Operator Theory
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Recent Progress in Operator Theory and Its Applications

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Founded in 1979 by Israel Gohberg

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Contents

Introduction

This volume contains the Proceedings of the Twentieth International Workshop on Operator Theory and Applications (IWOTA), held at Hotel Real de Minas in Guanajuato, Mexico, during September 21–25, 2009. This was the twentieth IWOTA; in fact, the workshop was held biannually since 1981, and annually in the recent years (starting 2002) rotating among eleven countries on three continents. Previous IWOTA meetings were held at:

Santa Monica, CA, USA (1981)
J.W. Helton, Chair

Rehovot, Israel (1983) – *Oper. Theory Adv. Appl.* 12;
H. Dym and I. Gohberg, Co-chairs

Amsterdam, Netherlands (1985) – *Oper. Theory Adv. Appl.* 19;
M.A. Kaashoek, Chair

Mesa, AZ, USA (1987) – *Oper. Theory Adv. Appl.* 35;
J.W. Helton and L. Rodman, Co-chairs

Rotterdam, Netherlands (1989) – *Oper. Theory Adv. Appl.* 50;
H. Bart, Chair

Sapporo, Japan (1991) – *Oper. Theory Adv. Appl.* 59;
T. Ando, Chair

Vienna, Austria (1993) – *Oper. Theory Adv. Appl.* 80;
H. Langer, Chair

Regensburg, Germany (1995) – *Oper. Theory Adv. Appl.* 102 and 103;
R. Mennicken, Chair

Bloomington, IN, USA (1996) – *Oper. Theory Adv. Appl.* 115;
H. Bercovici and C. Foiaş, Co-chairs

Groningen, Netherlands, (1998) – *Oper. Theory Adv. Appl.* 124;
A. Dijksma, Chair

Bordeaux, France (2000) – *Oper. Theory Adv. Appl.* 129;
N. Nikolskii, Chair

Faro, Portugal (2000) – *Oper. Theory Adv. Appl.* 142;
A.F. Dos Santos and N. Manojlovic, Co-chairs

Blacksburg, VA, USA (2002) – *Oper. Theory Adv. Appl.* 149;
J. Ball, Chair

Cagliari, Italy (2003) – *Oper. Theory Adv. Appl.* 160;
S. Seatzy and C. van der Mee, Co-chairs

Newcastle, UK (2004) – *Oper. Theory Adv. Appl.* 171;
M.A. Dritshel and N. Young, Co-chairs

Storrs, CT, USA (2005) – *Oper. Theory Adv. Appl.* 179;
V. Olshevsky, Chair

Seoul, Korea (2006) – *Oper. Theory Adv. Appl.* 187;
Woo Young Lee, Chair

Potchefstroom, South Africa (2007) – *Oper. Theory Adv. Appl.* 195;
K. Grobler and G. Groenewald, Co-chairs

Williamsburg, VA, USA, (2008) – *Oper. Theory Adv. Appl.* 202 and 203;
L. Rodman, Chair

Guanajuato, Mexico (2009) – *Oper. Theory Adv. Appl.* 220;
N. Vasilevski, Chair

Berlin, Germany (2010) – *Oper. Theory Adv. Appl.* 221;
J. Behrndt, K.-H. Förster and C. Trunk, Co-chairs

Seville, Spain (2011)
A. Montes Rodríguez, Chair.

Consistent with the topics of recent IWOTA meetings, IWOTA 2009 was designed as a comprehensive, inclusive conference covering all aspects of theoretical and applied operator theory, ranging from classical analysis, differential and integral equations, complex and harmonic analysis to mathematical physics, mathematical systems and control theory, signal processing and numerical analysis. The conference brought together international experts for a week-long stay at Hotel Real de Minas, in an atmosphere conducive to fruitful professional interactions. These Proceedings reflect the high quality of the papers presented at the conference. In addition to fourteen plenary sessions, IWOTA 2009 included the following special sessions:

Bergman and Segal-Bargmann spaces and Toeplitz operators

Factorization problems, Wiener-Hopf and Fredholm operators

Hypercomplex operator theory

Indefinite inner product spaces and spectral problems

Multivariable operator theory

Operators on function spaces

Pseudodifferential operators and related topics

Solution techniques for partial differential equations

Spectral theory and its applications

Toeplitz/rank structured tensors and matrices.

This volume contains twenty-one solicited articles by speakers at the workshop, ranging from expository surveys to original research papers, each carefully refereed. All contributions reflect recent developments in operator theory and its applications.

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CONACYT (Consejo Nacional de Ciencia y Tecnología, Mexico)
Sociedad Matemática Mexicana (Mexico)
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Exponential Decay of Semigroups for Second-order Non-selfadjoint Linear Differential Equations

Nikita Artamonov

Abstract. The Cauchy problem for second-order linear differential equation

$$u''(t) + Du'(t) + Au(t) = 0$$

in Hilbert space H with a sectorial operator A and an accretive operator D is studied. Sufficient conditions for exponential decay of the solutions are obtained.

Mathematics Subject Classification (2000). Primary 47D06, 34G10; Secondary 47B44, 35G15.

Keywords. Accretive operator, sectorial operator, C_0-semigroup, second-order linear differential equation, spectrum.

Many linearized equations of mechanics and mathematical physics can be reduced to a linear differential equation

$$u''(t) + Du'(t) + Au(t) = 0, \tag{0.1}$$

where $u(t)$ is a vector-valued function in an appropriate (finite- or infinite-dimensional) Hilbert space H, D and A are linear (bounded or unbounded) operators on H. Properties of the differential equation (0.1) are closely connected with spectral properties of a quadric pencil

$$L(\lambda) = \lambda^2 + \lambda D + A, \quad \lambda \in \mathbb{C}$$

which is obtained by substituting exponential functions $u(t) = \exp(\lambda t)x$, $x \in H$ into (0.1). In many applications A is a self-adjoint positive definite operator, D is a self-adjoint positive definite or an accretive operator (see definition in Section 1). In this case the differential equation (0.1) and spectral properties of the related quadric pencil $L(\lambda)$ are well studied, see [2, 6, 7, 8, 10, 11, 12, 13, 15] and references therein. It was obtained a localization of the pencil's spectrum, sufficient

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conditions of the completeness of eigen- and adjoint vectors of the pencil $L(\lambda)$ and it was proved, that all solutions of (0.1) exponentially decay. The exponential decay means, that the total energy exponentially decreases and corresponding mechanical system is stable. In paper [16] was studied spectral properties of the pencil $L(\lambda)$ for a self-adjoint non-positive definite operator A and an accretive operator D.

But some models of continuous mechanics are reduced to differential equation (0.1) with sectorial operator A, see [1, 9, 17] and references therein. In this cases methods, developed for self-adjoint operator A, cannot be applied.

The aim of this paper is the study of a Cauchy problem for second-order linear differential equation (0.1) in a Hilbert space H with initial conditions

$$u(0) = u_0 \quad u'(0) = u_1. \tag{0.2}$$

The shiffness operator A is assumed to be a sectorial operator, the damping operator D is assumed to be an accretive operator.

By $\mathcal{L}(H', H'')$ denote a space of bounded operators acting from a Hilbert space H' to a Hilbert space H''. $\mathcal{L}(H) = \mathcal{L}(H, H)$ is an algebra of bounded operators acting on Hilbert space H.

1. Preliminary results

First let us recall some definitions [4, 14].

Definition 1.1. Linear operator B with dense domain $\mathcal{D}(B)$ is called *accretive* if $\mathrm{Re}(Bx, x) \geq 0$ for all $x \in \mathcal{D}(B)$ and *m-accretive*, if the range of operator $B + \omega I$ is dense in H for some $\omega > 0$.

An accretive operator B is m-accretive iff B has not accretive extensions [14]. For m-accretive operator

$$\rho(B) \supset \{\lambda \in \mathbb{C} \ : \ \mathrm{Re}\,\lambda < 0\}.$$

Definition 1.2. An accretive operator B is called *sectorial* or *ω-accretive* if for some $\omega \in [0, \pi/2)$

$$\big|\mathrm{Im}(Bx, x)\big| \leq \tan(\omega)\,\mathrm{Re}(Bx, x) \quad x \in \mathcal{D}(B).$$

If a sectorial operator has not sectorial extensions, then it is called *m-sectorial* or *m-ω-accretive.*

The sectorial property means that the numerical range of the operator B belongs to a sector

$$\{z \in \mathbb{C} \mid |\,\mathrm{Im}\,z| \leq \tan(\omega)\,\mathrm{Re}\,z\}.$$

For a sectorial operator B there exist [14] a self-adjoint non-negative operator T_B and a self-adjoint operator $S_B \in \mathcal{L}(H)$, $\|S_B\| \leq \tan(\omega)$ such that

$$\mathrm{Re}(Bx, x) = (T_B^{1/2}x, T_B^{1/2}x), \quad B \subset T_B^{1/2}(I + iS_B)T_B^{1/2}$$

and $B = T_B^{1/2}(I + iS_B)T_B^{1/2}$ iff B is m-sectorial.

Throughout this paper we will assume, that

(A) Operator $A : \mathcal{D}(A) \subset H \to H$ is m-sectorial and for some positive a_0

$$\mathrm{Re}(Ax, x) \geq a_0(x, x) \quad x \in \mathcal{D}(A).$$

Since A is m-sectorial there exist a self-adjoint positive definite operator T and a self-adjoint $S \in \mathcal{L}(H)$, such that

$$\mathrm{Re}(Ax, x) = (T^{1/2}x, T^{1/2}x) \geq a_0(x, x), \quad x \in \mathcal{D}(A), \quad A = T^{1/2}(I + iS)T^{1/2}.$$

The operator A is invertible and

$$A^{-1} = T^{-1/2}(I + iS)^{-1}T^{-1/2}.$$

By H_s ($s \in \mathbb{R}$) denote a collection of Hilbert spaces generated by a self-adjoint operator $T^{1/2}$:

- for $s \geq 0$ $H_s = \mathcal{D}(T^{s/2})$ endowed with a norm $\|x\|_s = \|T^{s/2}x\|$;
- for $s < 0$ H_s is a closure of H with respect to the norm $\|\cdot\|_s$.

