ {}_a\mathbf{D}_x^{\alpha}\,{}_a\mathbf{D}_x^{-\beta}\phi = {}_a\mathbf{D}_x^{\alpha-\beta}\phi. \tag{1.24}$$

Composition Rules

$$ \frac{d^n}{dx^n}[{}_a\mathbf{D}_x^{\alpha}\phi] = {}_a\mathbf{D}_x^{n+\alpha}\phi, \quad \alpha \geq 0, \tag{1.25}$$

$$ {}_a\mathbf{D}_x^{\alpha}[\frac{d^n}{dx^n}\phi(x)] = {}_a\mathbf{D}_x^{n+\alpha}\phi(x) - \sum_{j=0}^{n-1}\frac{\phi^{(j)}(a)(x-a)^{j-\alpha-n}}{\Gamma(1+j-\alpha-n)}, \tag{1.26}$$

$$ n-1 \leq \alpha < n.$$

$\frac{d^n}{dx^n}$ and ${}_aD_x^{\alpha}$ are commutative only if $\phi^{(j)}(a) = 0$ for $j = 0, 1, ..., n-1$.

$$ {}_a\mathbf{D}_x^{\alpha}[{}_a\mathbf{D}_x^{\beta}\phi(x)] = {}_a\mathbf{D}_x^{\alpha+\beta}\phi(x) - \sum_{j=1}^{m}[{}_a\mathbf{D}_x^{\beta-j}\phi(x)]_{x=a}\frac{(x-t)^{-j-\alpha}}{\Gamma(1-j-\alpha)}, \tag{1.27}$$

$$ m-1 \leq \beta < m.$$

In particular, the fractional derivatives commute

$$ {}_a\mathbf{D}_x^{\alpha}[{}_a\mathbf{D}_x^{\beta}\phi] = {}_aD_x^{\beta}[{}_a\mathbf{D}_x^{\alpha}\phi] \tag{1.28}$$

if

$$ [{}_a\mathbf{D}_x^{\alpha-j}\phi(x)]_{x=a} = 0, \quad j = 1, ..., n \tag{1.29}$$

and

$$ [{}_a\mathbf{D}_x^{\beta-j}\phi(x)]_{x=a} = 0, \quad j = 1, ..., m. \tag{1.30}$$

Chain Rule has the following form:

For $\phi(x) = F(h(x))$ we obtain

$$ {}_a\mathbf{D}_x^{\alpha}F(h(x)) = \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}\phi(x) + \sum_{k=1}^{\infty}\binom{\alpha}{k}\frac{k!(x-a)^{k-\alpha}}{\Gamma(k-\alpha+1)}\frac{d^k}{dx^k}F(h(x)), \tag{1.31}$$

and it is obtained with the help of Faá di Bruno formula given below:

$$\frac{d^k}{dt^k}F(h(x)) = k!\sum_{m=1}^{k} F^{(m)}(h(x))\sum\prod_{r=1}^{k}\frac{1}{a_r!}\left(\frac{h^{(r)}(x)}{r!}\right)^{a_r}, \tag{1.32}$$

and

$$\sum_{r=1}^{k} ra_r = k, \quad \sum_{r=1}^{k} a_r = m. \tag{1.33}$$

Leibniz Rule has a complicated formula as given below

$${}_a\mathbf{D}_x^{\alpha}(\phi(x)\psi(x)) = \sum_{k=0}^{\infty}\binom{\alpha}{k}\phi^{(k)}(x)\,{}_a\mathbf{D}_x^{\alpha-k}\psi(x), \tag{1.34}$$

where $\phi(x)$ and $\psi(x)$ have continuous derivatives in $[a,t]$.

