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Saïd Abbas
Mouffak Benchohra
Gaston M. N'Guérékata

Topics in Fractional Differential Equations

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Saïd Abbas • Mouffak Benchohra
Gaston M. N'Guérékata

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Saïd Abbas
Laboratoire de Mathématiques
Université de Saïda
Saïda, Algeria

Mouffak Benchohra
Laboratoire de Mathématiques
Université de Sidi Bel-Abbès
Sidi Bel-Abbès
Algeria

Gaston M. N'Guérékata
Department of Mathematics
Morgan State University
Baltimore, MD, USA

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We dedicate this book to our family members. Inparticular, Saïd Abbas dedicates to the memory of his father, to his mother, his wife Zoubida and his children Mourad, Amina, and Ilyes; Mouffak Benchohra makes his dedication to the memory of his father Yahia Benchohra and Gaston N'Guérékata to the memory of his father Jean N'Guérékata.

Preface

Fractional calculus (FC) generalizes integrals and derivatives to non-integer orders. During the last decade, FC was found to play a fundamental role in the modeling of a considerable number of phenomena, in particular, the modeling of memory-dependent phenomena and complex media such as porous media. FC emerged as an important and efficient tool for the study of dynamical systems where classical methods reveal strong limitations. This book is devoted to the existence and uniqueness of solutions for various classes of Darboux problem for hyperbolic differential equations or inclusions involving the Caputo fractional derivative, the best fractional derivative of the time. Some equations present delay which may be finite, infinite, or state-dependent. Others are subject to impulsive effect. The tools used include classical fixed point theorems as well as sharp (new) ones such as the one by Dhage on ordered Banach algebras and the fixed point theorem for contraction multivalued maps due to Covitz and Nadier, as well as some generalizations of the Gronwall's lemma. Each chapter concludes with a section devoted to notes and bibliographical remarks and all abstract results are illustrated by examples.

The content of this book is new and complements the existing literature in fractional calculus. It is useful for researchers and graduate students for research, seminars, and advanced graduate courses, in pure and applied mathematics, engineering, biology, and all other applied sciences.

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Saïda, Algeria S. Abbas
Sidi Bel-Abbès, Algeria M. Benchohra
Baltimore, Maryland, USA G.M. N'Guérékata

Contents

Chapter 1
Introduction

Fractional calculus is a generalization of the ordinary differentiation and integration to arbitrary non-integer order. The subject is as old as the differential calculus and goes back to times when Leibniz and Newton invented differential calculus. One owes to Leibniz in a letter to L'Hôspital, dated September 30, 1695 [181], the exact birthday of the fractional calculus and the idea of the fractional derivative. L'Hôspital asked the question as to the meaning of $d^n y/\mathrm{d}x^n$ if $n = \frac{1}{2}$; i.e., what if n is fractional? Leibniz replied that $d^{\frac{1}{2}}x$ will be equal to $x\sqrt{\mathrm{d}x : x}$. In the letters to J. Wallis and J. Bernoulli (in 1697), Leibniz mentioned the possible approach to fractional-order differentiation in that sense that for non-integer values of n the definition could be the following: $\frac{d^n \mathrm{e}^{mx}}{\mathrm{d}x^n} = m^n \mathrm{e}^{mx}$. In 1730, Euler mentioned interpolating between integral orders of a derivative and suggested to use the following relationship: $\frac{d^n x^m}{\mathrm{d}x^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}$, where $\Gamma(.)$ is the (Euler's) Gamma function defined by $\Gamma(\xi) = \int_0^\infty t^{\xi-1}\mathrm{e}^{-t}\mathrm{d}t,\ \xi > 0$. Also for negative or non-integer (rational) values of n. Taking $m = 1$ and $n = \frac{1}{2}$, Euler obtained: $\frac{d^{\frac{1}{2}}x}{\mathrm{d}x^{\frac{1}{2}}} = \sqrt{\frac{4x}{\pi}} = \frac{2}{\sqrt{\pi}} x^{\frac{1}{2}}$. In 1812, Laplace [1820 vol. 3, 85 and 186] defined a fractional derivative by means of an integral, and in 1819 there appeared the first discussion of a derivative of fractional order in a calculus text written by Lacroix [171]. The first step to generalization of the notion of differentiation for arbitrary functions was done by Fourier (1822) [125]. After introducing his famous formula

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(z)\mathrm{d}z \int_{-\infty}^{+\infty} \cos(px - pz)\mathrm{d}p,$$

Fourier made a remark that

$$\frac{d^n f(x)}{\mathrm{d}x^n} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(z)\mathrm{d}z \int_{-\infty}^{+\infty} \cos\left(px - pz + n\frac{\pi}{2}\right)\mathrm{d}p,$$

S. Abbas et al., *Topics in Fractional Differential Equations*, Developments in Mathematics 27, DOI 10.1007/978-1-4614-4036-9_1,

and this relationship could serve as a definition of the nth order derivative for non-integer n. In 1823, Abel [38], considered the integral representation

$$\int_0^x \frac{S'(\eta)}{(x-\eta)^\alpha} \mathrm{d}\eta = \psi(x),$$

for arbitrary α and then wrote

$$S(x) = \frac{\sin(\pi\alpha)}{\pi} x^\alpha \int_0^1 \frac{\psi(xt)}{(1-t)^{1-\alpha}} \mathrm{d}t = \frac{1}{\Gamma(1-\alpha)} \frac{d^{-\alpha}\psi(x)}{sx^{-\alpha}}.$$

The first great theory of fractional derivation is due to Liouville (1832) [185].

I. In his first definition, according to exponential representation of a function

$f(x) = \sum_{n=0}^{\infty} c_n \mathrm{e}^{a_n x}$, he generalized the formula $\frac{d^\nu f(x)}{\mathrm{d}x^\nu} = \sum_{n=0}^{\infty} c_n a_n^\nu \mathrm{e}^{a_n x}$.