Obviously $H_0 = H$. The operator $T^{1/2}$ can be considered now as a unitary operator mapping H_s on H_{s-1}. A is a bounded operator $A \in \mathcal{L}(H_2, H_0)$ and it can be extended to a bounded operator $\tilde{A} \in \mathcal{L}(H_1, H_{-1})$. The inverse operator A^{-1} can be extended to a bounded operator $\tilde{A}^{-1} \in \mathcal{L}(H_{-1}, H_1)$.

By $(\cdot, \cdot)_{-1,1}$ denote a duality pairing on $H_{-1} \times H_1$. Note, that for all $x \in H_{-1}$ and $y \in H_1$ we have

$$\left|(x, y)_{-1,1}\right| \leq \|x\|_{-1} \cdot \|y\|_1$$

and $(x, y)_{-1,1} = (x, y)$ if $x \in H$. Further,

$$\mathrm{Re}(\tilde{A}x, x)_{-1,1} = (Tx, x)_{-1,1} = (T^{1/2}x, T^{1/2}x) = \|x\|_1^2, \quad x \in H_1 = \mathcal{D}(T^{1/2}).$$

Denote $\tilde{S} = T^{1/2}ST^{1/2} \in \mathcal{L}(H_1, H_{-1})$. Then, for the operator $\tilde{A}$ we have a representation $\tilde{A} = T + i\tilde{S}$ and

$$\mathrm{Im}(\tilde{A}x, x)_{-1,1} = (\tilde{S}x, x)_{-1,1} \quad x \in H_1.$$

Also $(\tilde{S}x, y)_{-1,1} = \overline{(\tilde{S}y, x)}_{-1,1}$ for all $x, y \in H_1$.

Following paper [11] we will assume

(B) D is a bounded operator $D \in \mathcal{L}(H_1, H_{-1})$, and

$$\beta = \inf_{x \in H_1, x \neq 0} \frac{\mathrm{Re}(Dx, x)_{-1,1}}{\|x\|^2} > 0. \tag{1.1}$$

Operator $T^{-1/2}$ is a unitary operator mapping H_s on H_{s+1}, therefore an operator $D' = T^{-1/2}DT^{-1/2}$, acting on H, is bounded. Let

$$D_1 = \frac{1}{2}T^{1/2}\Big(D' + (D')^*\Big)T^{1/2} \quad D_2 = \frac{1}{2i}T^{1/2}\Big(D' - (D')^*\Big)T^{1/2},$$

Obviously $D_1, D_2 \in \mathcal{L}(H_1, H_{-1})$, $D = D_1 + iD_2$ and for all $x \in H_1$

$$\mathrm{Re}(Dx, x)_{-1,1} = (D_1x, x)_{-1,1} \geq \beta\|x\|^2, \quad \mathrm{Im}(Dx, x)_{-1,1} = (D_2x, x)_{-1,1}.$$

Also $(D_jx, y)_{-1,1} = \overline{(D_jy, x)}_{-1,1}$ for all $x, y \in H_1$ ($j = 1, 2$).

2. Main result

Definition 2.1. A vector-valued function $u(t) \in H_1$ is called a solution of the differential equation (0.1) if $u'(t) \in H_1$, $u''(t) \in H$, $Du'(t) + \tilde{A}u(t) \in H$ and

$$u''(t) + Du'(t) + \tilde{A}u(t) = 0 \tag{2.1}$$

If $u(t)$ is a solution of (2.1), then a vector-function

$$\mathbf{x}(t) = \begin{pmatrix} u'(t) \\ u(t) \end{pmatrix}$$

(formally) satisfies a first-order differential equation

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) \tag{2.2}$$

with a block operator matrix

$$\mathbf{A} = \begin{pmatrix} -D & -\tilde{A} \\ I & 0 \end{pmatrix}.$$

From mechanical viewpoint it is most natural to consider the equation (2.2) in an "energy" space $\mathcal{H} = H \times H_1$ with a dense domain of the operator $\mathbf{A}$ [6, 7, 11, 16]

$$\mathcal{D}(\mathbf{A}) = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| \, x_1, x_2 \in H_1, \ Dx_1 + \tilde{A}x_2 \in H \right\}.$$

An inverse of $\mathbf{A}$ is formally defined by a block operator matrix

$$\mathbf{A}^{-1} = \begin{pmatrix} 0 & I \\ -\tilde{A}^{-1} & -\tilde{A}^{-1}D \end{pmatrix}.$$

Let $\mathbf{y} = (y_1, y_2)^\top \in \mathcal{H} = H \times H_1$, then

$$\mathbf{A}^{-1}\mathbf{y} = \begin{pmatrix} y_2 \\ -\tilde{A}^{-1}y_1 - \tilde{A}^{-1}Dy_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Since $\tilde{A}^{-1} \in \mathcal{L}(H_{-1}, H_1)$ and $D \in \mathcal{L}(H_1, H_{-1})$, then $\tilde{A}^{-1}D \in \mathcal{L}(H_1, H_1)$. Therefore $-\tilde{A}^{-1}y_1 - \tilde{A}^{-1}Dy_2 \in H_1$ and $\mathbf{A}^{-1}\mathbf{y} \in H_1 \times H_1$. Moreover,

$$Dx_1 + \tilde{A}x_2 = Dy_2 + \tilde{A}\left(-\tilde{A}^{-1}y_1 - \tilde{A}^{-1}Dy_2\right) = -y_1 \in H.$$

Thus $\mathbf{A}^{-1}\mathbf{y} \in \mathcal{D}(\mathbf{A})$. Since $I \in \mathcal{L}(H_1, H)$ the operator $\mathbf{A}^{-1}$ is bounded and therefore the operator $\mathbf{A}$ is closed and $0 \in \rho(\mathbf{A})$.

Let $(\mathbf{x}, \mathbf{y})_\mathcal{H}$ be a natural scalar product on $\mathcal{H} = H \times H_1$ and $\|\mathbf{x}\|^2_\mathcal{H} = (\mathbf{x}, \mathbf{y})_\mathcal{H}$.

If operator A is self-adjoint, the spectral properties of operator $\mathbf{A}$ are well studied: $-\mathbf{A}$ is an m-accretive operator in the Hilbert space $\mathcal{H} = H \times H_1$ (see [2, 6, 7, 8, 10, 11] and references therein) and, consequently, $\mathbf{A}$ is a generator of a C_0-semigroup. Thus, differential equation (2.2) (and equation (2.1)) is correctly solvable in the space $\mathcal{H}$ for all $\mathbf{x}(0) = (u_1, u_0)^\top \in \mathcal{D}(\mathbf{A})$. Moreover, in this case operator $\mathbf{A}$ is a generator of a contraction semigroup [7]. It implies, that all solutions of (2.2) (and (2.1)) exponentially decay, i.e., for some $C, \omega > 0$

$$\|\mathbf{x}(t)\|_\mathcal{H} \leq C \exp(-\omega t) \|\mathbf{x}(0)\|_\mathcal{H} \quad t \geq 0.$$

For non-selfadjoint A operator $(-\mathbf{A})$ is not longer accretive in the space $\mathcal{H}$ with respect to the standard scalar product. But, under some assumptions, one can define a new scalar product on $\mathcal{H}$, which is topologically equivalent to the given one, such that an operator $(-\mathbf{A}-qI)$ (for some $q \geq 0$) is m-accretive and therefore the operator $\mathbf{A}$ is a generator of a C_0-semigroup on $\mathcal{H}$. If $q>0$, then $\mathbf{A}$ is a generator of a contraction semigroup and all solutions of (2.2) exponentially decay.

Let $k \in (0,\beta)$ (β is defined by (1.1)). Consider on the space $\mathcal{H}$ a sesquilinear form

$$[\mathbf{x},\mathbf{y}]_{\mathcal{H}} = (T^{1/2}x_2, T^{1/2}y_2) + k(D_1x_2,y_2)_{-1,1} - k^2(x_2,y_2) + (x_1+kx_2, y_1+ky_2),$$
$$\mathbf{x}=(x_1,x_2)^\top,\ \mathbf{y}=(y_1,y_2)^\top \in \mathcal{H}.$$

Obviously, $[\mathbf{x},\mathbf{y}] = \overline{[\mathbf{y},\mathbf{x}]}$ and

$$[\mathbf{x},\mathbf{x}]_{\mathcal{H}} = \|x_2\|_1^2 + k(D_1x_2,x_2)_{-1,1} + \|x_1\|^2 + 2k\operatorname{Re}(x_1,x_2).$$

Since $(D_1x,x)_{-1,1} = \operatorname{Re}(Dx,x)_{-1,1} \geq \beta\|x\|^2$ and

$$2|\operatorname{Re}(x_1,x_2)| \leq 2|(x_1,x_2)| \leq 2\|x_1\|\cdot\|x_2\| \leq \frac{\|x_1\|^2}{\beta} + \beta\|x_2\|^2,$$

then

$$\begin{aligned}[\mathbf{x},\mathbf{x}]_{\mathcal{H}} &\geq \|x_2\|_1^2 + k\Big((D_1x,x)_{-1,1} - \beta\|x_2\|^2\Big) + \left(1-\frac{k}{\beta}\right)\|x_1\|^2 \\ &\geq \|x_2\|_1^2 + \left(1-\frac{k}{\beta}\right)\|x_1\|^2.\end{aligned}$$

Inequalities[1] $|(D_1x,x)_{-1,1}| \leq \|D_1x\|_{-1}\cdot\|x\|_1 \leq \|D_1\|\cdot\|x\|_1^2$ and $\|x\|_1^2 \geq a_0\|x\|^2$ imply

$$\begin{aligned}[\mathbf{x},\mathbf{x}]_{\mathcal{H}} &\leq \Big(1+k\|D_1\|\Big)\|x_2\|_1^2 + k\beta\|x_2\|^2 + \left(1+\frac{k}{\beta}\right)\|x_1\|^2 \\ &\leq \left(1+k\|D_1\|+\frac{k\beta}{a_0}\right)\|x_2\|_1^2 + \left(1+\frac{k}{\beta}\right)\|x_1\|^2.\end{aligned}$$

Thus,

$$\left(1-\frac{k}{\beta}\right)\|\mathbf{x}\|_{\mathcal{H}}^2 \leq [\mathbf{x},\mathbf{x}]_{\mathcal{H}} \leq \operatorname{const}\|\mathbf{x}\|_{\mathcal{H}}^2$$

and $[\cdot,\cdot]_{\mathcal{H}}$ is a scalar product on $\mathcal{H}$, which is topologically equivalent to the given one. Denote $|\mathbf{x}|_{\mathcal{H}}^2 = [\mathbf{x},\mathbf{x}]_{\mathcal{H}}$.