Definition 10. *Mittag-Leffler function* is defined as

$$E_\alpha = \sum_{k=0}^{\infty}\frac{t^k}{\Gamma(\alpha k+1)}, \quad \alpha > 0, \ \alpha \in \mathbb{R}. \tag{1.35}$$

By using Eq. (1.35) we observe that

$$E_1(t) = \exp(t), \tag{1.36}$$

and

$$E_2(t) = \cosh\sqrt{t}. \tag{1.37}$$

Definition 11. *The two-parameter Mittag-Lefller function* has the form:

$$E_{\alpha,\beta}(t) = \sum_{0}^{\infty}\frac{t^k}{\Gamma(\alpha k+\beta)}, \ \alpha > 0; \ \beta > 0, \ \alpha, \ \beta \in \mathbb{R}. \tag{1.38}$$

From (1.38) we conclude that

$$E_{1,2}(t) = \frac{e^t - 1}{t}, \tag{1.39}$$

$$E_{2,2}(t) = \frac{\sinh\sqrt{t}}{\sqrt{t}}, \tag{1.40}$$

$$E_{2,1}(t) = \cosh\sqrt{t}. \tag{1.41}$$

The Laplace transformations for several Mittag-Leffler functions are summarized in the following:

$$L(E_\alpha(-\lambda t^\alpha)) = \frac{s^{\alpha-1}}{s^\alpha+\lambda}, \tag{1.42}$$

$$L(t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha)) = \frac{1}{s^\alpha+\lambda}, \tag{1.43}$$

$$L(t^{\beta-1}E_{\alpha,\beta}(-\lambda t^{\alpha})) = \frac{s^{\alpha-\beta}}{s^{\alpha}+\lambda}. \tag{1.44}$$

In the following we are going to present a brief introduction of the generalized functions and they connect with the fractional derivatives.

Generalized functions have many interesting applications in science and engineering (Gelfand and Shilov, 1964).

Let us consider the Cauchy's integral formula as given below:

$$f^{(-n)} = \frac{1}{\Gamma(n)}\int_a^t f(\tau)(t-\tau)^{n-1}du. \tag{1.45}$$

Here n is a positive integer, $\Gamma(n)$ denotes the Gamma function and $a < t$. Let us consider $\Phi_n^+(t)$ as given below:

$$\Phi_n^+(t) = \frac{1}{\Gamma(n)}t^{n-1}, \;\; t > 0 \tag{1.46}$$

and zero for $t \leq 0$.

Letting f be zero for $t < a$ and by making use of (1.46), we obtain that (1.45) becomes

$$_a\mathbf{I}_t^n f(t) = f(t) * \Phi_n^+(t), \tag{1.47}$$

where $*$ denotes the convolution operation and it is given by

$$g(t) * f(t) = \int_{-\infty}^{+\infty} g(\tau)h(t-\tau)d\tau. \tag{1.48}$$

Equation (1.47) can be generalized for any $\alpha > 0$ as (Gelfand and Shilov, 1964)

$$_a\mathbf{I}_t^{\alpha} f(t) = f(t) * \Phi_{\alpha}^+(t), \tag{1.49}$$

where $\Phi_{\alpha}^+(t)$ is a generalized function or distribution (Gelfand and Shilov, 1964).

Having in mind to define the convolution of two generalized functions we have to defined first the test functions (Gelfand and Shilov, 1964). For these reasons we choose the set K of all real functions $\phi(x)$ with continuous derivatives of all orders and with bounded support. We denote these functions the test functions. We can add and multiply by a scalar a test function in order to get new test functions, as a result, K becomes a linear space. Another interesting property of these test functions is that the sequence $\phi_1(x), \ldots, \phi_{\nu}(x)$ of test functions converges to zero in K if all above mentioned functions vanish outside a given fixed common bounded region and converge uniformly to zero together with their derivatives of any order.