II. Second type of his definition was Fractional Integral

$$\int^\mu \Phi(x) \mathrm{d}x^\mu = \frac{1}{(-1)^\mu \Gamma(\mu)} \int_x^\infty (\tau - x)^{\mu-1} \Phi(\tau) \mathrm{d}\tau,$$

$$\int^\mu \Phi(x) \mathrm{d}x^\mu = \frac{1}{\Gamma(\mu)} \int_{-\infty}^x (x-\tau)^{\mu-1} \Phi(\tau) \mathrm{d}\tau.$$

III. Third definition includes Fractional derivative

$$\frac{d^\mu F(x)}{\mathrm{d}x^\mu} = \frac{(-1)^\mu}{h^\mu} \left(F(x) - \frac{\mu}{1} F(x+h) + \frac{\mu(\mu-1)}{1.2} F(x+2h) - \cdots \right),$$

$$\frac{d^\mu F(x)}{\mathrm{d}x^\mu} = \frac{1}{h^\mu} \left(F(x) - \frac{\mu}{1} F(x-h) + \frac{\mu(\mu-1)}{1.2} F(x-2h) - \cdots \right).$$

But the formula most often used today, called Riemann–Liouville integral, was given by Riemann (1847). His definition of Fractional Integral is

$$D^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_c^x (x-t)^{\nu-1} f(t) \mathrm{d}t + \psi(t).$$

On these historical concepts, one will be able to refer to work of Dugowson [117]. According to Riemann–Liouville the notion of fractional integral of order α, $\alpha > 0$ for a function $f(t)$, is a natural consequence of the well-known formula (Cauchy–Dirichlet), which reduces the calculation of the $n-$fold primitive of a function $f(t)$ to a single integral of convolution type.

$$I_{a+}^{n} f(t) = \frac{1}{(n-1)!} \int_{a}^{t} (t-\tau)^{n-1} f(\tau) \mathrm{d}\tau, \; n \in \mathbb{N},$$

vanishes at $t = a$ along with its derivatives of order $1, 2, \ldots, n-1$. One requires $f(t)$ and $I_{a+}^{n} f(t)$ to be causal functions, that is, vanishing for $t < 0$. Then one extends to any positive real value by using the Gamma function, $(n-1)! = \Gamma(n)$. So, the left-sided mixed Riemann–Liouville integral of order α of f is defined by

$$I_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{t} (t-\tau)^{\alpha-1} f(\tau) \mathrm{d}\tau.$$

The operator of fractional derivative $D^{\alpha} f(t)$ can be defined by the Transform of Laplace integrals, the derivative of order $\alpha < 0$ a causal function $f(t)$ is given by the Riemann–Liouville integral:

$$D^{\alpha} f(t) = \int_{0}^{t} \frac{\xi^{-\alpha-1}}{\Gamma(-\alpha)} f(t-\xi) \mathrm{d}\xi. \tag{1.1}$$

If $\alpha > 0$, we can pose

$$D^{\alpha} f(t) = PF \int_{0}^{t} \frac{\xi^{-\alpha-1}}{\Gamma(-\alpha)} f(t-\xi) \mathrm{d}\xi, \; \alpha > 0, \; \alpha \notin \mathbb{N},$$

where PF represents the finite part of the integral (Schwartz). In 1867, Grünwald and Letnikov joined this definition which is sometimes useful

$${}_{a}D_{t}^{\alpha} f(t) = \lim_{h \to 0} h^{-\alpha} \sum_{k=0}^{[\frac{t-a}{h}]} (-1)^{k} \binom{\alpha}{k} f(t-kh) = \sum_{k=0}^{\infty} (-1)^{k} \binom{\alpha}{k} f(t). \tag{1.2}$$

This definition of fractional derivative of a function $f(t)$ based on finite differences is obtained from the classical definition of integer order derivative (Grünwald [137]). We can get an idea of the equivalence of definitions (1.1) and (1.2) using the factorial function $\Gamma(\alpha)$ by Gauss: $\Gamma(\alpha) = \lim_{k \to \infty} \dfrac{k! k^{\alpha}}{\alpha(\alpha+1)\cdots(\alpha+k)}$. A list of mathematicians, who have provided important contributions up to the middle of the last century, includes N.Ya. Sonin (1869), A.V. Letnikov (1872), H. Laurent (1884), P.A. Nekrassov (1888), A. Krug (1890), J. Hadamard (1892) , O. Heaviside (1892–1912), S. Pincherle (1902), G.H. Hardy and J.E. Littlewood (1917–1928), H. Weyl (1917), P. Lévy (1923), A. Marchaud (1927), H.T. Davis (1924–1936), E. L. Post (1930), A. Zygmund (1935- 1945), E.R. Love (1938–1996), A. Erdélyi (1939–1965), H. Kober (1940), D.V. Widder (1941), M. Riesz (1949), W. Feller (1952), and K. Nishimoto (1987-). They considered the Cauchy Integral formula

$$f^{(n)}(z) = \frac{h!}{2\pi i}\int_c \frac{f(t)}{(t-z)^{n+1}}\mathrm{d}t,$$

and substituted n by ν to obtain

$$D^{\nu} f(x) = \frac{\Gamma(\nu+1)}{2\pi i}\int_c^{x^+} \frac{f(t)}{(t-z)^{\nu+1}}\mathrm{d}t.$$

The Riemann–Liouville definition of fractional calculus is the popular definition, it is this which shows joining of two previous definitions.

$${}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{\mathrm{d}t}\right)^n \int_a^t \frac{f(\tau)\mathrm{d}\tau}{(t-\tau)^{\alpha-n+1}};\ \ n-1 \le \alpha < n.$$

The Riemann–Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator ${}_a^cD_t^{\alpha}$ proposed by Caputo (1967) first in his work on the theory of viscoelasticity [91] and 2 years later in his book [92]. Caputo's definition can be written as

$${}_a^cD_t^{\alpha} = \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{f^{(n)}(\tau)\mathrm{d}\tau}{(t-\tau)^{\alpha-n+1}};\ \ n-1 \le \alpha < n.$$