Theorem 2.2. *Let the assumptions* (A) *and* (B) *hold and for some* $k\in(0,\beta)$ *and* $m\in(0,1]$

$$\omega_1 = \inf_{x\in H_1, x\neq 0} \frac{\frac{1}{k}(D_1x,x)_{-1,1} - \|x\|^2 - \frac{1}{4m}\left\|(\frac{1}{k}\tilde{S}-D_2)x\right\|_{-1}}{\|x\|^2} \geq 0. \qquad (2.3)$$

[1] $\|D_1\|$ is a norm of operator $D_1\in\mathcal{L}(H_1,H_{-1})$, i.e., $\|D_1\| = \sup_{x\in H_1,x\neq0}\|D_1x\|_{-1}/\|x\|_1$.

Then the operator $\mathbf{A}$ *is a generator of a* C_0*-semigroup* $\mathcal{T}(t) = \exp\{t\mathbf{A}\}$ $(t \geq 0)$ *and*

$$\left\|\mathcal{T}(t)\right\|_{\mathcal{H}} \leq \mathrm{const}\cdot\exp(-tk\theta)$$

where

$$\theta = \min\left\{\frac{\omega_1}{2}, \frac{1-m}{\omega_2}\right\} \geq 0$$

and[2]

$$\omega_2 = \sup_{x\in H_1, x\neq 0} \frac{\|x\|_1^2 + k(D_1 x, x)_{-1,1} + k^2\|x\|^2}{\|x\|_1^2} \tag{2.4}$$

Proof. For $\mathbf{x} = (x_1, x_2)^\top \in \mathcal{D}(\mathbf{A})$ let us consider a quadric form

$$\begin{aligned}
[\mathbf{A}\mathbf{x}, \mathbf{x}]_{\mathcal{H}} &= (T^{1/2}x_1, T^{1/2}x_2) + k(D_1x_1, x_2)_{-1,1} - k^2(x_1, x_2) \\
&\quad + (-Dx_1 - \tilde{A}x_2 + kx_1, x_1 + kx_2) \\
&= (Tx_1, x_2)_{-1,1} + k(D_1x_1, x_2)_{-1,1} - (Dx_1, x_1)_{-1,1} \\
&\quad - (\tilde{A}x_2, x_1)_{-1,1} + k(x_1, x_1) - k(Dx_1, x_2)_{-1,1} - k(\tilde{A}x_2, x_2)_{-1,1} \\
&= -(Dx_1, x_1)_{-1,1} + k(x_1, x_1) - k(\tilde{A}x_2, x_2)_{-1,1} - ik(D_2x_1, x_2)_{-1,1} \\
&\quad + (Tx_1, x_2)_{-1,1} - (Tx_2, x_1)_{-1,1} - i(\tilde{S}x_2, x_1)_{-1,1}
\end{aligned}$$

We used decompositions $\tilde{A} = T + i\tilde{S}$ and $D = D_1 + iD_2$. Consequently,

$$\begin{aligned}
\mathrm{Re}[\mathbf{A}\mathbf{x}, \mathbf{x}]_{\mathcal{H}} = &-(D_1x_1, x_1)_{-1,1} + k(x_1, x_1) - k(Tx_2, x_2)_{-1,1} \\
&- \mathrm{Re}\big(ik(D_2x_1, x_2)_{-1,1} + i(\tilde{S}x_2, x_1)_{-1,1}\big) \\
= &-(D_1x_1, x_1)_{-1,1} + k\|x_1\|^2 - k\|x_2\|_1^2 \\
&- \mathrm{Im}\big((\tilde{S}x_1, x_2)_{-1,1} - k(D_2x_1, x_2)_{-1,1}\big)
\end{aligned}$$

and

$$-\frac{1}{k}\,\mathrm{Re}[\mathbf{A}\mathbf{x}, \mathbf{x}]_{\mathcal{H}} = \frac{1}{k}(D_1x_1, x_1)_{-1,1} - \|x_1\|^2 + \|x_2\|_1^2 + \mathrm{Im}\left(\left(\frac{1}{k}\tilde{S} - D_2\right)x_1, x_2\right)_{-1,1}.$$

Since

$$\begin{aligned}
\left|\left(\left(\frac{1}{k}\tilde{S} - D_2\right)x_1, x_2\right)_{-1,1}\right| &\leq \left\|\left(\frac{1}{k}\tilde{S} - D_2\right)x_1\right\|_{-1} \cdot \|x_2\|_1 \\
&\leq \frac{1}{4m}\left\|\left(\frac{1}{k}\tilde{S} - D_2\right)x_1\right\|_{-1}^2 + m\|x_2\|_1^2,
\end{aligned}$$

then

$$\begin{aligned}
-\frac{1}{k}\mathrm{Re}[\mathbf{A}\mathbf{x},\mathbf{x}]_{\mathcal{H}} &\geq \frac{1}{k}(D_1x_1,x_1)_{-1,1} - \|x_1\|^2 - \frac{1}{4m}\left\|\left(\frac{1}{k}\tilde{S} - D_2\right)x_1\right\|_{-1}^2 + (1-m)\|x_2\|_1^2 \\
&\geq \omega_1\|x_1\|^2 + (1-m)\|x_2\|_1^2.
\end{aligned}$$

[2]Obviously, $\omega_2 \leq 1 + k\|D_1\| + k^2/a_0$.

Further, an inequality

$$2k|\operatorname{Re}(x_1, x_2)| \le 2|(x_1, kx_2)| \le 2\|x_1\| \cdot \|kx_2\| \le \|x_1\|^2 + k^2\|x_2\|^2$$

implies

$$[\mathbf{x}, \mathbf{x}]_{\mathcal{H}} \le 2\|x_1\|^2 + \|x_2\|_1^2 + k(D_1x_2, x_2)_{-1,1} + k^2\|x_2\|^2 \le 2\|x_1\|^2 + \omega_2\|x_2\|_1^2. \quad (2.5)$$

Thus

$$-\frac{1}{k}\operatorname{Re}[\mathbf{A}\mathbf{x}, \mathbf{x}]_{\mathcal{H}} \ge \omega_1\|x_1\|^2 + (1-m)\|x_2\|_1^2 \ge \theta(2\|x_1\|^2 + \omega_2\|x_2\|_1^2) \ge \theta[\mathbf{x}, \mathbf{x}]_{\mathcal{H}}$$

and an operator $(-\mathbf{A} - k\theta I)$ is accretive. Moreover, the operator $(-\mathbf{A} - k\theta I)$ is m-accretive (since $0 \in \rho(\mathbf{A})$) and[3]

$$\rho\left(-\mathbf{A} - k\theta I\right) \subset \{\lambda \in \mathbb{C},\ \operatorname{Re}\lambda < 0\} \Rightarrow \rho(-\mathbf{A}) \supset \{\lambda \in \mathbb{C},\ \operatorname{Re}\lambda < k\theta\}.$$

Therefore, the operator $\mathbf{A}$ is a generator of a C_0-semigroup [4, 5] $\mathcal{T}(t) = \exp\{t\mathbf{A}\}$, $t \ge 0$ and

$$\left|\mathcal{T}(t)\right|_{\mathcal{H}} \le \exp(-k\theta t), \quad t \ge 0.$$

On the space $\mathcal{H}$ norms $|\mathbf{x}|_{\mathcal{H}}$ and $\|\mathbf{x}\|_{\mathcal{H}}$ are equivalent and the inequality

$$\left\|\mathcal{T}(t)\right\|_{\mathcal{H}} \le \mathrm{const} \cdot \exp(-k\theta t), \quad t \ge 0$$

holds for some positive constant. □

Corollary 2.3. *Under the conditions of Theorem* 2.2 *for all* $\mathbf{x}_0 = (u_1, u_0)^\top \in \mathcal{D}(\mathbf{A})$ *vector-function*

$$\mathbf{x}(t) = \begin{pmatrix} w(t) \\ u(t) \end{pmatrix} = \mathcal{T}(t)\mathbf{x}_0 \in \mathcal{D}(\mathbf{A})$$

satisfies the first-order differential equation (2.2)*. $u(t)$ satisfies the second-order differential equation* (2.1) *with the initial conditions* (0.2) *and an inequality*

$$\|u(t)\|_1^2 + \|u'(t)\|^2 \le \mathrm{const} \cdot \exp\{-2k\theta t\}\left(\|u_0\|_1^2 + \|u_1\|^2\right)$$

holds for all $t \ge 0$.