We claim that f is a continuous linear functional on K if there exists some rule according to which we can associate with every $\phi(x)$ in K a real number (f, ϕ) such that (Gelfand and Shilov, 1964)

a) $(f, \alpha_1\phi_1 + \alpha_2\phi_2) = \alpha_1(f, \phi_1) + \alpha_2(f, \phi_2)$, for any real numbers α_1 and α_2 and for any two functions $\phi_1(x)$ and $\phi_2(x)$.

b) If the sequence $\phi_1, \phi_2, \ldots, \phi_\nu, \ldots$ converges to zero in K, then the sequence $(f, \phi_1), (f, \phi_2), \ldots, (f, \phi_\nu), \ldots$ converges to zero (Gelfand and Shilov, 1964).

The next step is to consider $k(t) = g(t) * h(t)$ and a test function $\phi(x)$. Therefore, we obtain the following (Podlubny, 1999; Gelfand and Shilov, 1964):

$$\begin{aligned}\langle k, \phi\rangle &= \int k(t)\phi(t)dt = it\int\left\{\int g(\xi)h(t-\xi)d\xi\right\}\phi(t)dt\\ &= \int\int g(\xi)h(\eta)\phi(\xi+\eta)d\xi d\eta. \qquad (1.50)\end{aligned}$$

Here the limits of the integrals are $-\infty$ and $+\infty$ respectively.

By making use of (1.50) we obtain the generalization of a convolution of two functions as mentioned in the following:

$$\langle g * h, \phi\rangle = \langle g(t), \langle h(\tau), \phi(t+\tau)\rangle\rangle. \qquad (1.51)$$

From (1.51) we obtain the following properties of the convolution operation:

$$g * h = h * g, \qquad (1.52)$$

$$f * (g * h) = (f * g) * h, \qquad (1.53)$$

$$\mathbf{D}(g * h) = (\mathbf{D}g) * h = g * (\mathbf{D}h), \qquad (1.54)$$

where $\mathbf{D}(\cdot)$ denotes the generalized derivative. The relation between the generalized derivative and the classical derivative becomes (Podlubny, 1999; Gelfand and Shilov, 1964)

$$\mathbf{D}^n f = f^{(n)} + \sum_{k=0}^{n-1}[\mathbf{D}^{n-k-1}\delta(t-a)]f^{(k)}(a). \qquad (1.55)$$

For $\alpha < 0$ and $\Phi_\alpha^+(t)$ as a generalized function we introduce the notion of left fractional derivative as given below:

$${}_a\mathbf{D}_t^{-\alpha} = {}_a\mathbf{I}_t^{\alpha}[f], \qquad (1.56)$$

or

$${}_a\mathbf{D}_t^{-\alpha}[f] = f(t) * \Phi_\alpha^+(t). \qquad (1.57)$$

In the same way, we define

$${}_a\mathbf{D}_t^{\alpha}[f] = f(t) * \Phi_{-\alpha}^+(t),\ \alpha > 0. \qquad (1.58)$$

The most interesting properties of the distributions $\Phi_\alpha^+(t)$ are (Podlubny, 1999; Gelfand and Shilov, 1964)

$$\mathbf{D}^{-n} = \Phi_n^+(t), n \in Z, \tag{1.59}$$

and

$$\Phi_\alpha^+(t-a) * \Phi_\beta^+(t) = \Phi_{\alpha+\beta}^+(t-a). \tag{1.60}$$

From (1.60) we obtain

$${}_a\mathbf{D}_t^\alpha({}_a\mathbf{D}_t^\beta(f)) = {}_a\mathbf{D}_t^{\alpha+\beta}(f). \tag{1.61}$$

For $0 \leq n-1 \leq \alpha < n$ we obtain the following important relations for the generalized fractional derivative

$$\begin{aligned} {}_a\mathbf{D}_t^\alpha(f) &= f(t) * \Phi_\alpha^+(t) = f(t) * (D^n \Phi_{n-\alpha}^+)(t) \\ &= (D^n f(t) * \Phi_{n-\alpha}^+(t)) = D^n(f(t) * \Phi_{n-\alpha}^+(t)). \end{aligned} \tag{1.62}$$

From (1.62) we observe that the distributional forms of Caputo and the Riemann-Liouville are the same.