The Mittag-Leffler function is a generalization of the exponential function that plays an important role in fractional calculus. The function was developed by the Scandinavian mathematician Mittag-Leffler (1846–1927) [195, 196], who was a contemporary of Oliver Heaviside(1850–1925). In 1993, Miller and Ross used differential operator D as $D^{\overline{\alpha}} f(t) = D^{\alpha_1} D^{\alpha_2} \cdots D^{\alpha_n} f(t);\ \ \overline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, in which D^{α_i} are Riemann–Liouville or Caputo definitions. The idea of fractional calculus and fractional order differential equations and inclusions has been a subject of interest not only among mathematicians but also among physicists and engineers. Indeed, we can find numerous applications in rheology, porous media, viscoelasticity, electrochemistry, electromagnetism, signal processing, dynamics of earthquakes, optics, geology, viscoelastic materials, biosciences, bioengineering, medicine, economics, probability and statistics, astrophysics, chemical engineering, physics, splines, tomography, fluid mechanics, electromagnetic waves, nonlinear control, control of power electronic, converters, chaotic dynamics, polymer science, proteins, polymer physics, electrochemistry, statistical physics, thermodynamics, neural networks, etc. [115,133,134,151,187,189,191,208,209,220,229,231,250]. The problem of the existence of solutions of Cauchy-type problems for ordinary differential equations of fractional order and without delay in spaces of integrable functions was studied in some works [164, 228]. The similar problem in spaces of continuous functions was studied in [243]. Recently several papers have been devoted to the study of hyperbolic partial integer order differential equations and inclusions with local and nonlocal conditions; see for instance [85–88, 176], the nonlocal conditions of this type can be applied in the theory of elasticity with better effect than the initial or Darboux conditions. For similar results with set-valued right-hand side we refer to [74–76,89,158,213]. During the last 10 years, hyperbolic ordinary and partial differential equations and inclusions of fractional order have been intensely studied by many mathematicians; see for instance [4–6,15,244–247].

In recent years, there has been a significant development in fractional calculus techniques in ordinary and partial functional differential equations and inclusions, some recent contributions can be seen in the monographs of Anastassiou [52], Baleanu et al. [61], Diethelm [113], Kaczorek [156], Kilbas et al. [166], Lakshmikantham et al. [175], Miller and Ross [192], Podlubny [214], Samko et al. [225], the papers of Abbas et al. [25, 30, 32, 35, 36], Abbas and Benchohra [5, 6, 9, 10], Agarwal et al. [39, 43, 45, 46], Ahmad and Nieto [47], Ait Dads et al. [49], Almeida and Torres [50, 51], Araya and Lizama [53], Arshad and Lupulescu [54], Balachandran et al. [59, 60], Baleanu and Vacaru [62], Bazhlekova [64], Belarbi et al. [66], Benchohra et al. [67–69, 71, 73], Burton [83], Chang and Nieto [94], Darwish et al. [100], Danca and Diethelm [99], Debbouche [102], Debbouche and Baleanu [103], Delbosco and Rodino [105], Denton and Vatsala [106], Diagana et al. [112], Diethelm [114, 115], Dong et al. [116], El-Borai [118, 119], El-Borai et al. [120, 121], El-Sayed [122–124], Furati and Tatar [131, 132], Henderson and Ouahab [144, 145], Herzallah et al. [149, 150], Ibrahim [154], Kadem and Baleanu [157], Kaufmann and Mboumi [163], Kilbas and Marzan [165], Kirane et al. [167], Kiryakova and Luchko [168], Li et al. [183], Labidi and Tatar [170], Lakshmikanthan [173], Lakshmikantham and Vatsala [178], Li and N'Guérékata [182], Luchko [186], Magin et al. [188], Mainardi [189], Moaddy et al. [197], Mophou [198], Mophou et al. [199–204], Muslih and Agrawal [205], Muslih et al. [206], Nieto [207], Podlubny et al. [216], Ramrez and Vatsala [217], Razminia et al. [218], Rivero et al. [219], Sabatier et al. [221], Salem [222–224], Samko et al. [226], Tarasov [232], Tarasov and Edelman [233], Tenreiro Machado [234–236], Tenreiro Machado et al. [237–239], Trigeassou et al. [240], Vzquez [241], Wang et al. [248], Vityuk [242], Vityuk and Golushkov [244], Yu and Gao [249], Zhang [251], Zhou et al. [253–255], and the references therein.

Applied problems require definitions of fractional derivative allowing the utilization of physically interpretable initial conditions. Caputo's fractional derivative, originally introduced by Caputo [90] and afterwards adopted in the theory of linear viscoelasticity, satisfies this demand. For a consistent bibliography on this topic, historical remarks, and examples, we refer to [41, 48, 49, 77, 214, 215].

The method of upper and lower solutions has been successfully applied to study the existence of multiple solutions for initial and boundary value problems of the first-and second-order partial differential equations. This method plays an important role in the investigation of solutions for differential and partial differential equations and inclusions. We refer to the monographs by Benchohra et al. [70], the papers of Abbas and Benchohra [7, 8, 12, 14], Heikkila and Lakshmikantham [143], Ladde et al. [172], Lakshmikantham and Pandit [176], Lakshmikantham et al. [177], Pandit [213], and the references cited therein.

The theory of impulsive integer order differential equations and inclusions has become important in some mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology, and economics. The study of impulsive fractional differential equations and inclusions was initiated in the 1960s by Milman and Myshkis [193, 194]. At present the foundations of the general theory are already laid, and many of them are investigated in detail in the books of Benchohra et al. [70], Lakshmikantham et al. [174], Samoilenko and Peresyuk [227], and the references therein. There was an intensive development of the impulse theory, especially in the area of impulsive differential equations and inclusions with fixed moments. The theory of impulsive differential equations and inclusions with variable time is relatively less developed due to the difficulties created by the state-dependent impulses. Some interesting extensions to impulsive differential equations with variable times have been done by Bajo and Liz [58], Abbas et al. [2, 26, 27], Abbas and Benchohra [13], Belarbi and Benchohra [65], Benchohra et al. [70, 72], Frigon and O'Regan [128–130], Kaul et al. [160], Kaul and Liu [161, 162], and the references cited therein. In the case of non-integer order derivative, impulsive differential equations and inclusions have been initiated in the papers [41, 77]. See also [40, 48, 49].

Functional differential equations with state-dependent delay appear frequently in applications as model of equations and for this reason the study of these types of equations has received great attention in the last year; see for instance [140, 141] and the references therein. The literature related to partial functional differential equations with state-dependent delay is limited; see for instance [11, 37, 148]. The literature related to ordinary and partial functional differential equations with delay for which $\rho(t,.) = t$ or $(\rho_1(x,y,.), \rho_2(x,y,.)) = (x,y)$ is very extensive; see for instance [5, 6, 139] and the references therein.

Implicit differential equations involving the regularized fractional derivative were analyzed by many authors, in the last year; see for instance [16, 17, 33, 34, 246, 247] and the references therein.