Consider now a more strong assumption on the operator D:

(C) $D \in \mathcal{L}(H_1, H_{-1})$ and

$$\delta = \inf_{x \in H_1, x \ne 0} \frac{\operatorname{Re}(Dx, x)_{-1,1}}{\|x\|_1^2} > 0.$$

It is easy to show that the assumption (C) implies (B) and $\beta > a_0\delta$.

By $\|\tilde{S}\|$ and $\|D_2\|$ denote norms of the bounded operators $\tilde{S} \in \mathcal{L}(H_1, H_{-1})$ and $D_2 \in \mathcal{L}(H_1, H_{-1})$. Then for all $x \in H_1$

$$\|\tilde{S}x\|_{-1} \le \|\tilde{S}\| \cdot \|x\|_1, \quad \|D_2x\|_{-1} \le \|D_2\| \cdot \|x\|_1$$

[3] Obviously, the operator $(-\mathbf{A})$ is m-accretive as well.

Theorem 2.4. *Let the assumptions* (A) *and* (C) *are fulfilled and for some* $k \in (0,\beta)$ *and some* $p,q>0$ *with* $p+q \le 1$

$$\omega_1' = a_0 \left(\frac{\delta}{k} - \frac{1}{4pk^2}\|\tilde S\|^2 - \frac{1}{4q}\|D_2\|^2 \right) \ge 1$$

Then the operator $\mathbf{A}$ *is a generator of a* C_0*-semigroup* $\mathcal{T}(t) = \exp\{t\mathbf{A}\}$ $(t \ge 0)$ *and*

$$\big\|\mathcal{T}(t)\big\|_{\mathcal{H}} \le \mathrm{const}\cdot \exp(-tk\theta')$$

where

$$\theta' = \min\left\{ \frac{\omega_1'-1}{2}, \frac{1-p-q}{\omega_2} \right\} \ge 0$$

and ω_2 *is defined by* (2.4).

Proof. Consider on Hilbert space $\mathcal{H} = H \times H_1$ the scalar product $[\mathbf{x},\mathbf{y}]_{\mathcal{H}}$. Then

$$-\frac{1}{k}\operatorname{Re}[\mathbf{A}\mathbf{x},\mathbf{x}]_{\mathcal{H}} = \frac{1}{k}(D_1x_1,x_1)_{-1,1} - \|x_1\|^2 + \|x_2\|_1^2 + \frac{1}{k}\operatorname{Im}(\tilde S x_1, x_2)_{-1,1} - \operatorname{Im}(D_2x_1,x_2)_{-1,1}$$

(see the proof of Theorem 2.2). Since

$$\begin{aligned} |\operatorname{Im}(D_2x_1,x_2)_{-1,1}| &\le |(D_2x_1,x_2)_{-1,1}| \le \|D_2x_1\|_{-1}\cdot\|x_2\|_1 \\ &\le \frac{1}{4q}\|D_2x_1\|_{-1}^2 + q\|x_2\|_1^2 \le \frac{1}{4q}\|D_2\|^2\cdot\|x_1\|_1^2 + q\|x_2\|_1^2 \end{aligned}$$

$$\begin{aligned} \frac{1}{k}|\operatorname{Im}(\tilde S x_1,x_2)_{-1,1}| &\le \left|\left(\frac{1}{k}\tilde S x_1, x_2\right)_{-1,1}\right| \le \left\|\frac{1}{k}\tilde S x_1\right\|_{-1}\cdot\|x_2\|_1 \\ &\le \frac{1}{4p}\left\|\frac{1}{k}\tilde S x_1\right\|_{-1}^2 + p\|x_2\|_1^2 \le \frac{1}{4pk^2}\|\tilde S\|^2\cdot\|x_1\|_1^2 + p\|x_2\|_1^2 \end{aligned}$$

and taking into account $(D_1x,x)_{-1,1} \ge \delta\|x\|_1^2$ and $\|x\|_1^2 \ge a_0\|x\|^2$ we obtain

$$\begin{aligned} &-\frac{1}{k}\operatorname{Re}[\mathbf{A}\mathbf{x},\mathbf{x}]_{\mathcal{H}} \\ &\ge \frac{1}{k}(D_1x_1,x_1)_{-1,1} - \|x_1\|^2 - \frac{\|\tilde S\|^2}{4pk^2}\cdot\|x_1\|_1^2 - \frac{\|D_2\|^2}{4q}\cdot\|x_1\|_1^2 + (1-p-q)\|x_2\|_1^2 \\ &\ge \left(\frac{\delta}{k} - \frac{\|\tilde S\|^2}{4pk^2} - \frac{\|D_2\|^2}{4q}\right)\|x_1\|_1^2 - \|x_1\|^2 + (1-p-q)\|x_2\|_1^2 \\ &\ge (\omega_1'-1)\|x_1\|^2 + (1-p-q)\|x_2\|_1^2. \end{aligned}$$

Using (2.5) we finally have

$$-\frac{1}{k}\operatorname{Re}[\mathbf{A}\mathbf{x},\mathbf{x}]_{\mathcal{H}} \ge \theta'[\mathbf{x},\mathbf{x}]_{\mathcal{H}}.$$

Thus an operator $(-\mathbf{A} - k\theta' I)$ in m-accretive (since $0 \in \rho(\mathbf{A})$) and

$$\rho(-\mathbf{A}) \supset \{\lambda \in \mathbb{C},\ \operatorname{Re}\lambda < k\theta'\}.$$

Therefore, the operator $\mathbf{A}$ is a generator of a C_0-semigroup [4, 5] $\mathcal{T}(t) = \exp\{t\mathbf{A}\}$ $(t \geq 0)$ and

$$\left|\mathcal{T}(t)\right|_{\mathcal{H}} \leq \exp(-k\theta' t), \quad t \geq 0.$$

Since the norms $|\mathbf{x}|_{\mathcal{H}}$ and $\|\mathbf{x}\|_{\mathcal{H}}$ are equivalent then we have an inequality

$$\left\|\mathcal{T}(t)\right\|_{\mathcal{H}} \leq \mathrm{const} \cdot \exp(-k\theta' t), \quad t \geq 0$$

for some positive constant. □

Corollary 2.5. *Under the conditions of Theorem* 2.4 *for all* $\mathbf{x}_0 = (u_1, u_0)^\top \in \mathcal{D}(\mathbf{A})$ *a vector-valued function*

$$\mathbf{x}(t) = \begin{pmatrix} w(t) \\ u(t) \end{pmatrix} = \mathcal{T}(t)\mathbf{x}_0 \in \mathcal{D}(\mathbf{A})$$

satisfies the first-order differential equation (2.2)*.* $u(t)$ *satisfies the second-order differential equation* (2.1) *with an initial conditions* (0.2) *and the inequality*

$$\|u(t)\|_1^2 + \|u'(t)\|^2 \leq \mathrm{const} \cdot \exp\{-2k\theta' t\}\left(\|u_0\|_1^2 + \|u_1\|^2\right)$$

holds for all $t \geq 0$*.*

3. Related spectral problem

Let us consider a quadric pencil associated with the differential equation (0.1)

$$L(\lambda) = \lambda^2 I + \lambda D + A \quad \lambda \in \mathbb{C}.$$

Since $D : H_1 \to H_{-1}$ it is more naturally to consider an extension of pencil

$$\tilde{L}(\lambda) = \lambda^2 I + \lambda D + \tilde{A}$$

mapping H_1 to H_{-1}. Moreover, $\tilde{L}(\lambda) \in \mathcal{L}(H_1, H_{-1})$ for all $\lambda \in \mathbb{C}$.

Definition 3.1. The resolvent set of the pencil $\tilde{L}(\lambda)$ is defined as

$$\rho(\tilde{L}) = \{\lambda \in \mathbb{C} \ : \ \exists \tilde{L}^{-1}(\lambda) \in \mathcal{L}(H_{-1}, H_1)\}$$

The spectrum of the pencil is $\sigma(\tilde{L}) = \mathbb{C} \backslash \rho(\tilde{L})$.

In [7, 16] it was proved that $\sigma(\tilde{L}) = \sigma(\mathbf{A})$ and for $\lambda \neq 0$

$$(\mathbf{A} - \lambda I)^{-1} = \begin{pmatrix} \lambda^{-1}\left(\tilde{L}^{-1}(\lambda)\tilde{A} - I\right) & -\tilde{L}^{-1}(\lambda) \\ \tilde{L}^{-1}(\lambda)\tilde{A} & -\lambda\tilde{L}^{-1}(\lambda) \end{pmatrix}$$

This result allows to obtain a localization of the pencil's spectrum in a half-plane.

Proposition 3.2.

1. *Under the conditions of Theorem* 2.2 *the spectrum of the pencil* $\tilde{L}(\lambda)$ *belongs to a half-plane*
$$\sigma(\tilde{L}) \subseteq \{\mathrm{Re} \leq -k\theta\}.$$
2. *Under the conditions of Theorem* 2.4 *the spectrum of the pencil* $\tilde{L}(\lambda)$ *belongs to a half-plane*
$$\sigma(\tilde{L}) \subseteq \{\mathrm{Re} \leq -k\theta'\}.$$

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Infinite Norm Decompositions of C*-algebras

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Abstract. In the given article the notion of infinite norm decomposition of a C*-algebra is investigated. The infinite norm decomposition is some generalization of Peirce decomposition. It is proved that the infinite norm decomposition of any C*-algebra is a C*-algebra. C*-factors with an infinite and a nonzero finite projection and simple purely infinite C*-algebras are constructed.

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Introduction

In the given article the notion of infinite norm decomposition of a C*-algebra is investigated. It is known that for any projection p of a unital C*-algebra A the next equality is valid $A = pAp \oplus pA(1-p) \oplus (1-p)Ap \oplus (1-p)A(1-p)$, where $\oplus$ is a direct sum of spaces. The infinite norm decomposition is some generalization of Peirce decomposition. First such infinite decompositions were introduced in [1] by the author.