The right fractional derivative can be define as follows

$${}_t\mathbf{D}_b^\alpha(f) = f(t) * \Phi_{-\alpha}^-, \tag{1.63}$$

where $\Phi_{-\alpha}^-$ is defined as

$$\begin{aligned} \Phi_{-\alpha}^-(t) &= \frac{(-t)^{\alpha-1}}{\Gamma(\alpha)}, \quad t < 0, \\ \Phi_{-\alpha}^-(t) &= 0, \qquad\qquad t \geqslant 0. \end{aligned} \tag{1.64}$$

In addition, we have

$$\Phi_n^-(t) = (-1)^n D^{-n}\delta(t^-), \tag{1.65}$$

for n being integer.

The integration by parts formula is valid for the generalized fractional derivatives, namely

$$\int {}_a\mathbf{D}_t^\beta[f]g(t)dt = \int {}_t\mathbf{D}_b^\beta[g]f(t)dt. \tag{1.66}$$

1.3 Fractional variational principles and their applications

The Lagrangian formulation of dynamical systems represents one of the most important principle in physics.

The corresponding Lagrangian for dissipative systems depends explicitly on time, therefore the Hamiltonian depends explicitly on time too.

One still open and important issue in this area is the fractional quantization procedure. The main obstacle for fractional calculus quantization is represented by its non-locality of the fractional derivatives.

The main advantage of this theory is that it incorporates, under certain limits, both the conservative and nonconservative systems.

1.3.1 *Fractional Euler-Lagrange equations for discrete systems*

The classical Euler-Lagrange differential equation is the fundamental equation of calculus of variations.

It states that J if is defined by an integral of the form

$$I = \int f(t, y, \dot{y}) dt, \tag{1.67}$$

where

$$\dot{y} = \frac{dy}{dt}, \tag{1.68}$$

then J has a stationary value if the Euler-Lagrange differential equation

$$\frac{\partial f}{\partial y} - \frac{d}{dt}\left(\frac{\partial f}{\partial y}\right) = 0. \tag{1.69}$$

Let us consider the following Lagrangian

$$L = \frac{1}{2}\dot{q}^2 - V(q). \tag{1.70}$$

As a result, the corresponding Euler-Lagrange equation becomes

$$\frac{\partial V(q)}{\partial q} + \ddot{q} = 0. \tag{1.71}$$

In the following we are giving the fractional generalization of the above results.

Let us assume that α_j $(j = 1, \ldots, n_1)$ and β_k $(k = 1, \ldots, n_2)$ are two sets of real numbers all greater than 0, $\alpha_{max} = \max(\alpha_1, \ldots, \alpha_{n1}, \beta_1, \ldots, \beta_{n2})$, and M is an integer such that $M - 1 \leq \alpha_{max} \leq M$. Let $J[q^\rho]$ be a functional of the type

$$\int_a^b L\left(t, q^\rho, {}_a\mathbf{D}_t^{\alpha_1} q^\rho, \ldots, {}_a\mathbf{D}_t^{\alpha_{n_1}} q^\rho,\ {}_t\mathbf{D}_b^{\beta_1} q^\rho, \ldots, {}_t\mathbf{D}_b^{\beta_{n_2}} q^\rho\right) dt, \tag{1.72}$$

defined on the set of n functions q^ρ, $\rho = 1, \ldots, n$ which have continuous left Riemann-Liouville fractional derivative of order α_j, $j = 1, \ldots, n_1$ and right Riemann-Liouville fractional derivative of order β_j, $j = 1, \ldots, n_2$ in $[a, b]$ and satisfy the boundary conditions $(q^\rho(a))^{(j)} = q^\rho_{aj}$ and $(q^\rho(b))^{(j)} = q^\rho_{bj}$, $j = 1, \ldots, M-1$. A necessary condition for $J[q^\rho]$ to admit an extremum for given functions $q^\rho(t)$, $\rho = 1, \ldots, n$ is that $q^\rho(t)$ satisfy Euler-Lagrange equations (Agrawal, 2002)

$$\frac{\partial L}{\partial q^\rho}+\sum_{j=1}^{n1}{}_t\mathbf{D}_b^{\alpha_j}\frac{\partial L}{\partial_a\mathbf{D}_t^{\alpha_j}q^\rho}+\sum_{j=1}^{n2}{}_a\mathbf{D}_t^{\beta_j}\frac{\partial L}{\partial_t\mathbf{D}_b^{\beta_j}q^\rho}=0. \tag{1.73}$$

