George E. Andrews
Bruce C. Berndt

Ramanujan's Lost Notebook

Part III

Springer

Ramanujan's Lost Notebook

George E. Andrews • Bruce C. Berndt

Ramanujan's Lost Notebook

Part III

Springer

George E. Andrews
Department of Mathematics
The Pennsylvania State University
University Park, PA
USA

Bruce C. Berndt
Department of Mathematics
University of Illinois at
Urbana–Champaign
Urbana, IL
USA

ISBN 978-1-4614-3809-0 ISBN 978-1-4614-3810-6 (eBook)
DOI 10.1007/978-1-4614-3810-6
Springer New York Heidelberg Dordrecht London

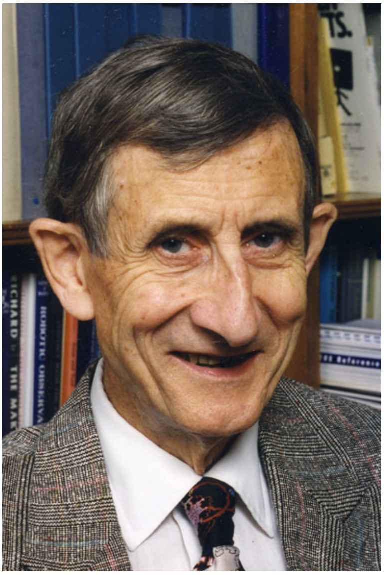
Library of Congress Control Number: 2005923547

Mathematics Subject Classification (2010): 11P82, 11P81, 11P84, 11P83, 33Dxx

Printed on acid-free paper

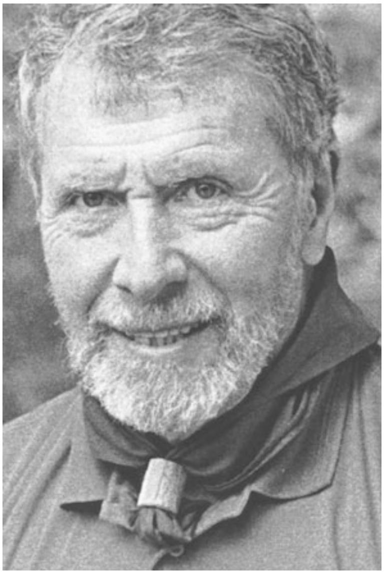
Springer is part of Springer Science+Business Media (www.springer.com)

The CEO's of Ranks and Cranks



Freeman Dyson

Frank Garvan



Oliver Atkin

H.P.F. Swinnerton-Dyer

I felt the joy of an explorer who suddenly discovers the key to the language lying hidden in the hieroglyphs which are beautiful in themselves.

–Rabindranath Tagore, *The Religion of Man*

Preface

This is the third of five volumes that the authors plan to write in their examination of all the claims made by S. Ramanujan in *The Lost Notebook and Other Unpublished Papers*, which was published by Narosa in 1988. This publication contains the “Lost Notebook,” which was discovered by the first author in the spring of 1976 at the library of Trinity College, Cambridge. Also included therein are other partial manuscripts, fragments, and letters that Ramanujan wrote to G.H. Hardy from nursing homes during 1917–1919. Our third volume contains ten chapters and focuses on some of the most important and influential material in *The Lost Notebook and Other Unpublished Papers.* At center stage is the partition function $p(n)$. In particular, three chapters are devoted to ranks and cranks of partitions. Ramanujan’s handwritten manuscript on the partition and tau functions is also examined.

Contents

1

Introduction

This is the third volume devoted to Ramanujan's lost notebook and to partial manuscripts, fragments, and letters published with the lost notebook [283]. The centerpiece of this volume is the partition function $p(n)$. Featured in this book are congruences for $p(n)$, ranks and cranks of partitions, the Ramanujan τ-function, the Rogers–Ramanujan functions, and the unpublished portion of Ramanujan's paper on highly composite numbers [274].

The first three chapters are devoted to ranks and cranks of partitions. In 1944, F. Dyson [127] defined the rank of a partition to be the largest part minus the number of parts. If $N(m,t,n)$ denotes the number of partitions of n with rank congruent to m modulo t, then Dyson conjectured that

$$N(k,5,5n+4)=\frac{p(5n+4)}{5}, \qquad 0\le k\le 4, \tag{1.0.1}$$

and

$$N(k,7,7n+5)=\frac{p(7n+5)}{7}, \qquad 0\le k\le 6. \tag{1.0.2}$$

Thus, if (1.0.1) and (1.0.2) were true, the partitions counted by $p(5n+4)$ and $p(7n+5)$ would fall into five and seven equinumerous classes, respectively, thus providing combinatorial explanations and proofs for Ramanujan's famous congruences $p(5n+4)\equiv 0 \pmod 5$ and $p(7n+5)\equiv 0 \pmod 7$. Dyson's conjectures were first proved by A.O.L. Atkin and H.P.F. Swinnerton-Dyer [28] in 1954.

Dyson observed that the corresponding analogue to (1.0.1) and (1.0.2) does not hold for the third famous Ramanujan congruence $p(11n+6)\equiv 0 \pmod{11}$, and so he conjectured the existence of a statistic that he called the *crank* that would combinatorially explain this congruence. In his doctoral dissertation [144], F.G. Garvan defined a crank for vector partitions, which became the forerunner of the *true crank*, which was discovered by Andrews and Garvan [17] during the afternoon of June 6, 1987, at Illinois Street Residence Hall, a student dormitory at the University of Illinois, following a meeting on June 1–5 to commemorate the centenary of Ramanujan's birth.

G.E. Andrews, B.C. Berndt, *Ramanujan's Lost Notebook: Part III*,

DOI 10.1007/978-1-4614-3810-6_1,

Although Ramanujan did not record any written text about ranks and cranks in his lost notebook [283], he did record theorems about their generating functions. Chapter 2 is devoted to the five and seven-dissections of each of these two generating functions. Cranks are the exclusive topic of Chapter 3, where dissections for the generating function for cranks are studied, but now in the context of congruences. A particular formula found in the lost notebook and proved in Chapter 4 is employed in our proofs in Chapter 3. As we argue in the following two paragraphs, it is likely that Ramanujan was working on cranks up to four days before his death on April 26, 1920.

In January 1984, the second author, Berndt, was privileged to have a very pleasant and exceptionally informative conversation with Ramanujan's widow, Janaki. In particular, this author asked her about the extent of papers that her late husband possessed at his death, and remarked that the only papers that have been passed down to us are those constituting the lost notebook of 138 pages. She claimed that Ramanujan had many more than 138 pages in his possession at his death. She related that as her husband "did his sums," he would deposit his papers in a large leather trunk beneath his bed, and that the number of pages in this trunk certainly exceeded 138. She told Berndt that during her husband's funeral, certain people, whom she named but whom we do not name here, came to her home and stole most of Ramanujan's papers and never returned them. She later donated those papers that were not stolen to the University of Madras. These papers certainly contain, or possibly are identical to, the lost notebook.