In this article a unital C*-algebra A with an infinite orthogonal set $\{p_i\}$ of equivalent projections such that $\sup_i p_i = 1$, and the set $\sum_{ij}^{o} p_iAp_j = \{\{a_{ij}\}$: for any indexes i, j, $a_{ij} \in p_iAp_j$, and $\| \sum_{k=1,\dots,i-1}(a_{ki} + a_{ik}) + a_{ii}\| \to 0$ at $i \to \infty\}$ are considered. Note that all infinite sets like $\{p_i\}$ are supposed to be countable.

The main results of the given article are the next:

- For any C*-algebra A with an infinite orthogonal set $\{p_i\}$ of equivalent projections such that $\sup_i p_i = 1$ the set $\sum_{ij}^{o} p_iAp_j$ is a C*-algebra with the componentwise algebraic operations, the associative multiplication and the norm.
- There exist a C*-algebra A and different countable orthogonal sets $\{e_i\}$ and $\{f_i\}$ of equivalent projections in A such that $\sup_i e_i = 1$, $\sup_i f_i = 1$, $\sum_{ij}^{o} e_iAe_j \neq \sum_{ij}^{o} f_iAf_j$.

- If A is a W*-factor of type II$_\infty$, then there exists a countable orthogonal set $\{p_i\}$ of equivalent projections in A such that $\sum_{ij}^o p_i A p_j$ is a C*-factor with a nonzero finite and an infinite projection. In this case $\sum_{ij}^o p_i A p_j$ is not a von Neumann algebra.
- If A is a W*-factor of type III, then for any countable orthogonal set $\{p_i\}$ of equivalent projections in A. The C*-subalgebra $\sum_{ij}^o p_i A p_j$ is simple and purely infinite. In this case $\sum_{ij}^o p_i A p_j$ is not a von Neumann algebra.
- There exists a C*-algebra A with an orthogonal set $\{p_i\}$ of equivalent projections such that $\sum_{ij}^o p_i A p_j$ is not a two-sided ideal of A.

1. Infinite norm decompositions

Lemma 1. *Let A be a C*-algebra, $\{p_i\}$ be an infinite orthogonal set of projections with the least upper bound 1 in the algebra A and let $\mathcal{A} = \{\{p_i a p_j\} : a \in A\}$. Then,*

1) *the set $\mathcal{A}$ is a vector space with the next componentwise algebraic operations*

$$\lambda\{p_i a p_j\} = \{p_i \lambda a p_j\}, \lambda \in \mathbb{C}$$
$$\{p_i a p_j\} + \{p_i b p_j\} = \{p_i(a+b)p_j\}, a, b \in A,$$

2) *the algebra A and the vector space $\mathcal{A}$ can be identified in the sense of the next map*

$$\mathcal{I} : a \in A \to \{p_i a p_j\} \in \mathcal{A}.$$

Proof. Item 1) of the lemma can be easily proved.

Proof of item 2): We assert that $\mathcal{I}$ is a one-to-one map. Indeed, it is clear, that for any $a \in A$ there exists a unique set $\{p_i a p_j\}$, defined by the element a.

Suppose that there exist different elements a and b in A such that $p_i a p_j = p_i b p_j$ for all i, j, i.e., $\mathcal{I}(a) = \mathcal{I}(b)$. Then $p_i(a-b)p_j = 0$ for all i and j. Observe that $p_i((a-b)p_j(a-b)^*) = ((a-b)p_j(a-b)^*)p_i = 0$ and $(a-b)p_j(a-b)^* \geq 0$ for all i, j. Therefore, the element $(a-b)p_j(a-b)^*$ commutes with every projection in $\{p_i\}$.

We prove $(a-b)p_j(a-b)^* = 0$. Indeed, there exists a maximal commutative $*$-subalgebra A_o of the algebra A, containing the set $\{p_i\}$ and the element $(a-b)p_j(a-b)^*$. Since $(a-b)p_j(a-b)^* p_i = p_i(a-b)p_j(a-b)^* = 0$ for any i, then the condition $(a-b)p_j(a-b)^* \neq 0$ contradicts the equality $\sup_i p_i = 1$.

Indeed, in this case $p_i \leq 1 - 1/\|(a-b)p_j(a-b)^*\|(a-b)p_j(a-b)^*$ for any i. Since by $(a-b)p_j(a-b)^* \neq 0$ we have $1 > 1 - 1/\|(a-b)p_j(a-b)^*\|(a-b)p_j(a-b)^*$, then we get a contradiction with $\sup_i p_i = 1$. Therefore $(a-b)p_j(a-b)^* = 0$.

Hence, since A is a C*-algebra, than $\|(a-b)p_j(a-b)^*\| = \|((a-b)p_j)((a-b)p_j)^*\| = \|((a-b)p_j)\|\|((a-b)p_j)^*\| = \|(a-b)p_j\|^2 = 0$ for any j. Therefore $(a-b)p_j = 0$, $p_j(a-b)^* = 0$ for any j. Analogously, we can get $p_j(a-b) = 0$, $(a-b)^* p_j = 0$ for any j. Hence the elements $a-b$, $(a-b)^*$ commute with every projection in $\{p_i\}$. Then there exists a maximal commutative $*$-subalgebra A_o of the algebra A, containing the set $\{p_i\}$ and the element $(a-b)(a-b)^*$. Since

$p_i(a-b)(a-b)^* = (a-b)(a-b)^* p_i = 0$ for any i, then the condition $(a-b)(a-b)^* \neq 0$ contradicts the equality $\sup_i p_i = 1$.

Therefore, $(a-b)(a-b)^* = 0$, $a-b=0$, i.e., $a=b$. Thus the map $\mathcal{I}$ is one-to-one. $\square$

Lemma 2. *Let A be a C*-algebra, $\{p_i\}$ be an infinite orthogonal set of projections with the least upper bound* 1 *in the algebra A and $a \in A$. Then, if $p_i a p_j = 0$ for all i, j, then $a = 0$.*

Proof. Let $p \in \{p_i\}$. Observe that $p_i a p_j a^* = p_i(a p_j a^*) = a p_j a^* p_i = (a p_j a^*) p_i = 0$ for all i, j and $a p_j a^* = a p_j p_j a^* = (a p_j)(p_j a^*) = (a p_j)(a p_j)^* \geq 0$. Therefore, the element $a p_j a^*$ commutes with all projections of the set $\{p_i\}$.

We prove $a p_j a^* = 0$. Indeed, there exists a maximal commutative $*$-subalgebra A_o of the algebra A, containing the set $\{p_i\}$ and the element $a p_j a^*$. Since $p_i(a p_j a^*) = (a p_j a^*) p_i = 0$ for any i, then the condition $a p_j a^* \neq 0$ contradicts the equality $\sup_i p_i = 1$ (see the proof of Lemma 1). Hence $a p_j a^* = 0$.

Hence, since A is a C*-algebra, then

$$\|a p_j a^*\| = \|(a p_j)(a p_j)^*\| = \|(a p_j)\| \|(a p_j)^*\| = \|a p_j\|^2 = 0$$

for any j. Therefore $a p_j = 0$, $p_j a^* = 0$ for any j. Analogously we have $p_j a = 0$, $a^* p_j = 0$ for any j. Hence the elements a, a^* commute with all projections of the set $\{p_i\}$. Then there exists a maximal commutative $*$-subalgebra A_o of the algebra A, containing the set $\{p_i\}$ and the element aa^*. Since $p_i a a^* = a a^* p_j = 0$ for any i, then the condition $aa^* \neq 0$ contradicts the equality $\sup_i p_i = 1$ (see the proof of Lemma 1). Hence $aa^* = 0$ and $a = 0$. $\square$

Lemma 3. *Let A be a C*-algebra on a Hilbert space H, $\{p_i\}$ be an infinite orthogonal set of projections in A with the least upper bound* 1 *in the algebra $B(H)$ and $a \in A$. Then $a \geq 0$ if and only if for any finite subset $\{p_k\}_{k=1}^n \subset \{p_i\}$ the inequality $pap \geq 0$ holds, where $p = \sum_{k=1}^n p_k$.*

Proof. By positivity of the operator $T : a \to bab, a \in A$ for any $b \in A$, if $a \geq 0$, then for any finite subset $\{p_k\}_{k=1}^n \subset \{p_i\}$ the inequality $pap \geq 0$ holds.

Conversely, let $a \in A$. Suppose that for any finite subset $\{p_k\}_{k=1}^n \subset \{p_i\}$ the inequality $pap \geq 0$ holds, where $p = \sum_{k=1}^n p_k$.

Let $a = c + id$ for some nonzero self-adjoint elements c, d in A. Then $(p_i + p_j)(c+id)(p_i+p_j) = (p_i+p_j)c(p_i+p_j) + i(p_i+p_j)d(p_i+p_j) \geq 0$ for all i, j. In this case the elements $(p_i+p_j)c(p_i+p_j)$ and $(p_i+p_j)d(p_i+p_j)$ are self-adjoint. Then $(p_i+p_j)d(p_i+p_j) = 0$ and $p_i d p_j = 0$ for all i, j. Hence by Lemma 2 we have $d = 0$. Therefore $a = c = c^* = a^*$, i.e., $a \in A_{sa}$. Hence, a is a nonzero self-adjoint element in A. Let $b_n^\alpha = \sum_{kl=1}^n p_k^\alpha a p_l^\alpha$ for all natural numbers n and finite subsets $\{p_k^\alpha\}_{k=1}^n \subset \{p_i\}$. Then the set (b_n^α) ultraweakly converges to the element a.