Integral equations are one of the useful mathematical tools in both pure and applied analysis. This is particularly true of problems in mechanical vibrations and the related fields of engineering and mathematical physics. There has been a significant development in ordinary and partial fractional integral equations in recent years; see the monographs of Miller and Ross [192], Podlubny [214], Abbas et al. [18–23, 28, 29], Banaś et al. [63], Darwish et al. [100], Dhage [107–111], and the references therein.

In this book we are interested by initial value problems (IVP for short) for partial hyperbolic functional differential equations and inclusions with Caputo's fractional derivative and partial hyperbolic implicit differential equations involving the regularized fractional derivative. Our results may be interpreted as extensions of previous results of Dawidowski and Kubiaczyk [101], Kamont [158], Kamont and Kropielnicka [159] obtained for "classical" hyperbolic differential equations and inclusions with integer order derivative and those of Kilbas and Marzan [165] considered with fractional derivative and without delay. In fact, in the proof of our theorems we essentially use several fixed-point techniques. This book is arranged and organized as follows:

In Chap. 2, we introduce notations, definitions, and some preliminary notions. In Sect. 2.1, we give some notations from the theory of Banach spaces and Banach algebras. Section 2.2 is concerned to recall some basic definitions and facts on partial fractional calculus theory. In Sect. 2.3, we give some properties of set-valued maps. Section 2.4 is devoted to fixed-points theory, here we give the main theorems that will be used in the following chapters. In Sect. 2.5, we give some generalizations of Gronwall's lemmas for two independent variables and singular kernel.

In Chap. 3, we shall be concerned by fractional order partial functional differential equations. In Sect. 3.2, we study initial value problem for a class of partial hyperbolic differential equations. We give two results, one based on Banach fixed-point theorem and the other based on the nonlinear alternative of Leray–Schauder type. We present two similar results to nonlocal problems. An example will be presented in the last illustrating the abstract theory. Section 3.3 is concerned to study a system of perturbed partial hyperbolic differential equations. We give two results, one based on Banach fixed-point theorem and other based on a fixed-point theorem due to Burton and Kirk for the sum of contraction and completely continuous operators. Also, we give similar results to nonlocal problems and we present an illustrative example. Section 3.4 is devoted to study initial value problem for partial neutral functional differential equations. We present some existence results using Krasnoselskii's fixed-point theorem. Also we present an example illustrating the applicability of the imposed conditions. In Sect. 3.5, we shall be concerned by partial hyperbolic differential equations in Banach algebras. We shall prove the existence of solutions, as well as the existence of extremal solutions. Our approach is based, for the existence of solutions, on a fixed-point theorem due to Dhage under Lipschitz and Carathéodory conditions, and for the existence of extremal solutions, on the concept of upper and lower solutions combined with a fixed-point theorem on ordered Banach algebras established by Dhage under certain monotonicity conditions. An example is presented in the last part of this section. In Sect. 3.6, we investigate the existence of solutions for a class of initial value problem for partial hyperbolic differential equations by using the lower and upper solutions method combined with Schauder's fixed-point theorem. In Sect. 3.7, we study a system of partial hyperbolic differential equations with infinite delay. We present two results, one based on Banach fixed-point theorem and the other based on the nonlinear alternative of Leray–Schauder type. Section 3.8 is devoted to study the existence and uniqueness of solutions of some classes of partial functional and neutral functional hyperbolic differential equations with state-dependent delay. Some examples will be presented in the last part of this section. In the last section of this paper, we shall be concerned by global uniqueness results for partial hyperbolic differential equations. We investigate the global existence and uniqueness of solutions of four classes of partial hyperbolic differential equations with finite and infinite delays and we present some illustrative examples.

In Chap. 4, we shall be concerned by functional partial differential inclusions. In Sect. 4.2, we investigate the existence of solutions of a class of partial hyperbolic differential inclusions with finite delay. We shall present two existence results, when the right-hand side is convex as well as nonconvex valued. The first result relies on the nonlinear alternative of Leray–Schauder type. In the second result, we shall use the fixed-point theorem for contraction multivalued maps due to Covitz and Nadler. In Sect. 4.3, we prove a Filippov-type existence result for a class of partial hyperbolic differential inclusions by applying the contraction principle in the space of selections of the multifunction instead of the space of solutions. The second result is about topological structure of the solution set, more exactly, we prove that the solution set is not empty and compact. Section 4.4 is devoted to an existence result of solutions for functional differential inclusions. Our proof relies on the nonlinear alternative of Leray–Schauder combined with lower and upper solutions method. Section 4.5 deals with the existence of solutions for the initial value problems for fractional-order hyperbolic and neutral hyperbolic functional differential inclusions with infinite delay by using the nonlinear alternative of Leray–Schauder type for multivalued operators. In Sect. 4.6, we investigate the existence of solutions for a system of integral inclusions of fractional order. Our approach is based on appropriate fixed-point theorems, namely Bohnenblust–Karlin fixed-point theorem for the convex case and Covitz-Nadler for the nonconvex case.

In Chap. 5, we shall be concerned with functional impulsive partial hyperbolic differential equations. Section 5.2 deals with the existence and uniqueness of solutions of a class of partial hyperbolic differential equations with fixed time impulses. We present two results, the first one is based on Banach's contraction principle and the second one on the nonlinear alternative of Leray–Schauder type. As an extension to nonlocal problems, we present two similar results. Finally we present an illustrative example. In Sect. 5.3, we investigate the existence and uniqueness of solutions of a class of partial hyperbolic differential equations with variable time impulses. We present two results, the first one is based on Schaefer's fixed-point and the second one on Banach's contraction principle. As an extension to nonlocal problems, we present two similar results. An example will be presented in the last illustrating the abstract theory. Section 5.4 deals with the existence of solutions and extremal solutions to partial hyperbolic differential equations of fractional order with impulses in Banach algebras under Lipschitz and Carathéodory conditions and certain monotonicity conditions. Finally we present an illustrative example. Section 5.5 deals with the existence of solutions to partial functional differential equations with impulses at variable times and infinite delay. Our works will be considered by using the nonlinear alternative of Leray–Schauder type and we present an illustrative example. Section 5.6 is devoted to study the existence and uniqueness of solutions of two classes of partial hyperbolic differential equations with fixed time impulses and state-dependent delay. We present two results for each of our problems, the first one is based on Banach's contraction principle and the second one on the nonlinear alternative of Leray–Schauder type. In Sect. 5.7, we investigate the existence and uniqueness of solutions of two classes of partial hyperbolic differential equations with variable time impulses and state-dependent delay, we present existence results for our problems based on Schaefer's fixed-point. In Sect. 5.8, we investigate the existence of solutions for a class of initial value problem for impulsive partial hyperbolic differential equations by using the lower and upper solutions method combined with Schauder's fixed-point theorem.