Here, if α_j is an integer, then ${}_a\mathbf{D}_t^{\alpha_j}$ and ${}_t\mathbf{D}_b^{\alpha_j}$ must be replaced with the ordinary derivatives $(d/dt)^{\alpha_j}$ and $(-d/dt)^{\alpha_j}$, respectively. The method initiated by Agrawal (Agrawal, 2002) was generalized and improved by Baleanu and coworkers (Baleanu et al., 2006; Baleanu and Avkar, 2004; Baleanu, 2004; Muslih and Baleanu, 2005c; Baleanu and Muslih, 2005b).

Let us start with the following classical Lagrangian

$$L(x,y,z)=\dot{x}\dot{z}+yz^3. \tag{1.74}$$

The classical solutions of Euler-Lagrange equations are given below

$$x(t)=at+b, z(t)=0. \tag{1.75}$$

We notice that $y(t)$ has an undetermined evolution and a and b are constants to be determined from the initial conditions.

The fractional generalization of (1.74) becomes

$$L_f=({}_a\mathbf{D}_t^\alpha x){}_a\mathbf{D}_t^\alpha z+yz^3. \tag{1.76}$$

As a result, the Euler-Lagrange equations of (1.76) are given below

$${}_t\mathbf{D}_b^\alpha({}_a\mathbf{D}_t^\alpha z)=0, z^3=0, \quad {}_t\mathbf{D}_b^\alpha({}_a\mathbf{D}_t^\alpha x)+3yz^2=0. \tag{1.77}$$

From (1.77) we notice that $z=0$, y is not determined. We note that x fulfills the following equation

$${}_t\mathbf{D}_b^\alpha({}_a\mathbf{D}_t^\alpha x)=0. \tag{1.78}$$

The solution of (1.78), under the assumption of $1<\alpha<2$, is given by

$$\begin{aligned}x(t)&=A(t-a)^{\alpha-1}+B(t-a)^{\alpha-2}\\&+C(t-a)^\alpha{}_2F_1\left(1,1-\alpha,1+\alpha,\frac{t-a}{b-a}\right)\\&+D(t-a)^\alpha{}_2F_1\left(1,2-\alpha,1+\alpha,\frac{t-a}{b-a}\right).\end{aligned} \tag{1.79}$$

Here ${}_2F_1$ represents Gauss hypergeometric function and $A, B, C,$ and D are real constants. When $\alpha\to 1^+$ and $a=0$, the classical linear solution of one-dimensional space is recovered, namely

$$x(t)=A+Ct. \tag{1.80}$$

1.3.2 Fractional Hamiltonian formulation

1.3.2.1 A direct method with Riemann-Liouville fractional derivatives

In the following we introduce the meaning of fractional Hamiltonian. For simplicity, in the following we consider the following form of the fractional Euler-Lagrange equations (Agrawal, 2002)

$$\frac{\partial L}{\partial q^{\rho}(t)} + {}_t\mathbf{D}_b^{\alpha} \frac{\partial L}{\partial_a \mathbf{D}_t^{\alpha} q^{\rho}(t)} = 0, \quad 0 < \alpha < 1\,, \rho = 1, \ldots, N. \tag{1.81}$$

In the following by using (1.81) we define the generalized momenta as (see, for example, Ref.(Rabei et al., 2007) for more details)

$$p_{\alpha_\rho} = \frac{\partial L}{\partial_a \mathbf{D}_t^{\alpha} q^{\rho}(t)}, \quad \rho = 1, \ldots, N. \tag{1.82}$$