It is our contention that Ramanujan kept at least two stacks of papers while doing mathematics in his last year. In one pile, he put primarily those pages containing the statements of his theorems, and in another stack or stacks he put papers containing his calculations and proofs. The one stack of papers containing the lost notebook was likely in a different place and missed by those taking his other papers. (Of course, it is certainly possible that more than one pile of papers contained statements of results that Ramanujan wanted to save.) Undoubtedly, Ramanujan produced scores of pages containing calculations, scratch work, and proofs, but the approximately twenty pages of scratch work in the lost notebook apparently pertain more to cranks than to any other topic. Our guess is that when Ramanujan ceased research four days prior to his death, he was thinking about cranks. His power series expansions, factorizations, preliminary tables, and scratch work were part of his deliberations and had not yet been put in a secondary pile of papers. Thus, these sheets were found with the papers that had been set aside for special keeping and so unofficially became part of his lost notebook. In particular, pages 58–89 in the lost notebook likely include some pages that Ramanujan intended to keep in his principal stack, but most of this work probably would have been relegated to a secondary pile if Ramanujan had lived longer. Further remarks can be found in [64].

Ramanujan's famous manuscript on the partition and tau functions is examined in Chapter 5. This chapter is a substantially revised and extended version of the original publication by the second author and K. Ono [67] appearing in a volume honoring the first author on his 60th birthday. Difficult decisions in the presentation of this manuscript were necessary. As readers peruse the manuscript, it will become immediately clear that Ramanujan left out many details, and that the frequency of omitted details increases as the manuscript progresses. Often, especially in beginning sections, it is not difficult to insert missing details. Thus, to augment readability, we have inserted such details in square brackets, so that readers can easily separate Ramanujan's exposition from that of the authors. However, other claims require considerably more amplification or are completely lacking in details. It was decided that such claims should be either proved or discussed in an appendix. Thus, further decisions needed to be made: Should all of the necessary arguments be presented, or should readers be referred to papers where complete proofs can be given. If details for all of Ramanujan's claims were to be supplied, because of the increased number of pages, this volume might necessarily be devoted *only* to this manuscript.

G.H. Hardy [280] extracted a portion of Ramanujan's manuscript and added several details in giving proofs of his aforementioned famous congruences for the partition function, namely,

$$p(5n+4) \equiv 0 \pmod 5, \; p(7n+5) \equiv 0 \pmod 7, \; p(11n+6) \equiv 0 \pmod{11}. \tag{1.0.3}$$

Thus, we feel that it is unnecessary to give any further commentary on these passages here; readers can proceed to [280] or [281, 232–238] for complete proofs. From the remarkable recent work of S. Ahlgren and M. Boylan [5], we now know that (1.0.3) are the only congruences for $p(n)$ in which the prime moduli of the congruences match the moduli of the arithmetic progressions in the arguments. We remark that we are also following the practice of Hardy, who placed additional details in square brackets, so that readers could see precisely what Ramanujan had recorded and what he had not.

These congruences (1.0.3) are the first cases of the infinite families of congruences

$$p(5^k n + \delta_{5,k}) \equiv 0 \pmod{5^k}, \tag{1.0.4}$$

$$p(7^k n + \delta_{7,k}) \equiv 0 \pmod{7^{[k/2]+1}}, \tag{1.0.5}$$

$$p(11^k n + \delta_{11,k}) \equiv 0 \pmod{11^k},$$

where $\delta_{p,k} :\equiv 1/24 \pmod{p^k}$. In Ramanujan's manuscript, he actually gives a complete proof of (1.0.4), but many of the details are omitted. These details were supplied by G.N. Watson [336], who unfortunately did not mention that his proof had its genesis in Ramanujan's unpublished manuscript. Ramanujan also began a proof of (1.0.5), but he did not finish it. If he had done so, then he would have seen that his original conjecture was incorrect and needed to be corrected as given in (1.0.5). Since proofs of (1.0.4) and (1.0.5) can now be found in several sources (which we relate in Chapter 5), there is no need to give proofs here.

It was surprising for us to learn that Ramanujan had also found congruences for $p(n)$ for the moduli 13, 17, 19, and 23 and had formulated a general conjecture about congruences for any prime modulus. However, unlike (1.0.3), these congruences do not give divisibility of $p(n)$ in any arithmetic progressions. In his doctoral dissertation, J.M. Rushforth [305] supplied all of the missing details for Ramanujan's congruences modulo 13, 17, 19, and 23. Since Rushforth's work has never been published and since his proofs are motivated by those found by Ramanujan, we have decided to publish them here for the first time. In fact, almost all of Rushforth's thesis is devoted to Ramanujan's unpublished manuscript on $p(n)$ and $\tau(n)$, and so we have extracted from it further proofs of results claimed by Ramanujan in this famous manuscript. Ramanujan's general conjecture on congruences for prime moduli was independently corrected, proved, and generalized in two distinct directions by H.H. Chan and J.-P. Serre and by Ahlgren and Boylan [5]. The proof by Chan and Serre is given here for the first time.

Many of the results in Ramanujan's manuscript are now more efficiently proved using the theory of modular forms. Indeed, much of this manuscript has given impetus for further work not only on $p(n)$ but also on the Fourier coefficients of other modular forms. Some of this work is briefly described in Chapter 5, but except for the proof by Chan and Serre, we have not employed the theory of modular forms in proofs within our commentary on Ramanujan's manuscript.

A series of Ramanujan's claims in the $p(n)/\tau(n)$ manuscript are wrong. Rushforth first noted and examined these mistakes in his thesis [305]. However, P. Moree has made a thorough examination of all these erroneous claims and corrected them in a particularly illuminating paper [228].

Lastly, we remark that the $p(n)/\tau(n)$ manuscript is found on pages 133–177, 238–243 of [283], with the latter portion, designated as Part II, in the handwriting of Watson. In fact, the original version of Part II in Ramanujan's own handwriting can be found in the library at Trinity College. One might therefore ask why Narosa published a facsimile of Watson's handwritten copy instead of Ramanujan's own version. There are two possible explanations. First, Watson's copy is closely written, while Ramanujan's more sprawling version would have required more pages in the published edition [283]. Second, the editors might not have been aware of Ramanujan's original manuscript in his *own* handwriting.

Having given an extensive account on our approach to the $p(n)/\tau(n)$ manuscript in Chapter 5, we turn to other chapters.

Chapter 6 is devoted to six entries on page 189 of the lost notebook [283], all of which are related to the content of Chapter 5, and to entries on page 182, which are related to Ramanujan's paper on congruences for $p(n)$ [276] and of course also to Chapter 5. In particular, we give proofs of two of Ramanujan's most famous identities, immediately yielding the first two congruences in (1.0.3). On page 182, we also see that Ramanujan briefly examined congruences for $p_r(n)$, where $p_r(n)$ is defined by

$$(q;q)_\infty^r = \sum_{n=0}^{\infty} p_r(n) q^n, \qquad |q| < 1.$$

Apparently, page 182 is page 5 from a manuscript, but unfortunately all of the remaining pages of this manuscript are likely lost forever. We have decided also to discuss in Chapter 6 various scattered, miscellaneous entries on $p(n)$. Most of this mélange can be found in Ramanujan's famous paper with Hardy establishing their asymptotic series for $p(n)$ [167].

In Chapter 7, we examine nine congruences that make up page 178 in the lost notebook. These congruences are on generalized tau functions and are in the spirit of Ramanujan's famous congruences for $\tau(n)$ discussed in Chapter 5.