In Chap. 6, we shall be concerned with impulsive partial hyperbolic functional differential inclusions. Section 6.2 deals with the existence of solutions of a class of partial hyperbolic differential inclusions with fixed time impulses. We shall present existence results when the right-hand side is convex as well as nonconvex valued. We present three results, the first one is based on the nonlinear alternative of Leray–Schauder type. In the second result, we shall use the fixed-point theorem for contraction multivalued maps due to Covitz and Nadler. The third result relies on the nonlinear alternative of Leray–Schauder type for single-valued map combined with a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued operators with closed and decomposable values. In Sect. 6.3, we investigate the existence of solutions of some classes of partial impulsive hyperbolic differential inclusions at variable times by using the nonlinear alternative of Leray–Schauder type. In Sect. 6.4, we use the upper and lower solutions method combined with fixed-point theorem of Bohnnenblust-Karlin for investigating the existence of solutions of a class of partial hyperbolic differential inclusions at fixed moments of impulse.

In Chap. 7, we shall be concerned with implicit partial hyperbolic differential equations.In Section 7.2 we investigate the existence and uniqueness of solutions for implicit partial hyperbolic functional differential equations. We present two results, the first one is based on Banach's contraction principle and the second one on the nonlinear alternative of Leray–Schauder type. Section 7.3 deals with a global uniqueness result for fractional-order implicit differential equations, we make use of the nonlinear alternative of Leray–Schauder type for contraction maps on Fréchet spaces. To illustrate the result an example is provided. In Sect. 7.4, we shall be concerned with implicit partial hyperbolic differential equations with finite delay, infinite delay, and with state-dependent delay. We present two results for each of our problems, the first one is based on Banach's contraction principle and the second one on the nonlinear alternative of Leray–Schauder type. We illustrate our results by some examples. Section 7.5 deals with the existence and uniqueness of solutions of a class of implicit impulsive partial hyperbolic differential equations. We present two results for our problem, the first one is based on Banach's contraction principle and the second one on the nonlinear alternative of Leray–Schauder type. To illustrate the results an example is provided. In Sect. 7.6, we shall be concerned with the existence and uniqueness of solutions of two classes of partial implicit impulsive hyperbolic differential equations with fixed time impulses and state-dependent delay. We present two results for each of our problems, the first one is based on Banach's contraction principle and the second one on the nonlinear alternative of Leray–Schauder type. Also, we present some illustrative examples.

In Chap. 8, we shall be concerned with Riemann–Liouville integral equations of fractional order. In Sect. 8.2 we study the existence and uniqueness of solutions of a certain Fredholm-type Riemann–Liouville integral equation of two variables by using Banach contraction principle. Section 8.3 deals with the existence and uniqueness of solutions for a system of integral equations of fractional order with multiple time delay by using some fixed-point theorems. We illustrate our results with some examples. In Sect. 8.4 we prove an existence result for a nonlinear quadratic Volterra integral equation of fractional order. Our technique is based on a fixed-point theorem due to Dhage [109]. Finally, an example illustrating the main existence result is presented in the last section. Section 8.5 deals with the existence and global asymptotic stability of solutions of a class of fractional order functional integral equations by using the Schauder fixed-point theorem. Also, we obtain some results about the asymptotic stability of solutions of the equation in question. Finally, we present an example illustrating the applicability of the imposed conditions. In Sect. 8.6 we prove the existence and local asymptotic attractivity of solutions for a functional integral equation of Riemann–Liouville fractional order in Banach algebras, by using a fixed-point theorem of Dhage [109]. Also, we present an example illustrating the applicability of the imposed conditions.

Chapter 2
Preliminary Background

In this chapter, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this book.

2.1 Notations and Definitions

Let $J := [0,a] \times [0,b]$; $a,b > 0$ and $\rho > 0$. Denote $L^{\rho}(J,\mathbb{R}^n)$ the space of Lebesgue-integrable functions $u : J \to \mathbb{R}^n$ with the norm

$$\|u\|_{L^{\rho}} = \left(\int_0^a \int_0^b \|u(x,y)\|^{\rho} \mathrm{d}y\mathrm{d}x\right)^{\frac{1}{\rho}},$$

where $\|.\|$ denotes a suitable complete norm on $\mathbb{R}^n$. Also $L^1(J,\mathbb{R}^n)$ is endowed with norm $\|.\|_{L^1}$ defined by

$$\|u\|_{L^1} = \int_0^a \int_0^b \|u(x,y)\|\mathrm{d}y\mathrm{d}x.$$

Let $L^{\infty}(J,\mathbb{R}^n)$ be the Banach space of measurable functions $u : J \to \mathbb{R}^n$ which are bounded, equipped with the norm

$$\|u\|_{L^{\infty}} = \inf\{c > 0 : \|u(x,y)\| \leq c,\ a.e.\ (x,y) \in J\}.$$

As usual, by $AC(J,\mathbb{R}^n)$ we denote the space of absolutely continuous functions from J into $\mathbb{R}^n$, and $C(J,\mathbb{R}^n)$ is the Banach space of all continuous functions from J into $\mathbb{R}^n$ with the norm

$$\|u\|_{\infty} = \sup_{(x,y)\in J} \|u(x,y)\|.$$

Also $C(J,\mathbb{R})$ is endowed with norm $\|.\|_{\infty}$ defined by $\|u\|_{\infty} = \sup_{(x,y)\in J} |u(x,y)|$.

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Define a multiplication "$\cdot$" by

$$(u \cdot v)(x, y) = u(x, y)v(x, y), \quad \text{for } (x, y) \in J.$$

Then $C(J, \mathbb{R})$ is a Banach algebra with above norm and multiplication.

If $u \in C([-\alpha, a] \times [-\beta, b], \mathbb{R}^n);\ a, b, \alpha, \beta > 0$ then for any $(x, y) \in J$ define $u_{(x,y)}$ by

$$u_{(x,y)}(s, t) = u(x + s, y + t),$$

for $(s, t) \in [-\alpha, 0] \times [-\beta, 0]$. Here $u_{(x,y)}(., .)$ represents the history of the state from time $(x - \alpha, y - \beta)$ up to the present time (x, y).