As a consequence of (1.81) and (1.82) a Hamiltonian function is defined as

$$H = p_{\alpha_\rho}{}_a\mathbf{D}_t^{\alpha} q^{\rho}(t) - L. \tag{1.83}$$

The canonical equations corresponding to (1.83) are given below

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}, \tag{1.84}$$

$$\frac{\partial H}{\partial p_{\alpha_\rho}} = {}_a\mathbf{D}_t^{\alpha} q^{\rho}, \tag{1.85}$$

$$\frac{\partial H}{\partial q^{\rho}} = {}_t\mathbf{D}_b^{\alpha} p_{\alpha_\rho}, \; 0 < \alpha < 1, \; \rho = 1, \ldots, N. \tag{1.86}$$

1.3.2.2 A direct method within Caputo fractional derivatives

In the following we present briefly the Hamiltonian formulation within Caputo's fractional derivatives (Baleanu and Agrawal, 2006). Let us consider the fractional Lagrangian as given below

$$L(q, {}_a^C D_t^{\alpha} q, t), \; 0 < \alpha < 1. \tag{1.87}$$

By using (1.87) we define the canonical momenta p_α as follows

$$p_\alpha = \frac{\partial L}{\partial_a^C D_t^{\alpha} q}. \tag{1.88}$$

We define the fractional canonical Hamiltonian as

$$H = p_\alpha \left({}_a^C D_t^{\alpha} q\right) - L. \tag{1.89}$$

Taking total differential of (1.89) and by using (1.88), we obtain

$$dH = dp_\alpha \, ({}_a^C D_t^\alpha q) - \frac{\partial L}{\partial q} dq - \frac{\partial L}{\partial t} dt. \tag{1.90}$$

Taking into account the fractional Euler-Lagrange equations we obtain

$$dH = ({}_a^C D_t^\alpha q) dp_\alpha + ({}_t\mathbf{D}_b^\alpha p_\alpha) dq - \frac{\partial L}{\partial t} dt. \tag{1.91}$$

Finally, after some simple manipulations, the fractional Hamilton equations are obtained as follows

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}, \tag{1.92}$$

$$\frac{\partial H}{\partial p_\alpha} = {}_a^C D_t^\alpha q, \tag{1.93}$$

$$\frac{\partial H}{\partial q} = {}_t\mathbf{D}_b^\alpha p_\alpha. \tag{1.94}$$

1.3.2.3 Fractional Ostrogradski's formulation

The higher-derivatives theories (Gitman and Tytin, 1990; Nesterenko, 1989) appear naturally as corrections to general relativity and cosmic strings as well (Birell and Davies, 1982). The unconstrained higher-order derivatives have more degree of freedom than lower-derivative theories, as a result a lack a lower-energy bound was reported. A method how to remove all these problems was suggested in (Simon, 1990). It was reported that the non-local formulation can be written as an infinite order Ostrogradski's formulation (Gitman and Tytin, 1990; Nesterenko, 1989). On the other hand the fractional derivatives are non-local objects and we have a decomposition formula for them. In conclusion, a natural question is how to formulate a theory corresponding to the fractional case.

Let us consider an ordinary local Lagrangian depending on a finite number of derivatives at a given time as (Bering, online)

$$L\left(q(t), \dot{q}(t), ..., q^{(n)}(t)\right). \tag{1.95}$$

Let us consider a Lagrangian depending on a piece of the trajectory $q(t, \lambda)$ for $\forall\lambda$ belonging to an interval $[a, b]$, namely

$$L^{non}(t) = L(q(t + \lambda)). \tag{1.96}$$

Here a, b represent real numbers. Therefore, we have created a non-local Lagrangian and the corresponding action function is given by

$$S(q) = \int dt L^{non}(t). \tag{1.97}$$

We are able to write the Euler-Lagrange equation corresponding to (1.97) as (Bering, online)