The Rogers–Ramanujan functions are the focus of Chapter 8, wherein Ramanujan's 40 famous identities for these functions are examined. Having been sent some, or possibly all, of the 40 identities in a letter from Ramanujan, L.J. Rogers [304] proved eight of them, with Watson [333] later providing proofs for six further identities as well as giving different proofs of two of the identities proved by Rogers. For several years after Ramanujan's death, the list of 40 identities was in the hands of Watson, who made a handwritten copy for himself, and it is this copy that is published in [283]. Fortunately, he did not discard the list in Ramanujan's handwriting, which now resides in the library at Trinity College, Cambridge. Approximately ten years after Watson's death, B.J. Birch [75] found Watson's copy in the library at Oxford University and published it in 1975, thus bringing it to the mathematical public for the first time. D. Bressoud [81] and A.J.F. Biagioli [74] subsequently proved several further identities from the list.

Our account of the 40 identities in Chapter 8 is primarily taken from a *Memoir* [65] by Berndt, G. Choi, Y.-S. Choi, H. Hahn, B.P. Yeap, A.J. Yee, H. Yesilyurt, and J. Yi. The goal of these authors was to provide proofs for as many of these identities as possible that were in the spirit of Ramanujan's mathematics. In doing so, they borrowed some proofs from Rogers, Watson, and Bressoud, while supplying many new proofs as well. After the publication of [65] in which proofs of 35 of the 40 identities were given in the spirit of Ramanujan, Yesilyurt [347], [348] devised ingenious and difficult proofs of the remaining five identities, and so these papers [347], [348] are the second primary source on which Chapter 8 is constructed.

Chapter 9 is devoted to one general theorem on certain sums of positive integral powers of theta functions, and five examples in illustration. Many offered original ideas about the entries in this chapter; in particular, Heng Huat Chan and Hamza Yesilyurt deserve special thanks. Ramanujan's primary theorem has inspired several generalizations, but it seems likely that Ramanujan's approach has not yet been discovered.

In 1915, the London Mathematical Society published Ramanujan's paper on highly composite numbers [274], [281, 78–128]. However, this is only part of the paper that Ramanujan submitted. The London Mathematical Society was in poor financial condition at that time, and to diminish expenses, they did not publish all of Ramanujan's paper. Fortunately, the remainder of the paper has not been lost and resides in the library at Trinity College, Cambridge. In its original handwritten form, it was photocopied along with Ramanujan's lost notebook in 1988 [283]. J.-L. Nicolas and G. Robin prepared an annotated version of the paper for the first volume of the *Ramanujan Journal* in 1997 [284]. In particular, they inserted text where gaps occurred, and at the end of the paper, they provided extensive commentary on research in the field of highly composite numbers accomplished since the publication of Ramanujan's original paper [274]. Chapter 10 contains this previously unpublished manuscript of Ramanujan on highly composite numbers, as completed by Nicolas and Robin, and a moderately revised and extended version of the commentary originally written by Nicolas and Robin.

The first author is grateful to Frank Garvan, whose ideas and insights permeate Chapter 2. The second author thanks Heng Huat Chan, Song Heng Chan, and Wen-Chin Liaw for their collaboration on the papers [62] and [63], from which Chapters 3 and 4 were prepared. The last section of the former paper, which corresponds to Section 3.8 of Chapter 3, is due to Garvan, whom we thank for the many valuable remarks and suggestions on ranks and cranks that he made to the authors of [62] and [63]. Atul Dixit read Chapters 2 and 9 in detail and offered several corrections and suggestions.

We thank Paul Bateman, Heng Huat Chan, Frank Garvan, Michael Hirschhorn, Pieter Moree, Robert A. Rankin, and Jean-Pierre Serre for helpful comments on Chapter 5. We are particularly grateful to Hirschhorn for reading several versions of Chapter 5 and providing insights that we would not have otherwise observed. In particular, the argument given in square brackets near the beginning of Section 5.21 is his. He showed us that Ramanujan's conjecture on the value of c_λ at the beginning of Section 5.23 is correct. He also provided the meaning of the four mysterious numbers that Ramanujan recorded at the end of Section 5.21, but which we moved to a more proper place at the end of Section 5.24. Lastly, he provided references that Ono and the second author had overlooked in our earlier version [67] of the $p(n)/\tau(n)$ manuscript.

We are grateful to the late Professor W.N. Everitt, the School of Mathematics, and the Library at the University of Birmingham for supplying us with a copy of Rushforth's dissertation and for permission to use material from it in this volume.

Our account of Chapter 6 originates primarily from two papers by the second author that he coauthored, the first with Ae Ja Yee and Jinhee Yi, and the second with Chadwick Gugg and Sun Kim. We thank all of them for their kind collaboration. One particular entry on page 331 that we discuss in Chapter 6 was particularly puzzling, and we are grateful to L. Bruce Richmond for helpful correspondence.

The authors thank Heng Huat Chan for informing us that the results on page 189 of the lost notebook were briefly discussed by K.G. Ramanathan [273, pp. 154–155], and for discussion on one of the incorrect entries on page

189. We are also pleased to thank Scott Ahlgren for his proof of another entry on page 189.

Chapter 7 is entirely due to Dennis Eichhorn, who completed this work as part of a research assistantship under the second author at the University of Illinois.

The second author is greatly indebted to his coauthors, G. Choi, Y.-S. Choi, H. Hahn, B.P. Yeap, A.J. Yee, H. Yesilyurt, and J. Yi, of the *Memoir* [65], which has been revised for Chapter 8 in this volume.

Chapter 9 has been significantly enhanced by correspondence that the second author had with Hamza Yesilyurt, who provided material from his forthcoming paper with A. Berkovich and Garvan [53].

It is our great pleasure to thank J.-L. Nicolas and G. Robin for their initial preparation of Chapter 10 and in particular for their insightful comments accompanying it. We thank K.S. Williams for providing several references for our commentary on Chapter 10.

We are indebted to J.P. Massias for calculating largely composite numbers and finding the meaning of the table appearing in [283, p. 280].

Heng Huat Chan, Atul Dixit, Byungchan Kim, Pieter Moree, Jaebum Sohn, and Michael Somos read in detail large portions of the manuscript for this volume and provided many useful comments and corrections.

David Kramer, who is likely the most careful and knowledgeable copy editor in the mathematics publishing realm, uncovered many errors and inconsistencies, and we thank him for his usually superb copy editing of our book. We are also indebted to Springer's TeXexpert, Rajiv Monsurate, for considerable advice.

We thank the librarian and staff at Trinity College Library, Cambridge, for providing the authors with copies of the lost notebook and several other manuscripts by Ramanujan.

The first author thanks the National Science Foundation for support while this book was being prepared. The second author thanks the National Security Agency for summer support. The second author is particularly grateful to the Sloan Foundation for a grant that relieved him of teaching duties for the 2011–2012 academic year and enabled him to complete the writing of this volume with his coauthor.

2

Ranks and Cranks, Part I

2.1 Introduction

This somewhat lengthy chapter concerns some of the most important formulas from the lost notebook [283], which are contained in only a few lines. We first introduce some standard notation that will be used throughout this chapter (and most of this book). Secondly, we record the two formulas listed at the top of page 20 (one of which is repeated in the middle of page 18). After stating these formulas, we provide history demonstrating that these entries are the genesis of some of the most important developments in the theory of partitions during the twentieth and twenty-first centuries. Next, we offer two further claims found in the lost notebook. Lastly, we provide proofs for all four claims.