2.2 Properties of Partial Fractional Calculus

In this section, we introduce notations, definitions, and preliminary lemmas concerning partial fractional calculus theory.

Definition 2.1 ([216, 225]). The Riemann–Liouville fractional integral of order $\alpha \in (0, \infty)$ of a function $h \in L^1([0, b], \mathbb{R}^n);\ b > 0$ is defined by

$$I_0^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} h(s) \mathrm{d}s.$$

Definition 2.2 ([216, 225]). The Riemann–Liouville fractional derivative of order $\alpha \in (0, 1]$ of a function $h \in L^1([0, b], \mathbb{R}^n)$ is defined by

$$\begin{aligned} D_0^\alpha h(t) &= \frac{\mathrm{d}}{\mathrm{d}t} I_0^{1-\alpha} h(t) \\ &= \frac{1}{\Gamma(1 - \alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t (t - s)^{-\alpha} h(s) \mathrm{d}s; \text{ for almost all } t \in [0, b]. \end{aligned}$$

Definition 2.3 ([216, 225]). The Caputo fractional derivative of order $\alpha \in (0, 1]$ of a function $h \in L^1([0, b], \mathbb{R}^n)$ is defined by

$$\begin{aligned} {}^c D_0^\alpha h(t) &= I_0^{1-\alpha} \frac{\mathrm{d}}{\mathrm{d}t} h(t) \\ &= \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - s)^{-\alpha} \frac{\mathrm{d}}{\mathrm{d}s} h(s) \mathrm{d}s; \text{ for almost all } t \in [0, b]. \end{aligned}$$

Definition 2.4 ([166, 225]). Let $\alpha \in (0,\infty)$ and $u \in L^1(J,\mathbb{R}^n)$. The partial Riemann–Liouville integral of order α of $u(x,y)$ with respect to x is defined by the expression

$$I_{0,x}^{\alpha}u(x,y) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}u(s,y)\mathrm{d}s;\ \text{for almost all } (x,y) \in J.$$

Analogously, we define the integral

$$I_{0,y}^{\alpha}u(x,y) = \frac{1}{\Gamma(\alpha)}\int_0^y (y-s)^{\alpha-1}u(x,s)\mathrm{d}s;\ \text{for almost all } (x,y) \in J.$$

Definition 2.5 ([166, 225]). Let $\alpha \in (0,1]$ and $u \in L^1(J,\mathbb{R}^n)$. The Riemann–Liouville fractional derivative of order α of $u(x,y)$ with respect to x is defined by

$$(D_{0,x}^{\alpha}u)(x,y) = \frac{\partial}{\partial x}I_{0,x}^{1-\alpha}u(x,y);\ \text{for almost all } (x,y) \in J.$$

Analogously, we define the derivative

$$(D_{0,y}^{\alpha}u)(x,y) = \frac{\partial}{\partial y}I_{0,y}^{1-\alpha}u(x,y);\ \text{for almost all } (x,y) \in J.$$

Definition 2.6 ([166, 225]). Let $\alpha \in (0,1]$ and $u \in L^1(J,\mathbb{R}^n)$. The Caputo fractional derivative of order α of $u(x,y)$ with respect to x is defined by the expression

$${}^{c}D_{0,x}^{\alpha}u(x,y) = I_{0,x}^{1-\alpha}\frac{\partial}{\partial x}u(x,y);\ \text{for almost all } (x,y) \in J.$$

Analogously, we define the derivative

$${}^{c}D_{0,y}^{\alpha}u(x,y) = I_{0,y}^{1-\alpha}\frac{\partial}{\partial y}u(x,y);\ \text{for almost all } (x,y) \in J.$$

Definition 2.7 ([244]). Let $r = (r_1, r_2) \in (0,\infty) \times (0,\infty)$ and $u \in L^1(J,\mathbb{R}^n)$. The left-sided mixed Riemann–Liouville integral of order r of u is defined by

$$(I_0^r u)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}u(s,t)\mathrm{d}t\mathrm{d}s.$$

In particular,

$$(I_0^0 u)(x,y) = u(x,y),\ (I_0^\sigma u)(x,y) = \int_0^x \int_0^y u(s,t)\mathrm{d}t\mathrm{d}s;\ \text{for almost all } (x,y) \in J,$$

where $\sigma = (1,1)$.

For instance, $I_0^r u$ exists for all $r_1, r_2 > 0$, when $u \in L^1(J,\mathbb{R}^n)$. Note also that when $u \in C(J,\mathbb{R}^n)$, then $(I_0^r u) \in C(J,\mathbb{R}^n)$; moreover,

$$(I_0^r u)(x,0) = (I_0^r u)(0,y) = 0;\ x \in [0,a],\ y \in [0,b].$$

Example 2.8. Let $\lambda, \omega \in (-1,\infty)$ and $r = (r_1, r_2) \in (0,\infty) \times (0,\infty)$, then

$$I_0^r x^\lambda y^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)} x^{\lambda+r_1} y^{\omega+r_2};\ \text{for almost all } (x,y) \in J.$$

By $1-r$ we mean $(1-r_1, 1-r_2) \in [0,1) \times [0,1)$. Denote by $D_{xy}^2 := \frac{\partial^2}{\partial x \partial y}$, the mixed second-order partial derivative.

Definition 2.9 ([244]). Let $r \in (0,1] \times (0,1]$ and $u \in L^1(J,\mathbb{R}^n)$. The Caputo fractional-order derivative of order r of u is defined by the expression $({}^c D_0^r u)(x,y) = (I_0^{1-r} D_{xy}^2 u)(x,y)$ and the mixed fractional Riemann–Liouville derivative of order r of u is defined by the expression $(D_0^r u)(x,y) = (D_{xy}^2 I_0^{1-r} u)(x,y)$.