Indeed, we have $A \subseteq B(H)$. Let $\{q_\xi\}$ be a maximal orthogonal set of minimal projections of the algebra $B(H)$ such, that $p_i = \sup_\eta q_\eta$ for some subset $\{q_\eta\} \subset \{q_\xi\}$ for any i. For arbitrary projections q and p in $\{q_\xi\}$ there exists a number $\lambda \in \mathbb{C}$ such, that $qap = \lambda u$, where u is an isometry in $B(H)$, satisfying the conditions $q = uu^*$, $p = u^*u$. Let $q_{\xi\xi} = q_\xi$, $q_{\xi\eta}$ be such element that $q_\xi = q_{\xi\eta}q_{\xi\eta}^*$, $q_\eta = q_{\xi\eta}^* q_{\xi\eta}$ for all different ξ and η. Then, let $\{\lambda_{\xi\eta}\}$ be a set of numbers such that $q_\xi a q_\eta = \lambda_{\xi\eta} q_{\xi\eta}$ for all ξ, η. In this case, since $q_\xi aa^* q_\xi = q_\xi(\sum_\eta \lambda_{\xi\eta}\bar{\lambda}_{\xi\eta})q_\xi < \infty$ we have the quantity of nonzero numbers of the set $\{\lambda_{\xi\eta}\}_\eta$ (ξth string of the infinite-dimensional matrix $\{\lambda_{\xi\eta}\}_{\xi\eta}$) is not greater then the countable cardinal number and the sequence (λ_n^ξ) of all these nonzero numbers converges to zero. Let v_{q_ξ} be a vector of the Hilbert space H which generates the minimal projection q_ξ. Then the set $\{v_{q_\xi}\}$ forms a complete orthonormal system of the space H. Let v be an arbitrary vector of the space H and μ_ξ be a coefficient of Fourier of the vector v, corresponding to v_{q_ξ} in relative to the complete orthonormal system $\{v_{q_\xi}\}$. Then, since $\sum_\xi \mu_\xi\bar{\mu}_\xi < \infty$ we have the quantity of all nonzero elements of the set $\{\mu_\xi\}_\xi$ is not greater then the countable cardinal number and the sequence (μ_n) of all these nonzero numbers converges to zero.

Let ν_ξ be the ξth coefficient of Fourier (corresponding to v_{q_ξ}) of the vector $a(v) \in H$ in relative to the complete orthonormal system $\{v_{q_\xi}\}$. Then $\nu_\xi = \sum_\eta \lambda_{\xi\eta}\mu_\eta$ and the scalar product $< a(v), v >$ is equal to the sum $\sum_\xi \nu_\xi\mu_\xi$. Since the element $a(v)$ belongs to H we have quantity of all nonzero elements in the set $\{\nu_\xi\}_\xi$ is not greater then the countable cardinal number and the sequence (ν_n) of all these nonzero numbers converges to zero.

Let ε be an arbitrary positive number. Then, since quantity of nonzero numbers of the sets $\{\mu_\xi\}_\xi$ and $\{\nu_\xi\}_\xi$ is not greater then the countable cardinal number and $\sum_\xi \nu_\xi\bar{\nu}_\xi < \infty$, $\sum_\xi \mu_\xi\bar{\mu}_\xi < \infty$ we have there exists $\{f_k\}_{k=1}^l \subset \{p_i\}$ such that for the set of indexes $\Omega_1 = \{\xi : \exists p \in \{f_k\}_{k=1}^l, q_\xi \leq p\}$ the next equality holds

$$\left|\sum_\xi \nu_\xi\mu_\xi - \sum_{\xi\in\Omega_1} \nu_\xi\mu_\xi\right| < \varepsilon.$$

Then, since quantity of nonzero numbers of the sets $\{\mu_\xi\}_\xi$ and $\{\lambda_{\xi\eta}\}_\eta$ is not greater then the countable cardinal number, and $\sum_\eta \lambda_{\xi\eta}\bar{\lambda}_{\xi\eta} < \infty$, $\sum_\xi \mu_\xi\bar{\mu}_\xi < \infty$ we have there exists $\{e_k\}_{k=1}^m \subset \{p_i\}$ such that for the set of indexes $\Omega_2 = \{\xi : \exists p \in \{e_k\}_{k=1}^m, q_\xi \leq p\}$ the next equality holds

$$\left|\sum_\eta \lambda_{\xi\eta}\mu_\eta - \sum_{\eta\in\Omega_2} \lambda_{\xi\eta}\mu_\eta\right| < \varepsilon.$$

Hence for the finite set $\{p_k\}_{k=1}^n = \{f_k\}_{k=1}^l \cup \{e_k\}_{k=1}^m$ and the set $\Omega = \{\xi : \exists p \in \{p_k\}_{k=1}^n, q_\xi \leq p\}$ of indexes we have

$$\left|\sum_\xi \nu_\xi\mu_\xi - \sum_{\xi\in\Omega}\left(\sum_{\eta\in\Omega} \lambda_{\xi\eta}\mu_\eta\right)\mu_\xi\right| < \varepsilon.$$

At the same time, $\langle(\sum_{kl=1}^n p_k a p_l)(v), v\rangle = \sum_{\xi\in\Omega}(\sum_{\eta\in\Omega} \lambda_{\xi\eta}\mu_\eta)\mu_\xi$. Therefore,

$$\left|\langle a(v), v\rangle - \left\langle\left(\sum_{kl=1}^n p_k a p_l\right)(v), v\right\rangle\right| < \varepsilon.$$

Hence, since the vector v and the number ε are chosen arbitrarily we have the net (b_n^α) ultraweakly converges to the element a.

Now there exists a maximal orthogonal set $\{e_\xi\}$ of minimal projections of the algebra $B(H)$ of all bounded linear operators on H such that the element a and the set $\{e_\xi\}$ belong to some maximal commutative $*$-subalgebra A_o of the algebra $B(H)$. We have for any finite subset $\{p_k\}_{k=1}^n \subset \{p_i\}$ and $e \in \{e_\xi\}$ the inequality $e(\sum_{kl=1}^n p_k a p_l)e \geq 0$ holds by the positivity of the operator $T : b \to ebe, b \in A$.

By the previous part of the proof the net $(e_\xi b_n^\alpha e_\xi)_{\alpha n}$ ultraweakly converges to the element $e_\xi a e_\xi$ for any index ξ. Then we have $e_\xi b_n^\alpha e_\xi \geq 0$ for all n and α. Therefore, the ultraweak limit $e_\xi a e_\xi$ of the net $(e_\xi b_n^\alpha e_\xi)_{\alpha n}$ is a nonnegative element. Hence $e_\xi a e_\xi \geq 0$. Therefore, since e_ξ is chosen arbitrarily we have $a \geq 0$. □

Lemma 4. *Let A be a C*-algebra on a Hilbert space H, $\{p_i\}$ be an infinite orthogonal set of projections in A with the least upper bound 1 in the algebra $B(H)$ and $a \in A$. Then*

$$\|a\| = \sup\left\{\left\|\sum_{kl=1}^n p_k a p_l\right\| : n \in N, \{p_k\}_{k=1}^n \subseteq \{p_i\}\right\}.$$

Proof. The inequality $-\|a\|1 \leq a \leq \|a\|1$ holds. Then $-\|a\|p \leq pap \leq \|a\|p$ for all natural numbers n and finite subsets $\{p_k^\alpha\}_{k=1}^n \subset \{p_i\}$, where $p = \sum_{k=1}^n p_k$. Therefore

$$\|a\| \geq \sup\left\{\left\|\sum_{kl=1}^n p_k a p_l\right\| : n \in N, \{p_k\}_{k=1}^n \subseteq \{p_i\}\right\}.$$

At the same time, since the finite subset $\{p_k\}_{k=1}^n$ of $\{p_i\}$ is chosen arbitrarily and by Lemma 6 we have

$$\|a\| = \sup\left\{\left\|\sum_{kl=1}^n p_k a p_l\right\| : n \in N, \{p_k\}_{k=1}^n \subseteq \{p_i\}\right\}.$$

Otherwise, if

$$\|a\| > \lambda = \sup\left\{\left\|\sum_{kl=1}^n p_k a p_l\right\| : n \in N, \{p_k\}_{k=1}^n \subseteq \{p_i\}\right\}$$

then by Lemma 3 $-\lambda 1 \leq a \leq \lambda 1$. But the last inequality is a contradiction. □

Lemma 5. *Let A be a C*-algebra on a Hilbert space H, $\{p_i\}$ be an infinite orthogonal set of projections in A with the least upper bound 1 in the algebra $B(H)$, and let $\mathcal{A} = \{\{p_i a p_j\} : a \in A\}$. Then,*

1) *the vector space $\mathcal{A}$ is a unit-order space with respect to the order $\{p_i a p_j\} \geq 0$ ($\{p_i a p_j\} \geq 0$ if for any finite subset $\{p_k\}_{k=1}^n \subset \{p_i\}$ the inequality $pap \geq 0$ holds, where $p = \sum_{k=1}^n p_k$) and the norm*

$$\|\{p_i a p_j\}\| = \sup\left\{\left\|\sum_{kl=1}^n p_k a p_l\right\| : n \in N, \{p_k\}_{k=1}^n \subseteq \{p_i\}\right\}.$$

2) *the algebra A and the unit-order space $\mathcal{A}$ can be identified as unit-order spaces in the sense of the map*

$$\mathcal{I} : a \in A \to \{p_i a p_j\} \in \mathcal{A}.$$

Proof. This lemma follows by Lemmas 1, 3 and 4. □

Remark. Observe that by Lemma 4 the order and the norm in the unit-order space $\mathcal{A} = \{\{p_i a p_j\} : a \in A\}$ can be defined as follows to: $\{p_i a p_j\} \geq 0$ if $a \geq 0$; $\|\{p_i a p_j\}\| = \|a\|$. By Lemmas 3 and 4 they are equivalent to the order and the norm, defined in Lemma 5, correspondingly.