For each nonnegative integer n, set

$$(a)_n := (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \qquad (a)_\infty := (a;q)_\infty := \lim_{n\to\infty}(a;q)_n, \quad |q|<1.$$

Also, set

$$(a_1,\dots,a_m;q)_n := (a_1;q)_n \cdots (a_m;q)_n$$

and

$$(a_1,\dots,a_m;q)_\infty := (a_1;q)_\infty \cdots (a_m;q)_\infty. \tag{2.1.1}$$

Ramanujan's general theta function $f(a,b)$ is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2}, \qquad |ab|<1. \tag{2.1.2}$$

It satisfies the well-known Jacobi triple product identity [60, p. 10, Theorem 1.3.3], [12, p. 21, Theorem 2.8]

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{2.1.3}$$

G.E. Andrews, B.C. Berndt, *Ramanujan's Lost Notebook: Part III*,
DOI 10.1007/978-1-4614-3810-6_2,

Also recall that [55, p. 34, Entry 18(iv)] for any integer n,

$$f(a,b) = a^{n(n+1)/2}b^{n(n-1)/2}f(a(ab)^n, b(ab)^{-n}). \tag{2.1.4}$$

We now state the first of the two aforementioned remarkable entries from the lost notebook.

Entry 2.1.1 (pp. 18, 20). *Let ζ_5 be a primitive fifth root of unity, and let*

$$F_5(q) := \frac{(q;q)_\infty}{(\zeta_5 q;q)_\infty(\zeta_5^{-1}q;q)_\infty}. \tag{2.1.5}$$

Then

$$\begin{aligned} F_5(q) = A(q^5) &- (\zeta_5+\zeta_5^{-1})^2 qB(q^5) \\ &+ (\zeta_5^2+\zeta_5^{-2})q^2C(q^5) - (\zeta_5+\zeta_5^{-1})q^3D(q^5), \end{aligned} \tag{2.1.6}$$

where

$$A(q) := \frac{(q^5;q^5)_\infty G^2(q)}{H(q)}, \tag{2.1.7}$$

$$B(q) := (q^5;q^5)_\infty G(q), \tag{2.1.8}$$

$$C(q) := (q^5;q^5)_\infty H(q), \tag{2.1.9}$$

$$D(q) := \frac{(q^5;q^5)_\infty H^2(q)}{G(q)}, \tag{2.1.10}$$

with

$$G(q) := \frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty} \tag{2.1.11}$$

and

$$H(q) := \frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}. \tag{2.1.12}$$

We remark that by the famous Rogers–Ramanujan identities [15, Chapter 10],

$$G(q) = \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n} \qquad \text{and} \qquad H(q) = \sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q;q)_n}.$$

The identity (2.1.6) is an example of a *dissection*. Since this and the following chapter are devoted to dissections, we offer below their definition.

Definition 2.1.1. *Let $P(q)$ denote any power series in q. Then the t-dissection of P is given by*

$$P(q) =: \sum_{k=0}^{t-1} q^k P_k(q^t). \tag{2.1.13}$$

Note that (2.1.6) provides a 5-dissection for $F_5(q)$, i.e., (2.1.6) separates $F_5(q)$ into power series according to the residue classes modulo 5 of their powers. In analogy with (2.1.6), we see that (2.1.17) in the next entry provides a 5-dissection for $f_5(q)$.

Of the dissections offered by Ramanujan in his lost notebook, some, such as (2.1.6), are given as equalities in terms of roots of unity; others are given as congruences in terms of a variable a. In Chapter 3, we establish Ramanujan's dissections in terms of congruences, while in this chapter we prove 5- and 7-dissections in the form of equalities for each of the rank and crank generating functions, whose representations are given, respectively, in (2.1.24) and (2.1.27) below. The precise definitions of the *rank* and *crank* of a partition will be given after we record the second of the two aforementioned fundamental identities.

In order to explicate our remark about congruences in the preceding paragraph, following Ramanujan in his lost notebook, we define the more general function

$$F_a(q) := \frac{(q;q)_\infty}{(aq;q)_\infty (q/a;q)_\infty}. \tag{2.1.14}$$

(Note that the notation (2.1.14) conflicts with that of (2.1.5); the right-hand side of (2.1.5) would be $F_{\zeta_5}(q)$ in the notation (2.1.14).) Set

$$A_n := a^n + a^{-n} \qquad \text{and} \qquad S_n := \sum_{k=-n}^{n} a^k.$$

Then [62, p. 105, Theorem 5.1]

$$F_a(q) \equiv A(q^5) + (A_1 - 1)qB(q^5) + A_2 q^2 C(q^5) - A_1 q^3 D(q^5) \pmod{S_2}. \tag{2.1.15}$$

Thus, we have replaced the primitive root ζ_5 by the general variable a. The congruence (2.1.15) is then a generalization of (2.1.6), because if we set $a = \zeta_5$ in (2.1.15), the congruence is transformed into an identity. An advantage of (2.1.15) over (2.1.6) is that we can put $a = 1$ in (2.1.15) and so immediately deduce the Ramanujan congruence

$$p(5n+4) \equiv 0 \pmod 5,$$

where $p(n)$ is the number of partitions of n. Although (2.1.15) appears to be more general than (2.1.6), in fact, *it is not.* It is shown in [62, pp. 118–119] that (2.1.15) can be derived from (2.1.6). In Section 3.8 of the following chapter we reproduce that argument, which is due to F.G. Garvan.

Entry 2.1.2 (p. 20). *Let ζ_5 be a primitive fifth root of unity, and let*

$$f_5(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(\zeta_5 q;q)_n \left(\zeta_5^{-1} q;q\right)_n}. \tag{2.1.16}$$

Then

$$f_5(q) = A(q^5) + \left(\zeta_5 + \zeta_5^{-1} - 2\right)\phi(q^5) + qB(q^5) + \left(\zeta_5 + \zeta_5^{-1}\right) q^2 C(q^5) \\ - \left(\zeta_5 + \zeta_5^{-1}\right) q^3 \left\{ D(q^5) - \left(\zeta_5^2 + \zeta_5^{-2} - 2\right) \frac{\psi(q^5)}{q^5} \right\}, \qquad (2.1.17)$$

where $A(q)$, $B(q)$, $C(q)$, *and* $D(q)$ *are given in* (2.1.7)–(2.1.10), *and where*

$$\phi(q) := \sum_{n=0}^{\infty} \phi_n q^n := -1 + \sum_{n=0}^{\infty} \frac{q^{5n^2}}{(q;q^5)_{n+1}(q^4;q^5)_n} \qquad (2.1.18)$$

and

$$\frac{\psi(q)}{q} := -\frac{1}{q} + \sum_{n=0}^{\infty} \psi_n q^n := \sum_{n=0}^{\infty} \frac{q^{5n^2-1}}{(q^2;q^5)_{n+1}(q^3;q^5)_n}. \qquad (2.1.19)$$

Corollaries of the preceding entry appear in the middle of page 184 in the lost notebook. Since their proofs are immediate consequences of Entry 2.1.2, we offer them here.