The case $\sigma = (1,1)$ is included and we have

$$({}^c D_0^\sigma u)(x,y) = (D_0^\sigma u)(x,y) = (D_{xy}^2 u)(x,y);\ \text{for almost all } (x,y) \in J.$$

Example 2.10. Let $\lambda, \omega \in (-1,\infty)$ and $r = (r_1, r_2) \in (0,1] \times (0,1]$, then

$$D_0^r x^\lambda y^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda-r_1)\Gamma(1+\omega-r_2)} x^{\lambda-r_1} y^{\omega-r_2};\ \text{for almost all } (x,y) \in J.$$

Definition 2.11 ([247]). For a function $u : J \to \mathbb{R}^n$, we set

$$q(x,y) = u(x,y) - u(x,0) - u(0,y) + u(0,0).$$

By the mixed regularized derivative of order $r = (r_1, r_2) \in (0,1] \times (0,1]$ of a function $u(x,y)$, we name the function

$$\overline{D}_0^r u(x,y) = D_0^r q(x,y).$$

The function

$$\overline{D}_{0,x}^{r_1} u(x,y) = D_{0,x}^{r_1}[u(x,y) - u(0,y)],$$

is called the partial r_1−order regularized derivative of the function $u(x,y) : J \to \mathbb{R}^n$ with respect to the variable x. Analogously, we define the derivative

$$\overline{D}_{0,y}^{r_2} u(x,y) = D_{0,y}^{r_2}[u(x,y) - u(x,0)].$$

Let $a_1 \in [0,a],\ z^+ = (a_1,0) \in J,\ J_z = [a_1,a] \times [0,b]$. For $w \in L^1(J_z,\mathbb{R}^n)$, the expression

$$(I_{z+}^r w)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{a_1}^{x} \int_{0}^{y} (x-s)^{r_1-1}(y-t)^{r_2-1} w(s,t)\mathrm{d}t\mathrm{d}s,$$

is called the left-sided mixed Riemann–Liouville integral of order r of w. The Caputo fractional-order derivative of order r of w is defined by $({}^cD_{z+}^r w)(x,y) = (I_{z+}^{1-r} D_{xy}^2 w)(x,y)$ and the mixed fractional Riemann–Liouville derivative of order r of w is defined by $(D_{z+}^r w)(x,y) = (D_{xy}^2 I_{z+}^{1-r} w)(x,y)$.

Let $f,g \in L^1(J,\mathbb{R}^n)$.

Lemma 2.12 ([5,6]). *A function $u \in AC(J,\mathbb{R}^n)$ such that its mixed derivative D_{xy}^2 exists and is integrable on J is a solution of problems*

$$\begin{cases} ({}^cD_0^r u)(x,y) = f(x,y);\ (x,y) \in J, \\ u(x,0) = \varphi(x);\ x \in [0,a],\ u(0,y) = \psi(y);\ y \in [0,b], \\ \varphi(0) = \psi(0), \end{cases}$$

if and only if $u(x,y)$ satisfies

$$u(x,y) = \mu(x,y) + (I_0^r f)(x,y);\ (x,y) \in J,$$

where

$$\mu(x,y) = \varphi(x) + \psi(y) - \varphi(0).$$

Lemma 2.13 ([35]). *A function $u \in AC(J,\mathbb{R}^n)$ such that the mixed derivative $D_{xy}^2(u-g)$ exists and is integrable on J is a solution of problems*

$$\begin{cases} {}^cD_0^r[u(x,y) - g(x,y)] = f(x,y);\ (x,y) \in J, \\ u(x,0) = \varphi(x);\ x \in [0,a],\ u(0,y) = \psi(y);\ y \in [0,b], \\ \varphi(0) = \psi(0), \end{cases}$$

if and only if $u(x,y)$ satisfies

$$u(x,y) = \mu(x,y) + g(x,y) - g(x,0) - g(0,y) + g(0,0) + I_0^r(f)(x,y);\ (x,y) \in J.$$

Let $h \in C([x_k, x_{k+1}] \times [0, b], \mathbb{R}^n),\ z_k = (x_k, 0),\ 0 = x_0 < x_1 < \cdots < x_m < x_{m+1} = a$ and

$$\mu_k(x, y) = u(x, 0) + u(x_k^+, y) - u(x_k^+, 0);\ \ k = 0, \ldots, m.$$

Lemma 2.14 ([7, 8]). *A function $u \in AC([x_k, x_{k+1}] \times [0, b], \mathbb{R}^n);\ k = 0, \ldots, m$ whose r-derivative exists on $[x_k, x_{k+1}] \times [0, b],\ k = 0, \ldots, m$ is a solution of the differential equation*

$$({}^c D^r_{z_k} u)(x, y) = h(x, y);\ (x, y) \in [x_k, x_{k+1}] \times [0, b],$$

if and only if $u(x, y)$ satisfies

$$u(x, y) = \mu_k(x, y) + (I^r_{z_k} h)(x, y);\ (x, y) \in [x_k, x_{k+1}] \times [0, b].$$

Let $J_0 = [0, x_1] \times [0, b],\ J_k = (x_k, x_{k+1}] \times [0, b];\ k = 1, \ldots, m,\ I_k : \mathbb{R}^n \to \mathbb{R}^n;$ $k = 0, 1, \ldots, m$ and denote $\mu(x, y) := \mu_0(x, y);\ (x, y) \in J$.

Lemma 2.15 ([7, 8]). *Let $h : J \to \mathbb{R}^n$ be continuous. A function u whose r-derivative exists on $J_k;\ k = 0, \ldots, m$ is a solution of the fractional integral equation*

$$u(x, y) = \begin{cases} \mu(x, y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x - s)^{r_1 - 1}(y - t)^{r_2 - 1} h(s, t)\mathrm{d}t\mathrm{d}s; \\ \text{if } (x, y) \in [0, x_1] \times [0, b], \\ \mu(x, y) + \sum_{i=1}^{k} (I_i(u(x_i^-, y)) - I_i(u(x_i^-, 0))) \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} \int_0^y (x_i - s)^{r_1 - 1}(y - t)^{r_2 - 1} h(s, t)\mathrm{d}t\mathrm{d}s \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^{x} \int_0^y (x - s)^{r_1 - 1}(y - t)^{r_2 - 1} h(s, t)\mathrm{d}t\mathrm{d}s; \\ \text{if } (x, y) \in (x_k, x_{k+1}] \times [0, b],\ k = 1, \ldots, m, \end{cases}$$

if and only if u is a solution of the fractional IVP

$$\begin{cases} {}^c D^r_{x_k} u(x, y) = h(x, y); \quad (x, y) \in J_k, \quad k = 0, \ldots, m. \\ u(x_k^+, y) = u(x_k^-, y) + I_k(u(x_k^-, y));\ y \in [0, b]; \quad k = 1, \ldots, m. \end{cases}$$

