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The Beltrami Equation

A Geometric Approach

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Dedicated to Professor Bogdan Bojarski who is an important contributor to the Beltrami equation theory

Preface

The book is a summation of many years' work on the study of general Beltrami equations with singularities. This is not only a summary of our own long-term collaboration but also with that of many other authors in the field. We show that our geometric approach based on the modulus and capacity developed by us makes it possible to derive the main known existence theorems, including sophisticated and more general existence theorems that have been recently established.

The Beltrami equation plays a significant role in geometry, analysis, and physics, and, in particular, in the theory of quasiconformal mappings and their generalizations, Kleinian groups, and Teichmüller spaces. There has been renewed interest and activity in these areas and, in particular, in the study of degenerate and alternating Beltrami equations since the early 1990s.

In this monograph, we restrict ourselves to the study of very basic properties of solutions in the degenerate and in the alternating cases like existence, uniqueness, distortion, boundary behavior, and mapping problems that can be derived by extremal length methods. The monograph can serve as a textbook for a one- or two-semester graduate course.

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Chapter 1
Introduction

1.1 The Beltrami Equation

Let $\mathbb{C}$ be the complex plane. In the complex notation $w = u + \mathrm{i}v$ and $z = x + \mathrm{i}y$, the *Beltrami equation* in a domain $D \subset \mathbb{C}$ has the form

$$w_{\bar{z}} = \mu(z) w_z, \tag{B}$$

where $\mu : D \to \mathbb{C}$ is a measurable function and

$$w_{\bar{z}} = \bar{\partial} w = \frac{1}{2}(w_x + \mathrm{i}w_y) \quad \text{and} \quad w_z = \partial w = \frac{1}{2}(w_x - \mathrm{i}w_y)$$

are formal derivatives of w in $\bar{z}$ and z, while w_x and w_y are partial derivatives of w in the variables x and y, respectively. For the geometric interpretation of μ, see Appendix A.4.5.

In the real variables x, y, u, and v, (B) can be written in the form of the system

$$\begin{cases} v_y &= \alpha u_x + \beta u_y \\ -v_x &= \beta u_x + \gamma u_y, \end{cases} \tag{B'}$$

where α, β, and γ are given measurable functions in x and y; see, e.g., [263]. For $\mu \equiv 0$, (B′) reduces to the Cauchy–Riemann system, i.e., (B′) with $\beta \equiv 0$ and $\alpha = \gamma \equiv 1$.

This book is devoted mainly to the Beltrami (B). In addition to the theory of the Beltrami (B), there is a theory of the *Beltrami equation of the second kind*

$$w_{\bar{z}} = \nu(z) \cdot \overline{w_z}, \tag{S}$$

with applications to many problems of mathematical physics; see, for instance, [136]. The Beltrami equation of the second type also plays a significant role in the theory of harmonic mappings in the plane; see, e.g., [59, 201]. Hence, we give also

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some results on the Beltrami equation with two characteristics:

$$w_{\bar{z}} \;=\; \mu(z)\cdot w_z \;+\; \nu(z)\cdot \overline{w_z}\,. \tag{T}$$

The existence problem for degenerate Beltrami (B) when

$$K_\mu(z) \;:\; = \;\frac{1+|\mu(z)|}{1-|\mu(z)|} \;\notin\; L^\infty$$

is currently an active area of research; see, e.g., [26, 48–50, 56, 57, 65, 66, 70, 76, 77, 99, 117, 140, 147, 159, 160, 174, 212–214, 238, 254, 271]. The study of such homeomorphisms started from the theory of the so-called mean quasiconformal mappings; see, e.g., [5, 41, 94, 95, 100, 133, 134, 138, 139, 141, 185, 187, 204, 206, 243, 244, 257, 273, 274], and related to the modern theory of mappings with finite distortion; see, e.g., [25, 42, 104, 105, 112–114, 116, 118, 125–130, 132, 155, 161–165, 181–183, 192–195, 207–209, 225].

1.2 Historical Remarks

The system (B′) first appeared in [86] in connection with finding isothermal coordinates on a surface. Local coordinates u and v on a given surface are called *isothermal* if the curves $u=$ const are orthogonal to the curves $v=$ const or, equivalently, if the length element ds is given by

$$\mathrm{d}s^2 = \Lambda(u,v)\;(\mathrm{d}u^2+\mathrm{d}v^2).$$

The transition from given local coordinates x and y to isothermal coordinates u and v is an injective mapping $(x,y)\to(u,v)$ satisfying

$$a(x,y)\,\mathrm{d}x^2+2b(x,y)\,\mathrm{d}x\mathrm{d}y+c(x,y)\,\mathrm{d}y^2=\Lambda(\mathrm{d}u^2+\mathrm{d}v^2),\quad \Lambda>0,$$

where u and v are solutions of the Beltrami system (B′) with

$$\alpha=\frac{b}{\sqrt{\Delta}},\;\beta=\frac{a}{\sqrt{\Delta}},\gamma=\frac{c}{\sqrt{\Delta}}\qquad\text{and}\qquad \Delta=ac-b^2\ge\Delta_0>0.$$