Entry 2.1.3 (p. 184). *Write*

$$\sum_{n=0}^{\infty} \lambda_n q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1 + \frac{\sqrt{5}+1}{2} q + q^2) \cdots (1 + \frac{\sqrt{5}+1}{2} q^n + q^{2n})}. \qquad (2.1.20)$$

Then,

$$\sum_{n=0}^{\infty} \lambda_{5n+1} q^n = \frac{(q^5;q^5)_\infty}{(q;q^5)_\infty (q^4;q^5)_\infty} = (q^5;q^5)_\infty G(q), \qquad (2.1.21)$$

$$\sum_{n=0}^{\infty} \lambda_{5n+2} q^n = -\frac{\sqrt{5}+1}{2} \frac{(q^5;q^5)_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty} = -\frac{\sqrt{5}+1}{2} (q^5;q^5)_\infty H(q), \qquad (2.1.22)$$

λ_{5n-1} *is identically zero.* (2.1.23)

Proof. In the definition (2.1.16), set $\zeta_5 = e^{4\pi i/5}$; therefore, $\zeta_5 + \zeta_5^{-1} = -\frac{\sqrt{5}+1}{2}$. Using then the notation (2.1.20), equate coefficients of q^{5n+1} on both sides of (2.1.17). Divide both sides by q and lastly replace q^5 by q in the resulting identity to establish (2.1.21). Similarly, to prove (2.1.22), equate coefficients of q^{5n+2} on both sides of (2.1.17). Divide both sides by q^2 and replace q^5 by q. Finally, we note that the dissection (2.1.17) does not have any powers of the form q^{5n-1}, and so (2.1.23) is immediate. □

Before presenting the third and fourth entries for this chapter, as remarked above, it is appropriate to say something about these results, which lay hidden during one of the most interesting developments in the theory of partitions during the twentieth century.

In 1944, F. Dyson [127] published a paper filled with fascinating conjectures from the theory of partitions. Namely, Dyson began by defining the *rank* of a partition to be the largest part minus the number of parts. Dyson's objective was to provide a purely combinatorial description of Ramanujan's theorem that 5 divides $p(5n+4)$. In particular, Dyson conjectured that the partitions of $5n+4$ classified by their rank modulo 5 did, indeed, produce five sets of equal cardinality, namely $p(5n+4)/5$. He was also led to conjecture that the partitions of $7n+5$, classified by rank, split into seven sets each of cardinality $p(7n+5)/7$. This would prove the second Ramanujan congruence, namely, that 7 divides $p(7n+5)$. He also conjectured a generating function for ranks. If $N(m,n)$ denotes the number of partitions of n with rank m, then Dyson's observations make clear he knew that

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} N(m,n)z^m q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq;q)_n(z^{-1}q;q)_n}. \tag{2.1.24}$$

Observe that if we take $z=1$ in (2.1.24), then (2.1.24) reduces to the well-known generating function for $p(n)$,

$$\sum_{n=0}^{\infty} p(n)q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n^2},$$

which is due to Euler. If we set $z=-1$ in (2.1.24), we obtain Ramanujan's mock theta function $f(q)$.

Unfortunately, it turned out that the Ramanujan congruence

$$p(11n+6) \equiv 0 \pmod{11} \tag{2.1.25}$$

was *not* explicable in the same way that worked for $p(5n+4)$ and $p(7n+5)$. So Dyson conjectured the existence of an unknown parameter of partitions, which he whimsically called "the crank," to explain (2.1.25).

In 1954, A.O.L. Atkin and H.P.F. Swinnerton-Dyer [28] proved all of Dyson's conjectures; however, the crank remained undiscovered.

The real breakthrough in this study was made by Garvan in his Ph.D. thesis [146] at Pennsylvania State University in 1986. Garvan's thesis is primarily devoted to the Entries 2.1.1 and 2.1.2 given above. Observe that Entry 2.1.2 is devoted to a special case of the generating function (2.1.24) for ranks. Not only was Garvan able to prove these two entries, but he also deduced all of the Atkin and Swinnerton-Dyer results for the modulus 5 from Entry 2.1.2. As for Entry 2.1.1, Garvan defined a "vector crank," which did provide a combinatorial explanation for 11 dividing $p(11n+6)$, but did this via certain triples of partitions, i.e., vector partitions. Subsequently, Garvan and Andrews [17] found the actual crank. Namely, for any given partition π, let $\ell(\pi)$ denote the largest part of π, $\omega(\pi)$ the number of ones appearing in π, and $\mu(\pi)$ the number of parts of π larger than $\omega(\pi)$. Then the crank, $c(\pi)$, is given by

$$c(\pi) = \begin{cases} l(\pi), & \text{if } \omega(\pi) = 0, \\ \mu(\pi) - \omega(\pi), & \text{if } \omega(\pi) > 0. \end{cases} \tag{2.1.26}$$

For $n > 1$, let $M(m,n)$ denote the number of partitions of n with crank m, while for $n \leq 1$ we set

$$M(m,n) = \begin{cases} -1, & \text{if } (m,n) = (0,1), \\ 1, & \text{if } (m,n) = (0,0), (1,1), (-1,1), \\ 0, & \text{otherwise.} \end{cases}$$

The generating function for $M(m,n)$ is given by

$$\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} M(m,n) a^m q^n = \frac{(q;q)_\infty}{(aq;q)_\infty (q/a;q)_\infty}. \tag{2.1.27}$$

As shown by Andrews and Garvan [17], the combinatorial equivalent of (2.1.27) is given by (2.1.26). Note that if we set $a = 1$ in (2.1.27), we obtain Euler's original generating function for $p(n)$,

$$\sum_{n=0}^{\infty} p(n) q^n = \frac{1}{(q;q)_\infty}.$$

Observe that Entry 2.1.1 provides an identity for a special instance of the generating function for cranks.

Thus, although Ramanujan did not record combinatorial definitions of the rank and crank in his lost notebook (in fact, there are hardly any words at all in the lost notebook), he had discovered their generating functions. From the entries on ranks and cranks in this and the following two chapters, it is clear that Ramanujan placed considerable importance on these ideas, and it is regrettable indeed that we do not know Ramanujan's motivations and thoughts on these two fundamental concepts in the theory of partitions.

We finally record the last two results to be included in this chapter. Actually in each entry below, Ramanujan gives only the left-hand side or hints at it. However, the analogies with Entries 2.1.1 and 2.1.2 are so clear that we have filled in what was clearly intended for the right-hand sides. For Entry 2.1.4, Garvan has supplied the right-hand side in [146, p. 62].

Entry 2.1.4 (p. 19). *Let ζ_7 be a primitive seventh root of unity, and let*

$$F_7(q) := \frac{(q;q)_\infty}{(\zeta_7 q;q)_\infty (\zeta_7^{-1} q;q)_\infty}. \tag{2.1.28}$$

Then

$$F_7(q) = (q^7;q^7)_\infty \Big\{ X^2(q^7) + \left(\zeta_7 + \zeta_7^{-1} - 1\right) qX(q^7)Y(q^7) \tag{2.1.29}$$
$$+ \left(\zeta_7^2 + \zeta_7^{-2}\right) q^2 Y^2(q^7) + \left(\zeta_7^3 + \zeta_7^{-3} + 1\right) q^3 X(q^7)Z(q^7)$$
$$- \left(\zeta_7 + \zeta_7^{-1}\right) q^4 Y(q^7)Z(q^7) - \left(\zeta_7^2 + \zeta_7^{-2} + 1\right) q^6 Z^2(q^7) \Big\},$$

where

$$X(q) := \prod_{\substack{n=1 \\ n \not\equiv 0, \pm 3 \,(\mathrm{mod}\, 7)}}^{\infty} (1-q^n)^{-1}, \tag{2.1.30}$$

$$Y(q) := \prod_{\substack{n=1 \\ n \not\equiv 0, \pm 2 \,(\mathrm{mod}\, 7)}}^{\infty} (1-q^n)^{-1}, \tag{2.1.31}$$

$$Z(q) := \prod_{\substack{n=1 \\ n \not\equiv 0, \pm 1 \,(\mathrm{mod}\, 7)}}^{\infty} (1-q^n)^{-1}. \tag{2.1.32}$$

There are series representations for $X(q)$, $Y(q)$, and $Z(q)$ that yield analogues of the Rogers-Ramanujan identities for $G(q)$ and $H(q)$ [12, p. 117, Exercise 10].