$$\int dt \frac{\delta L^{non}(t)}{\delta(q(t))} = 0. \tag{1.98}$$

We observed that Equations (1.98) are functional relations to be satisfied by a Lagrangian constraint. Another observation is that there is no dynamics except the displacement inside the trajectory

$$q(t) \to q(t + \lambda). \tag{1.99}$$

The following step is to introduce the dynamical variable $Q(t, \lambda)$ as

$$Q(t, \lambda) = q(t + \lambda). \tag{1.100}$$

Let us consider a field $Q(t, \lambda)$ instead of a trajectory $q(t)$, namely

$$\dot{Q}(t, \lambda) = Q'(t, \lambda), \tag{1.101}$$

where $\dot{Q} = \frac{\partial Q(t,\lambda)}{\partial t}$ and $Q'(t, \lambda) = \frac{\partial Q(t,\lambda)}{\partial \lambda}$. In such a way we obtain a $1 + 1$ dimensional formulation of non-local Lagrangian (Bering, online).

The coordinates and momenta are suppose to have the following forms

$$Q(t, \lambda) = \sum_{m=0}^{\infty} e_m(\lambda) q^{(m)}(t), P(t, \lambda) = \sum_{m=0}^{\infty} e^m(\lambda) p_{(m)}(t), \tag{1.102}$$

where

$$\{q^{(n)}(t), p_{(m)}(t)\} = \delta_m^n \tag{1.103}$$

and

$$e_m(\lambda) = \frac{\lambda^m}{m!}, \ e^m(\lambda) = (-\partial_\lambda)^m \delta(\lambda). \tag{1.104}$$

In conclusion, the Hamiltonian for $1 + 1$ dimensional field has the form

$$H(t, [Q, P]) = \int d\lambda P(t, \lambda) Q'(t, \lambda) - \tilde{L}(t, [Q]), \tag{1.105}$$

where P denotes the canonical momentum of Q. The phase space is $T * J$ equipped with the fundamental Poisson brackets

$$\{Q(t, \lambda), P(t, \lambda')\} = \delta(\lambda - \lambda'). \tag{1.106}$$

The functional $\tilde{L}(t, [Q])$ is defined as below

$$\tilde{L}(t, [Q]) = \int d\lambda \delta(\lambda) L(t, \lambda). \tag{1.107}$$

By analyzing (1.107), the primary constraint becomes

$$\phi(t,\lambda,[q,P]) = P(t,\lambda) - \int d\sigma \chi(\lambda,-\sigma)\varepsilon(t;\sigma,\lambda) \approx 0. \tag{1.108}$$

$\varepsilon(t;\sigma,\lambda)$ and $\chi(\lambda,-\sigma)$ have the following definitions:

$$\varepsilon(t;\sigma,\lambda) = \frac{\partial L(t,\sigma)}{\partial Q(t,\lambda)},\ \ \chi(\lambda,-\sigma) = \frac{\varepsilon(\lambda)-\varepsilon(\sigma)}{2}, \tag{1.109}$$

where $\varepsilon(\lambda)$ denotes the sigma distribution. By using this construction the Euler-Lagrange equation is guaranteed by itself

$$\dot{\phi} \sim \psi = \int d\sigma \xi(t;\sigma,\lambda). \tag{1.110}$$

In the following we would like to derive both the Lagrangian and the Hamiltonian formalisms for non singular Lagrangian with fractional order derivatives starting from the Hamiltonian formalism of non local-theories (Baleanu et al., 2006). Let us consider the following Lagrangian to start with

$$L(q,t) = L(t,q^{\alpha_m}), \tag{1.111}$$

where the generalized coordinates are defined as

$$q^{\alpha_m} = {}_a\mathbf{D}_t^{\alpha_m} x(t), \tag{1.112}$$

where m is a natural number.

To obtain the reduced phase space quantization, we start with the infinite dimensional phase space $T * J(t) = \{Q(t,\lambda), P(t,\lambda)\}$.