Let A be a C*-algebra, $\{p_i\}$ be a countable orthogonal set of equivalent projections in A such that $\sup_i p_i = 1$ and

$$\sum_{ij}^{o} p_i A p_j = \Bigg\{\{a_{ij}\} : \text{for any indexes}\, i, j,\; a_{ij} \in p_i A p_j, \text{ and } \left\| \sum_{k=1,\dots,i-1} (a_{ki} + a_{ik}) + a_{ii} \right\| \to 0 \,\text{at}\; i \to \infty \Bigg\}.$$

If we introduce a componentwise algebraic operations in this set then $\sum_{ij}^{o} p_i A p_j$ becomes a vector space. Also, note that $\sum_{ij}^{o} p_i A p_j$ is a vector subspace of $\mathcal{A}$. Observe that $\sum_{ij}^{o} p_i A p_j$ is a normed subspace of the algebra $\mathcal{A}$ and $\|\sum_{i,j=1}^{n} a_{ij} - \sum_{i,j=1}^{n+1} a_{ij}\| \to 0$ at $n \to \infty$ for any $\{a_{ij}\} \in \sum_{ij}^{o} p_i A p_j$.

Let $\sum_{ij}^{o} a_{ij} := \lim_{n\to\infty} \sum_{i,j=1}^{n} a_{ij}$ for any $\{a_{ij}\} \in \sum_{ij}^{o} p_i A p_j$ and

$$C^*(\{p_i A p_j\}_{ij}) := \Bigg\{\sum_{ij}^{o} a_{ij} : \{a_{ij}\} \in \sum_{ij}^{o} p_i A p_j\Bigg\}.$$

Then $C^*(\{p_i A p_j\}_{ij}) \subseteq A$. By Lemma 5 A and $\mathcal{A}$ can be identified. We observe that, the normed spaces $\sum_{ij}^{o} p_i A p_j$ and $C^*(\{p_i A p_j\}_{ij})$ can also be identified. Further, without loss of generality we will use these identifications.

Theorem 6. *Let A be a unital C*-algebra, $\{p_i\}$ be a countable orthogonal set of equivalent projections in A and $\sup_i p_i = 1$. Then $\sum_{ij}^{o} p_i A p_j$ is a C*-subalgebra of A with the componentwise algebraic operations, the associative multiplication and the norm.*

Proof. We have $\sum_{ij}^{o} p_i A p_j$ is a normed subspace of the algebra A.

Let (a_n) be a sequence of elements in $\sum_{ij}^{o} p_i A p_j$ such that (a_n) norm converges to some element $a \in A$. We have $p_i a_n p_j \to p_i a p_j$ at $n \to \infty$ for all i and j. Hence $p_i a p_j \in p_i A p_j$ for all i, j. Let $b^n = \sum_{k=1}^{n} (p_{n-1} a p_k + p_k a p_{n-1}) + p_n a p_n$ and $c_m^n = \sum_{k=1}^{n} (p_{n-1} a_m p_k + p_k a_m p_{n-1}) + p_n a_m p_n$ for any n. Then $c_m^n \to b^n$ at $m \to \infty$. It should be proven that $\|b_n\| \to 0$ at $n \to \infty$.

Let $\varepsilon \in \mathbb{R}_+$. Then there exists m_o such that $\|a - a_m\| < \varepsilon$ for any $m > m_o$. Also for all n and $\{p_k\}_{k=1}^{n} \subset \{p_i\}$ $\|(\sum_{k=1}^{n} p_k)(a - a_m)(\sum_{k=1}^{n} p_k)\| < \varepsilon$. Hence $\|b^n - c_m^n\| < 2\varepsilon$ for any $m > m_o$. At the same time, $\|b^n - c_{m_1}^n\| < 2\varepsilon$, $\|b^n - c_{m_2}^n\| < 2\varepsilon$

for all $m_o < m_1$, m_2. Since $(a_n) \subset \sum_{ij}^{o} p_iAp_j$ then for any m $\|c_m^n\| \to 0$ at $n \to \infty$. Hence, since $\|c_{m_1}^n\| \to 0$ and $\|c_{m_2}^n\| \to 0$ at $n \to \infty$ we have there exists n_o such that $\|c_{m_1}^n\| < \varepsilon$, $\|c_{m_2}^n\| < \varepsilon$ and $\|c_{m_1}^n + c_{m_2}^n\| < 2\varepsilon$ for any $n > n_o$. Then $\|2b_n\| = \|b^n - c_{m_1}^n + c_{m_1}^n + c_{m_2}^n + b^n - c_{m_2}^n\| \leq \|b^n - c_{m_1}^n\| + \|c_{m_1}^n + c_{m_2}^n\| + \|b^n - c_{m_2}^n\| < 2\varepsilon + 2\varepsilon + 2\varepsilon = 6\varepsilon$ for any $n > n_o$, i.e., $\|b_n\| < 3\varepsilon$ for any $n > n_o$. Since ε is chosen arbitrarily we have $\|b_n\| \to 0$ at $n \to \infty$. Therefore $a \in \sum_{ij}^{o} p_iAp_j$. Since the sequence (a_n) is chosen arbitrarily we have $\sum_{ij}^{o} p_iAp_j$ is a Banach space.

Let $\{a_{ij}\}$, $\{b_{ij}\}$ be arbitrary elements of the Banach space $\sum_{ij}^{o} p_iAp_j$. Let $a_m = \sum_{kl=1}^{m} a_{kl}$, $b_m = \sum_{kl=1}^{m} b_{kl}$ for all natural numbers m. We have the sequence (a_m) converges to $\{a_{ij}\}$ and the sequence (b_m) converges to $\{b_{ij}\}$ in $\sum_{ij}^{o} p_iAp_j$. Also for all n and m $a_mb_n \in \sum_{ij}^{o} p_iAp_j$. Then for any n the sequence (a_mb_n) converges to $\{a_{ij}\}b_n$ at $m \to \infty$. Hence $\{a_{ij}\}b_n \in \sum_{ij}^{o} p_iAp_j$. Note that $\sum_{ij}^{o} p_iAp_j \subseteq A$. Therefore for any $\varepsilon \in \mathbb{R}_+$ there exists n_o such that $\|\{a_{ij}\}b_{n+1} - \{a_{ij}\}b_n\| \leq \|\{a_{ij}\}\|\|b_{n+1} - b_n\| \leq \varepsilon$ for any $n > n_o$. Hence the sequence $(\{a_{ij}\}b_n)$ converges to $\{a_{ij}\}\{b_{ij}\}$ at $n \to \infty$. Since $\sum_{ij}^{o} p_iAp_j$ is a Banach space then $\{a_{ij}\}\{b_{ij}\} \in \sum_{ij}^{o} p_iAp_j$. Since $\sum_{ij}^{o} p_iAp_j \subseteq A$ we have $\sum_{ij}^{o} p_iAp_j$ is a C*-algebra. □

Let H be an infinite-dimensional Hilbert space, $B(H)$ be the algebra of all bounded linear operators. Let $\{p_i\}$ be a countable orthogonal set of equivalent projections in $B(H)$ and $\sup_i p_i = 1$. Let $\{\{p_j^i\}_j\}_i$ be the set of infinite subsets of $\{p_i\}$ such that for all distinct ξ and η $\{p_j^\xi\}_j \cap \{p_j^\eta\}_j = \oslash$, $|\{p_j^\xi\}_j| = |\{p_j^\eta\}_j|$ and $\{p_i\} = \cup_i\{p_j^i\}_j$. Then let $q_i = \sup_j p_j^i$ in $B(H)$ for all i. Then $\sup_i q_i = 1$ and $\{q_i\}$ be a countable orthogonal set of equivalent projections. Then we say that the countable orthogonal set $\{q_i\}$ of equivalent projections *is defined by the set* $\{p_i\}$ in $B(H)$. We have the next corollary.

Corollary 7. *Let A be a unital C*-algebra on a Hilbert space H, $\{p_i\}$ be a countable orthogonal set of equivalent projections in A and $\sup_i p_i = 1$. Let $\{q_i\}$ be a countable orthogonal set of equivalent projections in $B(H)$ defined by the set $\{p_i\}$ in $B(H)$. Then $\sum_{ij}^{o} q_iAq_j$ is a C*-subalgebra of the algebra A.*

Proof. Let $\{\{p_j^i\}_j\}_i$ be the set of infinite subsets of $\{p_i\}$ such that for all distinct ξ and η $\{p_j^\xi\}_j \cap \{p_j^\eta\}_j = \oslash$, $|\{p_j^\xi\}_j| = |\{p_j^\eta\}_j|$ and $\{p_i\} = \cup_i\{p_j^i\}_j$. Then let $q_i = \sup_j p_j^i$ in $B(H)$ for all i. Then we have for all i and j $q_iAq_j = \{\{p_\xi^i a p_\eta^j\}_{\xi\eta} : a \in A\}$. Hence $q_iAq_j \subset A$ for all i and j.