In order to state the last major entry of this chapter, we need considerable notation. First, introducing the notation of Atkin and Swinnerton-Dyer [28, p. 94], we let

$$\Sigma(z,\zeta,q) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n \zeta^n q^{3n(n+1)/2}}{1 - zq^n}. \tag{2.1.33}$$

Furthermore, to simplify future considerations, in particular to state and prove Entry 2.1.5 below, we make the conventions

$$P_7(a) := \left(q^{7a}, q^{49-7a}; q^{49}\right)_\infty \quad (a \neq 0), \tag{2.1.34}$$
$$P_7(0) := \left(q^{49}; q^{49}\right)_\infty, \tag{2.1.35}$$
$$\Sigma_7(a,b) := \Sigma\left(q^{7a}, q^{7b}, q^{49}\right) \quad (a \neq 0), \tag{2.1.36}$$
$$\Sigma_7(0,b) := \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{(-1)^n q^{147n(n+1)/2 + 7bn}}{1 - q^{49n}}. \tag{2.1.37}$$

We note in passing that by (2.1.30)–(2.1.32),

$$P_7(1) = \frac{(q^7;q^7)_\infty Z(q^7)}{(q^{49};q^{49})_\infty}, \tag{2.1.38}$$
$$P_7(2) = \frac{(q^7;q^7)_\infty Y(q^7)}{(q^{49};q^{49})_\infty}, \tag{2.1.39}$$
$$P_7(3) = \frac{(q^7;q^7)_\infty X(q^7)}{(q^{49};q^{49})_\infty}. \tag{2.1.40}$$

Finally, we are ready to supply the right-hand side for the analogue of Entry 2.1.2 for the modulus 7.

Entry 2.1.5 (p. 19). *Let ζ_7 be a primitive seventh root of unity, and let*

$$f_7(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(\zeta_7 q;q)_n(\zeta_7^{-1}q;q)_n}. \tag{2.1.41}$$

Then

$$\begin{aligned} f_7(q) &= \left(2-\zeta_7-\zeta_7^{-1}\right)\left(1-A_7(q^7)+q^7Q_1(q^7)\right)+A_7(q^7) \\ &\quad + qT_1(q^7)+q^2\left\{\left(\zeta_7+\zeta_7^{-1}\right)B_7(q^7)+q^{14}Q_3(q^7)(\zeta_7+\zeta_7^{-1}-\zeta_7^{-2}-\zeta_7^{2}\right\} \\ &\quad + q^3T_2(q^7)\left(1+\zeta_7^2+\zeta_7^{-2}\right)-q^4\left(\zeta_7^2+\zeta_7^{-2}\right)T_3(q^7) \\ &\quad + q^6\left\{q^7Q_2(q^7)\left(\zeta_7^2+\zeta_7^{-2}-\zeta_7^3-\zeta_7^{-3}\right)-C_7(q^7)\left(1+\zeta_7^3+\zeta_7^{-3}\right)\right\}, \end{aligned} \tag{2.1.42}$$

where

$$A_7(q) := \frac{(q^7,q^3,q^4;q^7)_\infty}{(q,q^2,q^5,q^6;q^7)_\infty}, \tag{2.1.43}$$

$$B_7(q) := \frac{(q^7,q^2,q^5;q^7)_\infty}{(q,q^3,q^4,q^6;q^7)_\infty}, \tag{2.1.44}$$

$$C_7(q) := \frac{(q^7,q,q^6;q^7)_\infty}{(q^2,q^3,q^4,q^5;q^7)_\infty}, \tag{2.1.45}$$

and for $m = 1, 2, 3$,

$$Q_m(q^7) := \frac{\Sigma_7(m,0)}{P_7(0)} \tag{2.1.46}$$

and

$$T_m(q^7) := \frac{P_7(0)}{P_7(m)}. \tag{2.1.47}$$

We remark that the functions $Q_m(q^7)$ in (2.1.46) can be expressed in terms of the generating function for ranks. By a result of Garvan [146, p. 68, Lemma (7.9)], for $|q| < |z| < 1/|q|$ and $z \neq 1$,

$$-1+\frac{1}{1-z}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(zq;q)_n(q/z;q)_n} = \frac{z}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{3n(n+1)/2}}{1-zq^n}.$$

Hence, after modest rearrangement, we find that

$$\Sigma_7(m,0) = \frac{(q^{49};q^{49})_\infty}{q^{7m}}\left\{-1+\sum_{n=0}^{\infty}\frac{q^{49n^2}}{(q^{7m};q^{49})_{n+1}(q^{49-7m};q^{49})_n}\right\}.$$

Throughout this chapter our work will follow closely the marvelous papers by Atkin and Swinnerton-Dyer [28] and Garvan [146].

2.2 Proof of Entry 2.1.1

Here we shall follow the elegant proof given by Garvan [146]. Throughout this section ζ_5 is a primitive fifth root of unity. We begin with the observation [146, p. 58, Lemma (3.9)]

$$\frac{1}{(\zeta_5 q;q)_\infty(\zeta_5^{-1}q;q)_\infty} = G(q^5) + q(\zeta_5+\zeta_5^{-1})H(q^5), \tag{2.2.1}$$

where $G(q)$ is defined in (2.1.11) and $H(q)$ is defined in (2.1.12). We prove the identity (2.2.1). Using the Jacobi triple product identity (2.1.3) twice, we find that

$$\begin{aligned}
\frac{1}{(\zeta_5 q;q)_\infty(\zeta_5^{-1}q;q)_\infty} &= \frac{(q,\zeta_5^2 q,\zeta_5^{-2};q)_\infty}{(q,\zeta_5 q,\zeta_5^{-1}q,\zeta_5^2 q,\zeta_5^{-2};q)_\infty}\\
&= \frac{(q,\zeta_5^2 q,\zeta_5^{-2};q)_\infty}{(1-\zeta_5^{-2})(q^5;q^5)_\infty}\\
&= \frac{1}{(1-\zeta_5^{-2})(q^5;q^5)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n\zeta_5^{2n}q^{(n^2+n)/2}\\
&= \frac{1}{(1-\zeta_5^{-2})(q^5;q^5)_\infty}\sum_{\nu=-2}^{2}\sum_{m=-\infty}^{\infty}(-1)^{5m+\nu}\zeta_5^{10m+2\nu}q^{(5m+\nu)(5m+\nu+1)/2}\\
&= \frac{1}{(1-\zeta_5^{-2})(q^5;q^5)_\infty}\sum_{\nu=-2}^{2}(-1)^\nu\zeta_5^{2\nu}q^{\nu(\nu+1)/2}\sum_{m=-\infty}^{\infty}(-1)^m q^{\left(25m^2+(10\nu+5)m\right)/2}\\
&= \frac{1}{(1-\zeta_5^{-2})}\sum_{\nu=-2}^{2}(-1)^\nu\zeta_5^{2\nu}q^{\nu(\nu+1)/2}\frac{f(-q^{15+5\nu},-q^{10-5\nu})}{(q^5;q^5)_\infty}\\
&= \frac{1}{(1-\zeta_5^{-2})}\sum_{\nu=-2}^{2}(-1)^\nu\zeta_5^{2\nu}q^{\nu(\nu+1)/2}\frac{\left(q^{15+5\nu},q^{10-5\nu},q^{25};q^{25}\right)_\infty}{(q^5;q^5)_\infty}.
\end{aligned}$$

Now, by (2.1.11) and (2.1.12),

$$\frac{\left(q^{15+5\nu},q^{10-5\nu},q^{25};q^{25}\right)_\infty}{(q^5;q^5)_\infty} = \begin{cases} G(q^5), & \text{if } \nu=0,-1,\\ H(q^5), & \text{if } \nu=1,-2,\\ 0, & \text{if } \nu=2.\end{cases}$$

Hence,

$$\begin{aligned}
\frac{1}{(\zeta_5 q;q)_\infty(\zeta_5^{-1}q;q)_\infty} &= \frac{1}{(1-\zeta_5^{-2})}G(q^5)\left(1-\zeta_5^{-2}\right)\\
&\quad+\frac{1}{(1-\zeta_5^{-2})}H(q^5)\left(-\zeta_5^2 q+\zeta_5^{-4}q\right)\\
&= G(q^5)+q\left(\zeta_5+\zeta_5^{-1}\right)H(q^5),
\end{aligned}$$

which is (2.2.1).