Lemma 2.16 ([2]). *Let $h : J \to \mathbb{R}^n$ be continuous. A function u whose r-derivative exists on J_k; $k = 0, \ldots, m$ is a solution of the fractional integral equation*

$$u(x,y) = \begin{cases} \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} h(s,t) \mathrm{d}t \mathrm{d}s; \\ \text{if } (x,y) \in [0,x_1] \times [0,b], \\ \varphi(x) + I_k(u(x_k,y)) - I_k(u(x_k,0)) \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} h(s,t) \mathrm{d}t \mathrm{d}s; \\ \text{if } (x,y) \in (x_k, x_{k+1}] \times [0,b],\ k = 1, \ldots, m, \end{cases}$$

if and only if u is a solution of the fractional IVP

$$\begin{cases} {}^cD^r_{x_k} u(x,y) = h(x,y); \quad (x,y) \in J_k;\ k = 0, \ldots, m, \\ u(x_k^+, y) = I_k(u(x_k,y));\ y \in [0,b], \quad k = 1, \ldots, m. \end{cases}$$

Let $f \in C(J, \mathbb{R}^*)$, $g \in L^1(J, \mathbb{R})$ and $\mu_0(x,y) = \frac{\varphi(x)}{f(x,0)} + \frac{\psi(y)}{f(0,y)} - \frac{\varphi(0)}{f(0,0)}$.

Lemma 2.17 ([32]). *A function $u \in AC(J, \mathbb{R})$ such that the mixed derivative $D^2_{xy}(\frac{u}{f})$ exists and is integrable on J is a solution of problems*

$$\begin{cases} {}^cD^r_0\left(\frac{u(x,y)}{f(x,y)}\right) = g(x,y),\ (x,y) \in J, \\ u(x,0) = \varphi(x);\ x \in [0,a],\ u(0,y) = \psi(y);\ y \in [0,b], \\ \varphi(0) = \psi(0), \end{cases}$$

if and only if $u(x,y)$ satisfies

$$u(x,y) = f(x,y)\Big(\mu_0(x,y) + (I^r_0 g)(x,y)\Big);\ (x,y) \in J.$$

Let $f \in C([x_k, x_{k+1}] \times [0,b], \mathbb{R}^*)$, $g \in L^1([x_k, x_{k+1}] \times [0,b], \mathbb{R})$, $z_k = (x_k, 0)$, and

$$\mu_{0,k}(x,y) = \frac{u(x,0)}{f(x,0)} + \frac{u(x_k^+, y)}{f(x_k^+, y)} - \frac{u(x_k^+, 0)}{f(x_k^+, 0)};\ k = 0, \ldots, m.$$

Lemma 2.18 ([3]). *A function $u \in AC([x_k, x_{k+1}] \times [0,b], \mathbb{R})$, $k = 0, \ldots, m$ such that the mixed derivative $D^2_{xy}(\frac{u}{f})$ exists and is integrable on $[x_k, x_{k+1}] \times [0,b]$, $k = 0, \ldots, m$ is a solution of the differential equation*

$${}^cD^r_{z_k}\left(\frac{u}{f}\right)(x,y) = g(x,y);\ (x,y) \in [x_k, x_{k+1}] \times [0,b],$$

if and only if $u(x, y)$ *satisfies*

$$u(x, y) = f(x, y)\Big(\mu_{0,k}(x, y) + (I^r_{z_k} g)(x, y)\Big); \; (x, y) \in [x_k, x_{k+1}] \times [0, b].$$

Let $\mu' := \mu_{0,0}$.

Lemma 2.19 ([3]). *Let* $f : J \to \mathbb{R}^*$, $g : J \to \mathbb{R}$ *be continuous. A function* u *such that the mixed derivative* $D^2_{xy}(\frac{u}{f})$ *exists and is integrable on* J_k; $k = 0, \ldots, m$ *is a solution of the fractional integral equation*

$$u(x, y) = \begin{cases} f(x, y)[\mu'(x, y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int\limits_0^x \int\limits_0^y (x - s)^{r_1 - 1}(y - t)^{r_2 - 1} g(s, t) \mathrm{d}t \, \mathrm{d}s]; \\ \text{if } (x, y) \in [0, x_1] \times [0, b], \\ f(x, y)[\mu'(x, y) + \sum\limits_{i=1}^{k} \left(\frac{I_i(u(x_i^-, y))}{f(x_i^+, y)} - \frac{I_i(u(x_i^-, 0))}{f(x_i^+, 0)} \right) \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum\limits_{i=1}^{k} \int\limits_{x_{i-1}}^{x_i} \int\limits_0^y (x_i - s)^{r_1 - 1}(y - t)^{r_2 - 1} g(s, t) \mathrm{d}t \, \mathrm{d}s \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int\limits_{x_k}^{x} \int\limits_0^y (x - s)^{r_1 - 1}(y - t)^{r_2 - 1} g(s, t) \mathrm{d}t \, \mathrm{d}s]; \\ \text{if } (x, y) \in (x_k, x_{k+1}] \times [0, b], \; k = 1, \ldots, m, \end{cases}$$

if and only if u *is a solution of the fractional IVP*

$$\begin{cases} {}^c D^r_{x_k} \left(\frac{u}{f}\right)(x, y) = g(x, y); \quad (x, y) \in J_k; \; k = 0, \ldots, m, \\ u\left(x_k^+, y\right) = u\left(x_k^-, y\right) + I_k\left(u\left(x_k^-, y\right)\right); \; y \in [0, b]; \quad k = 1, \ldots, m. \end{cases}$$

2.3 Properties of Set-Valued Maps

Let $(X, \|\cdot\|)$ be a Banach space. Denote $\mathcal{P}(X) = \{Y \in X : Y \neq \emptyset\}$, $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$ l, and $\mathcal{P}_{cp,cv}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact and convex}\}$.

Definition 2.20. A multivalued map $T : X \to \mathcal{P}(X)$ is convex (closed) valued if $T(x)$ is convex (closed) for all $x \in X$. T is bounded on bounded sets if $T(B) = \cup_{x \in B} T(x)$ is bounded in X for all $B \in \mathcal{P}_b(X)$ (i.e. $\sup_{x \in B} \sup_{y \in T(x)} \|y\| < \infty$). T is called upper semicontinuous (u.s.c.) on X if for each $x_0 \in X$, the set $T(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $T(x_0)$, there exists an open neighborhood N_0 of x_0 such that $T(N_0) \subseteq N$. T is