The rest part of the proof is the repeating of the proof of Theorem 6. □

Example. 1. Let $\mathcal{M}$ be the closure on the norm of the inductive limit $\mathcal{M}_o$ of the inductive system

$$C \to M_2(C) \to M_3(C) \to M_4(C) \to \cdots,$$

where $M_n(C)$ is mapped into the upper left corner of $M_{n+1}(C)$. Then $\mathcal{M}$ is a C*-algebra ([1]). The algebra $\mathcal{M}$ contains the minimal projections of the form e_{ii}, where e_{ij} is an infinite-dimensional matrix, whose (i,i)th component is 1 and the rest components are zeros. These projections form the countable orthogonal set $\{e_{ii}\}_{i=1}^{\infty}$ of minimal projections. Let

$$M_n^o(\mathbb{C}) = \Bigg\{ \sum_{ij} \lambda_{ij} e_{ij} : \lambda_{ij} \in \mathbb{C} \,\text{for any indexes}\, i,j \,\text{and}\\ \Bigg\| \sum_{k=1,\ldots,i-1} (\lambda_{ki}e_{ki} + \lambda_{ik}e_{ik}) + \lambda_{ii}e_{ii} \Bigg\| \to 0 \,\text{at}\, i \to \infty \Bigg\}.$$

Then $\mathbb{C}\cdot 1 + M_n^o(\mathbb{C}) = \mathcal{M}$ (see [2]) and by Theorem 6 $M_n^o(\mathbb{C})$ is a simple C*-algebra. Note that there exists a mistake in the formulation of Theorem 3 in [2]. $\mathbb{C}\cdot 1 + M_n^o(\mathbb{C})$ is a C*-algebra. But the algebra $\mathbb{C}\cdot 1 + M_n^o(\mathbb{C})$ is not simple. Because $\mathbb{C}\cdot 1 + M_n^o(\mathbb{C}) \neq M_n^o(\mathbb{C})$ and $M_n^o(\mathbb{C})$ is an ideal of the algebra $\mathbb{C}\cdot 1 + M_n^o(\mathbb{C})$, i.e., $[\mathbb{C}\cdot 1 + M_n^o(\mathbb{C})]\cdot M_n^o(\mathbb{C}) \subseteq M_n^o(\mathbb{C})$.

2. There exist a C*-algebra A and different countable orthogonal sets $\{e_i\}$ and $\{f_i\}$ of equivalent projections in A such that $\sup_i e_i = 1$, $\sup_i f_i = 1$, $\sum_{ij}^o e_iAe_j \neq \sum_{ij}^o f_iAf_j$. Indeed, let H be an infinite-dimensional Hilbert space, $B(H)$ be the algebra of all bounded linear operators. Let $\{p_i\}$ be a countable orthogonal set of equivalent projections in $B(H)$ and $\sup_i p_i = 1$. Then $\sum_{ij}^o p_iB(H)p_j \subset B(H)$. Let $\{\{p_j^i\}_j\}_i$ be the set of infinite subsets of $\{p_i\}$ such that for all distinct ξ and η $\{p_j^\xi\}_j \cap \{p_j^\eta\}_j = \oslash$, $|\{p_j^\xi\}_j| = |\{p_j^\eta\}_j|$ and $\{p_i\} = \cup_i\{p_j^i\}_j$. Then let $q_i = \sup_j p_j^i$ for all i. Then $\sup_i q_i = 1$ and $\{q_i\}$ be a countable orthogonal set of equivalent projections. We assert that $\sum_{ij}^o p_iB(H)p_j \neq \sum_{ij}^o q_iB(H)q_j$. Indeed, let $\{x_{ij}\}$ be a set of matrix units constructed by the infinite set $\{p_j^1\}_j \in \{\{p_j^i\}_j\}_i$, i.e., for all i, j, $x_{ij}x_{ij}^* = p_i^1$, $x_{ij}^*x_{ij} = p_j^1$, $x_{ii} = p_i^1$. Then the von Neumann algebra $\mathcal{N}$ generated by the set $\{x_{ij}\}$ is isometrically isomorphic to $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. We note that $\mathcal{N}$ is not a subset of $\sum_{ij}^o p_iB(H)p_j$. At the same time, $\mathcal{N} \subseteq \sum_{ij}^o q_iB(H)q_j$ and $\sum_{ij}^o p_i^1\mathcal{N}p_j^1 \subseteq \sum_{ij}^o p_iB(H)p_j$.

Theorem 8. *Let A be a unital simple C*-algebra on a Hilbert space H, $\{p_i\}$ be a countable orthogonal set of equivalent projections in A and $\sup_i p_i = 1$. Let $\{q_i\}$ be a countable orthogonal set of equivalent projections in $B(H)$ defined by the set $\{p_i\}$ in $B(H)$. Then $\sum_{ij}^o q_iAq_j$ is a simple C*-algebra.*

Proof. By Theorem 6 $\sum_{ij}^o p_iAp_j$ is a C*-algebra. Let $\{\{p_j^i\}_j\}_i$ be the set of infinite subsets of $\{p_i\}$ such that for all distinct ξ and η $\{p_j^\xi\}_j \cap \{p_j^\eta\}_j = \oslash$, $|\{p_j^\xi\}_j| = |\{p_j^\eta\}_j|$ and $\{p_i\} = \cup_i\{p_j^i\}_j$. Then let $q_i = \sup_j p_j^i$ in $B(H)$, for all i. Then we have $q_iAq_j = \{\{p_\xi^i a p_\xi^j\} : a \in A\}$ for all i and j. Hence $q_iAq_j \subset A$ for all i and j. By Corollary 7 $\sum_{ij}^o q_iAq_j$ is a C*-algebra.

Since projections of the set $\{p_i\}$ are pairwise equivalent we have the projection q_i is equivalent to $1 \in A$ for any i. Hence $q_iAq_i \cong A$ and q_iAq_i is a simple C*-algebra for any i.

Let q be an arbitrary projection in $\{q_i\}$. Then qAq is a C*-subalgebra of $\sum_{ij}^{o} q_iAq_j$. Let I be a closed two-sided ideal of the algebra $\sum_{ij}^{o} q_iAq_j$. Then $IqAq \subset I$ and $Iq \cdot qAq \subset Iq$. Therefore $qIqqAq \subseteq qIq$, that is qIq is a closed two-sided ideal of the subalgebra qAq. Since qAq is simple then $qIq = qAq$.

Let q_1, q_2 be arbitrary projections in $\{q_i\}$. We assert that $q_1Iq_2 = q_1Aq_2$ and $q_2Iq_1 = q_2Aq_1$. Indeed, we have the projection $q_1 + q_2$ is equivalent to $1 \in A$. Let $e = q_1 + q_2$. Then $eAe \cong A$ and eAe is a simple C*-algebra. At the same time we have eAe is a subalgebra of $\sum_{ij}^{o} q_iAq_j$ and I is a two-sided ideal of $\sum_{ij}^{o} q_iAq_j$. Hence $IeAe \subset I$ and $Ie \cdot eAe \subset Ie$. Therefore $eIeeAe \subseteq eIe$, that is eIe is a closed two-sided ideal of the subalgebra eAe. Since eAe is simple then $eIe = eAe$. Hence $q_1Iq_2 = q_1Aq_2$ and $q_2Iq_1 = q_2Aq_1$. Therefore $q_iIq_j = q_iAq_j$ for all i and j. We have I is norm closed. Hence $I = \sum_{ij}^{o} q_iAq_j$, i.e., $\sum_{ij}^{o} q_iAq_j$ is a simple C*-algebra. □

2. Applications

Definition. A C*-algebra is called a C*-factor, if it does not have nonzero proper two-sided ideals I and J such that $I \cdot J = \{0\}$, where $I \cdot J = \{ab : a \in I, b \in J\}$.

Theorem 9. *Let $\mathcal{N}$ be a W*-factor of type II_∞ on a Hilbert space H, $\{p_i\}$ be a countable orthogonal set of equivalent projections in $\mathcal{N}$ and* $\sup_i p_i = 1$. *Then for any countable orthogonal set $\{q_i\}$ of equivalent projections in $B(H)$ defined by the set $\{p_i\}$ in $B(H)$ the C*-algebra $\sum_{ij}^{o} q_i\mathcal{N}q_j$ is a C*-factor with a nonzero finite and an infinite projection. In this case $\sum_{ij}^{o} q_i\mathcal{N}q_j$ is not a von Neumann algebra.*

Proof. By the definition of the set $\{q_i\}$ we have $\sup_i q_i = 1$ and $\{q_i\}$ be a countable orthogonal set of equivalent *infinite* projections. By Theorem 6 we have $\sum_{ij}^{o} q_i\mathcal{N}p_j$ is a C*-subalgebra of $\mathcal{N}$. Let q be a nonzero finite projection of $\mathcal{N}$. Then there exists a projection $p \in \{q_i\}$ such that $qp \neq 0$. We have $q\mathcal{N}q$ is a finite von Neumann algebra. Let $x = pq$. Then $x\mathcal{N}x^*$ is a weakly closed C*-subalgebra. Note that the algebra $x\mathcal{N}x^*$ has a center-valued faithful trace. Let e be a nonzero projection of the algebra $x\mathcal{N}x^*$. Then $ep = e$ and $e \in p\mathcal{N}p$. Hence $e \in \sum_{ij}^{o} q_i\mathcal{N}q_j$. We have the weak closure of $\sum_{ij}^{o} q_i\mathcal{N}q_j$ in the algebra $\mathcal{N}$ coincides with this algebra $\mathcal{N}$. Then by the weak continuity of the multiplication $\sum_{ij}^{o} q_i\mathcal{N}q_j$ is a C*-factor. Note since $1 \notin \sum_{ij}^{o} q_i\mathcal{N}q_j$ then $\sum_{ij}^{o} q_i\mathcal{N}q_j$ is not weakly closed in $\mathcal{N}$. Hence the C*-factor $\sum_{ij}^{o} q_i\mathcal{N}q_j$ is not a von Neumann algebra. □

Remark. Note that, in the article [3] a simple C*-algebra with an infinite and a nonzero finite projection have been constructed by M.Rørdam. In the next corollary we construct a simple purely infinite C*-algebra. Note that simple purely infinite C*-algebras are considered and investigated, in particular, in [4] and [5].

Theorem 10. *Let $\mathcal{N}$ be a W*-factor of type III on a Hilbert space H. Then for any countable orthogonal set $\{p_i\}$ of equivalent projections in $\mathcal{N}$ such that* $\sup_i p_i = 1$, *$\sum_{ij}^{o} p_i\mathcal{N}p_j$ is a simple purely infinite C*-algebra. In this case $\sum_{ij}^{o} p_i\mathcal{N}p_j$ is not a von Neumann algebra.*