Next, we continue to follow Garvan in [146, p. 60, Lemma 3.18] and so employ the identity

$$(q;q)_\infty = (q^{25};q^{25})_\infty \left(\frac{G(q^5)}{H(q^5)} - q - q^2 \frac{H(q^5)}{G(q^5)} \right), \tag{2.2.2}$$

which is one of the famous identities for the Rogers–Ramanujan continued fraction [15, p. 11, equation (1.1.10)]

$$\frac{1}{1} \, \frac{}{+} \, \frac{q}{1} \, \frac{}{+} \, \frac{q^2}{1} \, \frac{}{+} \, \frac{q^3}{1} \, \frac{}{+} \cdots = \frac{H(q)}{G(q)}.$$

We now multiply together (2.2.1) and (2.2.2) to obtain

$$\begin{aligned}
&\frac{(q;q)_\infty}{(\zeta_5 q;q)_\infty(\zeta_5^{-1}q;q)_\infty} = \left(G(q^5) + q(\zeta_5+\zeta_5^{-1})H(q^5)\right) \\
&\quad \times (q^{25};q^{25})_\infty \left(\frac{G(q^5)}{H(q^5)} - q - q^2 \frac{H(q^5)}{G(q^5)} \right) \\
&= (q^{25};q^{25})_\infty \left\{ \frac{G^2(q^5)}{H(q^5)} + q\left(-1+\zeta_5+\zeta_5^{-1}\right) G(q^5) \right. \\
&\qquad\qquad \left. + q^2\left(-1-(\zeta_5+\zeta_5^{-1})\right) H(q^5) + q^3\left(-(\zeta_5+\zeta_5^{-1})\right) \frac{H^2(q^5)}{G(q^5)} \right\} \\
&= A(q^5) - q(\zeta_5+\zeta_5^{-1})^2 B(q^5) \\
&\quad + q^2\left(\zeta_5^2+\zeta_5^{-2}\right) C(q^5) - \left(\zeta_5+\zeta_5^{-1}\right) q^3 D(q^5),
\end{aligned}$$

and Entry 2.1.1 is proved.

2.3 Background for Entries 2.1.2 and 2.1.4

As was mentioned in Section 2.1, Atkin and Swinnerton-Dyer [28] proved the conjectures of Dyson [127]. Garvan [146] proved that their work for the modulus 5 was in fact equivalent to Entry 2.1.2. Our proof here relies completely on Garvan's observation. We will modify the work of Atkin and Swinnerton-Dyer to the extent that we will eschew using their Lemma 2, which we state below.

Lemma 2.3.1. *Let $f(z)$ be a single-valued analytic function of z, except possibly for a finite number of poles, in every region $0 \le z_1 \le |z| \le z_2$; and suppose that for some constants A and w with $0 < |w| < 1$, and some (positive, zero, or negative) integer n, we have*

$$f(zw) = Az^n f(z)$$

identically in z. Then either $f(z)$ has exactly n more poles than zeros in

$$|w| \le |z| \le 1,$$

or $f(z)$ vanishes identically.

While this is a beautiful, powerful, and useful result, it is unlikely to have been the type of result that Ramanujan would have utilized.

The principal idea is to transform (2.1.16), (2.1.18), and (2.1.19) into certain bilateral series, which are called higher-level Appell series [355]. In particular, see Lemma 2.4.1 and the functions (2.1.33) and (2.3.11), which we define and develop in the next several pages.

The next identity does not appear in the lost notebook. However, it is effectively a generalization of Entries 12.4.4 (as restated in (12.4.15)) and 12.5.3 (as restated in (12.5.14)) in our first book [15, pp. 276, 283]. Consequently, it is a partial fraction decomposition of precisely the sort that Ramanujan often considered.

Lemma 2.3.2. [28, p. 94, Lemma 7] *For $\Sigma(z,\zeta,q)$ defined by* (2.1.33),

$$\zeta^3\Sigma(z\zeta,\zeta^3,q)+\Sigma(z\zeta^{-1},\zeta^{-3},q)-\zeta\frac{(\zeta^2,q/\zeta^2;q)_\infty}{(\zeta,q/\zeta;q)_\infty}\Sigma(z,1,q)$$
$$=\frac{(\zeta,q/\zeta,\zeta^2,q/\zeta^2,q,q;q)_\infty}{(z/\zeta,q\zeta/z,z,q/z,z\zeta,q/(z\zeta);q)_\infty}. \quad (2.3.1)$$

This formula was first proved by G.N. Watson [335], and we shall follow his proof. M. Jackson [185] has given a third proof from the theory of q-hypergeometric series, and S.H. Chan [105] has established a considerable generalization of Lemma 2.3.2.

Proof. Let us fix a positive integer N and consider the partial fraction decomposition with respect to z of the rational function

$$F_N(z):=\frac{(\zeta,q/\zeta,\zeta^2,q/\zeta^2,q,q;q)_N}{(z/\zeta,q\zeta/z,z,q/z,z\zeta,q/(z\zeta);q)_N}. \quad (2.3.2)$$

This function has simple poles at $z=\zeta q^m$, q^m, and $\zeta^{-1}q^m$ for $-(N-1)\le m\le N$. Hence, we see that

$$F_N(z):=\sum_{m=-N}^{N-1}\frac{A_m(N)}{1-z\zeta q^m}+\sum_{m=-N}^{N-1}\frac{B_m(N)}{1-zq^m/\zeta}+\sum_{m=-N}^{N-1}\frac{C_m(N)}{1-zq^m}. \quad (2.3.3)$$

Now for any integer m, algebraic simplification reveals that

$$(xq^{-m},q^{1+m}/x;q)_N=(-1)^mq^{-m(m+1)/2}x^m(q/x;q)_{N+m}(x;q)_{N-m}. \quad (2.3.4)$$