The key issue is to find an appropriate generalization of (1.104) for the fractional case (Baleanu et al., 2006). As it was pointed out in (Bering, online) the coordinates and the momenta are considered as a Taylor series. Therefore, the first step is to generalize the classical series to the fractional case. A natural extension is to use factorial instead of the Gamma function. In this way we introduce naturally the generalized functions instead of $e_m(\lambda)$ and $e^m(\lambda)$ given by (1.104).

As it is already known several fractional Taylor's series expansions were developed (Trujillo, 1999; Hardy, 1945), therefore we have to decide which one is appropriate for our generalization. Since we are dealing with fractional Riemann-Liouville derivatives we choose the following generalization proposed, namely

$$\begin{aligned} Q(t,\lambda) &= \sum_{m=-\infty}^{\infty} e_{\alpha_m}(\lambda) q^{(\alpha_m)}(t), \\ P(t,\lambda) &= \sum_{m=-\infty}^{\infty} e^{\alpha_m}(\lambda) p_{(\alpha_m)}(t), \end{aligned} \tag{1.113}$$

where

$$e_{\alpha_m}(\lambda) = \frac{(\lambda - \lambda_0)^{\alpha_m}}{\Gamma(\alpha_m + 1)}, \quad e^{\alpha_m}(\lambda) = \mathbf{D}_{\lambda}^{\alpha_m} \delta(\lambda - \lambda_0), \tag{1.114}$$

and $\alpha_m = m + \alpha$, with $0 \leq \alpha < 1$. Here λ_0 is a constant. The coefficients in (1.113) are new canonical variables:

$$\{q^{(\alpha_m)}, p_{(\alpha_{m'})}\} = \delta_{\alpha_m}^{\alpha_{m'}}. \tag{1.115}$$

By using (1.115) we obtain that

$$\sum_{m=-\infty}^{\infty} e^{\alpha_m}(\lambda) e_{\alpha_m}(\lambda') = \delta(\lambda - \lambda'), \tag{1.116}$$

and

$$\int_{-\infty}^{+\infty} d\lambda e^{\alpha_m}(\lambda) e_{\alpha_{m'}}(\lambda) = \delta_{\alpha_{m'}}^{\alpha_m}. \tag{1.117}$$

Therefore, $e^{\alpha_m}(\lambda)$ and $e_{\alpha_m}(\lambda)$ form an orthonormal basis. We stress on the fact that (1.116) and (1.117) involve the generalized functions and the relations have the meaning in the sense of generalized functions approach (Gelfand and Shilov, 1964; Hardy, 1945).

The fractional Hamiltonian is now given by

$$H = \sum_{m=-\infty}^{\infty} p^{\alpha_m} q^{\alpha_{m+1}} - L(q^0, q^{\alpha_m}). \tag{1.118}$$

The momenta constraints become an infinite set of constraints

$$\phi_n = p_{\alpha_n}(t) - \sum_{m=n}^{\infty} {}_t\mathbf{D}_b^{\alpha_{m-n}} \frac{\partial L}{\partial q^{(\alpha_{m+1})}(t)} = 0. \tag{1.119}$$

The fractional Euler-Lagrange equations are as follows

$$\sum_{l=-\infty}^{\infty} {}_t\mathbf{D}_b^{\alpha_l} \frac{\partial L(t)}{\partial q^{\alpha_l}(t)} = 0. \tag{1.120}$$

An interesting property of the fractional series proposed by Riemann and discussed by Hardy in (Hardy, 1945) is that when α_m becomes integers the usual form of Taylor series is obtained. Therefore one should notice that for integer values of α_m we have

$$p_{\alpha_m}(t) - \sum_{l=0}^{n-m-1} \left(-\frac{d}{dt}\right)^l \frac{\partial L(t)}{\partial(\partial_t^{l+m+1} q(t))} = 0, \tag{1.121}$$

which is the definition of Ostrogradski's momenta (Gitman and Tytin, 1990).