First, after three applications of (2.3.4), with $x=\zeta^{-2},\zeta^{-1},1$, respectively, we find that

$$A_m(N)=\lim_{z\to\zeta^{-1}q^{-m}}(1-z\zeta q^m)F_N(z)=\frac{(-1)^mq^{3m(m+1)/2}\zeta^{3m+3}}{(q/\zeta^2;q)_{N-m-1}(\zeta^2;q)_{N+m+1}}$$
$$\times\frac{(\zeta,q/\zeta,\zeta^2,q/\zeta^2,q,q;q)_N}{(q/\zeta;q)_{N-m-1}(\zeta;q)_{N+m+1}(q;q)_{N-m-1}(q;q)_{N+m}}, \quad (2.3.5)$$

and

$$\lim_{N\to\infty} A_m(N) = (-1)^m q^{3m(m+1)/2}\zeta^{3m+3}. \tag{2.3.6}$$

Second, applying (2.3.4) three times once again, but now with $x = 1, \zeta, \zeta^2$, respectively, we find that

$$\begin{aligned} B_m(N) &= \lim_{z\to\zeta q^{-m}} (1 - z\zeta^{-1}q^m)F_N(z) \qquad (2.3.7)\\ &= \frac{(-1)^m q^{3m(m+1)/2}\zeta^{-3m}(\zeta, q/\zeta, \zeta^2, q/\zeta^2, q, q; q)_N}{(\zeta;q)_{N-m}(q/\zeta;q)_{N+m}(\zeta^2;q)_{N-m}(q/\zeta^2;q)_{N+m}(q;q)_{N-m-1}(q;q)_{N+m}}, \end{aligned}$$

and

$$\lim_{N\to\infty} B_m(N) = (-1)^m q^{3m(m+1)/2}\zeta^{-3m}. \tag{2.3.8}$$

Third, applying (2.3.4) with $x = \zeta^{-1}, 1, \zeta$, respectively, we find that

$$\begin{aligned} C_m(N) &= \lim_{z\to q^{-m}} (1 - zq^m)F_N(z) \qquad (2.3.9)\\ &= \frac{-\zeta(-1)^m q^{3m(m+1)/2}(\zeta, q/\zeta, \zeta^2, q/\zeta^2, q, q; q)_N}{(\zeta;q)_{N-m}(q/\zeta;q)_{N+m}(\zeta;q)_{N+m+1}(q/\zeta;q)_{N-m-1}(q;q)_{N-m-1}(q;q)_{N+m}}, \end{aligned}$$

and

$$\lim_{N\to\infty} C_m(N) = \frac{-\zeta(\zeta^2, q/\zeta^2;q)_\infty(-1)^m q^{3m(m+1)/2}}{(\zeta, q/\zeta;q)_\infty}. \tag{2.3.10}$$

We can now easily deduce (2.3.1). Clearly $F_N(z)$ converges uniformly to the right-hand side of (2.3.1) as $N \to \infty$.

Equations (2.3.6), (2.3.8), and (2.3.10) when applied to (2.3.3) yield the left-hand side of (2.3.1), provided we are allowed to take the limit $N \to \infty$ inside the summation signs, and indeed this interchange of limit and summation is legitimate because the convergence is uniformly independent of m, and the resulting series, after letting $N \to \infty$, is convergent as long as $|q| < 1$ and z is restricted away from the poles. Thus (2.3.1) is proved. □

Following Atkin and Swinnerton-Dyer [28, p. 96], we now define

$$\begin{aligned} g(z,q) &:= z\frac{(z^2, q/z^2;q)_\infty}{(z, q/z;q)_\infty}\Sigma(z,1,q) - z^3\Sigma(z^2,z^3,q) \qquad (2.3.11)\\ &\quad - \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \frac{(-1)^n z^{-3n} q^{3n(n+1)/2}}{1-q^n}. \qquad (2.3.12) \end{aligned}$$

Now the definition of $g(z,q)$ is motivated as follows. We would like to set $\zeta = z$ in (2.3.1); however, this would produce an undefined term at $n = 0$ in $\sum(1, z^{-3}, q)$ in (2.3.1). Note that $g(z,q)$ is the negative of the left-hand side of (2.3.1), with $\zeta = z$ and the one offending term at $n = 0$ in $\sum(1, z^{-3}, q)$ removed. Thus,

$$g(z,q) = \lim_{\zeta \to z} \left(\frac{1}{1 - z/\zeta} - \frac{(\zeta, q/\zeta, \zeta^2, q/\zeta^2, q, q; q)_\infty}{(z/\zeta, q\zeta/z, z, q/z, z\zeta, q/(z\zeta); q)_\infty} \right). \tag{2.3.13}$$

It is now a straightforward exercise to prove the next lemma, which is the second half of Lemma 8 in [28, p. 96].

Lemma 2.3.3. *We have*

$$g(z,q) + g(q/z,q) = 1. \tag{2.3.14}$$

Proof. We proceed as follows:

$$\begin{aligned}
g(z,q) + g(q/z,q) &= \lim_{\zeta \to z} \left(\frac{1}{1 - z/\zeta} - \frac{(\zeta, q/\zeta, \zeta^2, q/\zeta^2, q, q; q)_\infty}{(z/\zeta, q\zeta/z, z, q/z, z\zeta, q/(z\zeta); q)_\infty} \right) \\
&\quad + \lim_{\zeta \to q/z} \left(\frac{1}{1 - q/(z\zeta)} - \frac{(\zeta, q/\zeta, \zeta^2, q/\zeta^2, q, q; q)_\infty}{(q/(z\zeta), z\zeta, q/z, z, q\zeta/z, z/\zeta; q)_\infty} \right) \\
&= \lim_{\zeta \to z} \left(\frac{1}{1 - z/\zeta} - \frac{(\zeta, q/\zeta, \zeta^2, q/\zeta^2, q, q; q)_\infty}{(z/\zeta, q\zeta/z, z, q/z, z\zeta, q/(z\zeta); q)_\infty} \right. \\
&\quad \left. + \frac{1}{1 - \zeta/z} + \frac{(\zeta, q/\zeta, \zeta^2, q/\zeta^2, q, q; q)_\infty}{(z/\zeta, q\zeta/z, z, q/z, z\zeta, q/(z\zeta); q)_\infty} \right) \\
&= \lim_{\zeta \to z} \left(\frac{1}{1 - z/\zeta} + \frac{1}{1 - \zeta/z} \right) \\
&= 1,
\end{aligned}$$

where in the antepenultimate line we replaced ζ by q/ζ in the second limit and algebraically simplified the second infinite product into the first product with opposite sign. This then completes the proof of (2.3.14). □

Our next objective is to establish a second component of Lemma 8 of Atkin and Swinnerton-Dyer [28, p. 96].

Lemma 2.3.4. *We have*

$$g(z,q) + g(z^{-1},q) = -2. \tag{2.3.15}$$

Proof. Replacing ζ by $1/\zeta$ in the second equality below, we find that

$$\begin{aligned}
&g(z,q) + g(z^{-1},q) \\
&= \lim_{\zeta \to z} \left(\frac{1}{1 - z/\zeta} + \frac{(\zeta, q/\zeta, \zeta^2, q/\zeta^2, q, q; q)_\infty}{(z/\zeta, q\zeta/z, z, q/z, z\zeta, q/(z\zeta); q)_\infty} \right) \\
&\quad + \lim_{\zeta \to 1/z} \left(\frac{1}{1 - 1/(z\zeta)} - \frac{(\zeta, q/\zeta, \zeta^2, q/\zeta^2, q, q; q)_\infty}{(1/(z\zeta), q\zeta z, 1/z, qz, \zeta/z, qz/\zeta; q)_\infty} \right) \\
&= \lim_{\zeta \to z} \left(\frac{1}{1 - z/\zeta} - \frac{(\zeta, q/\zeta, \zeta^2, q/\zeta^2, q, q; q)_\infty}{(z/\zeta, q\zeta/z, z, q/z, z\zeta, q/(z\zeta); q)_\infty} \right.
\end{aligned}$$