Gauss in [86] proved the existence and uniqueness of a solution in the case of real analytic α,β, and γ, or, equivalently, when μ is real-analytic. The equation then appeared in the Beltrami studies on surface theory; see [32]. There is a long list of names associated with proofs of existence and uniqueness theorems for more general classes of μ's. Among them we mention A. Korn [124] and L. Lichtenstein [154] (Hölder continuous μ's, 1916) and M.A. Lavrent'ev [144] (continuous μ's, 1935). C.B. Morrey [176] was the first to prove the existence and uniqueness of a homeomorphic solution for a measurable μ (1939). Morrey's proof was based on PDE methods. In the mid-1950s, L. Ahlfors [6], B. Bojarski [43, 44], and I.N. Vekua [263] proved the existence in the measurable case by singular integral methods. L. Ahlfors and L. Bers [7] established the analytic dependence of solutions on parameters; cf. the paper [44] and its English translation in [47], and the corresponding discussion in the monograph [51].

1.3 Applications of Beltrami Equations

The Beltrami equation was first used in various areas such as differential geometry on surfaces (see Sect. 1.2), hydrodynamics, and elasticity. Most applications of the Beltrami equation are based on the close relation to quasiconformal (qc) mappings. Plane qc mappings appeared already implicitly in the late 1920s in papers by Grötzsch [97]. The relation between qc mappings and the Beltrami equation was noticed by Ahlfors [4] and Lavrent'ev [144] in the 1930s. The significant connection between the theory of the Beltrami equation and the theory of plane qc mappings has stimulated intensive study and enriched both theories. Most notable is the contribution of qc theory to the modern development of Teichmüller spaces and Kleinian groups.

The Beltrami equation turned out to be useful in the study of Riemann surfaces, Teichmüller spaces, Kleinian groups, meromorphic functions, low dimensional topology, holomorphic motion, complex dynamics, Clifford analysis, control theory, and robotics. The following list is only a partial list of references: [2,4–6,8–11, 22,23,26,27,33–38,45,60,72,78,80–82,90,117,135–137,143,146,151,157,232, 242, 245, 246, 263]. Part of the list consists of books and expository papers where further references can be found. For the classical theory of the Beltrami equation and plane qc mappings, we refer to [9,30,44,152].

1.4 Classification of Beltrami Equations

We say that μ is *bounded* in D if $||\mu||_\infty < 1$ and that μ is *locally bounded* in D if $\mu|A$ is bounded whenever A is a relatively compact subdomain of D. The study of the Beltrami equation is divided into three cases according to the nature of $\mu(z)$ in D:

(1) *The classical case*: $||\mu||_\infty < 1$.
(2) *The degenerate case*: $|\mu| < 1$ almost everywhere (a.e.) and $||\mu||_\infty = 1$.
(3) *The alternating case*: $|\mu| < 1$ a.e. in a part of D and $1/|\mu| < 1$ a.e. in the remaining part of D.

1.5 ACL Solutions

By writing $f : D \to \mathbb{C}$, we assume that D is a domain in $\mathbb{C}$, which is an open and connected set and that f is continuous. A mapping $f : D \to \mathbb{C}$ is *absolutely continuous on lines (ACL)*, $f \in$ ACL, if for every rectangle $R, \overline{R} \subset D$, whose sides are parallel to the coordinate axes, f is absolutely continuous on almost every horizontal and almost every vertical line; see, e.g., [9] or [152]. A function $f: D \to \mathbb{C}$ is a *solution of (B)*, if f is ACL in D, and its ordinary partial derivatives, which exist a.e. in D, satisfy (B) a.e. in D.

Some authors (cf. [99,116]) include in the definition of a solution the assumption that f belongs to the Sobolev class $W^{1,1}_{\mathrm{loc}}$ and that (B) holds in the sense of distributions. If $f \in W^{1,1}_{\mathrm{loc}}$ and f is continuous, then $f \in$ ACL, and the generalized (distributional) partial derivatives coincide with the ordinary partial derivatives. In general, an ACL function need not belong to $W^{1,1}_{\mathrm{loc}}$. For some μ's, however, every ACL solution is just a $W^{1,1}_{\mathrm{loc}}$ solution.

A solution $f: D \to \mathbb{C}$ of (B) which is a homeomorphism of D into $\mathbb{C}$ is called a *μ-homeomorphism* or *μ-conformal* mapping. In the above cases (1) and (2), a solution $f: D \to \mathbb{C}$ of (B) will be called *elementary* if f is open and discrete, meaning that f maps every open set onto an open set and that the preimage of every point in D consists of isolated points. Elementary solutions of (B) are also called by us *μ-regular* mappings.

If $f: D \to \mathbb{C}$ is open and has partial derivatives a.e. in D, then by a result of Gehring and Lehto (see [9, 91, 152]), see also with the earlier Menchoff result for homeomorphisms in [172,255,256], f is differentiable a.e. in D. It thus follows that every elementary solution is differentiable a.e.

Let $f: D \to \mathbb{C}$ be an elementary solution. The *complex dilatation* of f is defined by

$$\mu_f(z) = \mu(z) = \overline{\partial} f(z) / \partial f(z), \tag{1.5.1}$$

if $\partial f(z) \neq 0$ and by $\mu(z) = 0$ if $\partial f(z) = 0$. For such a mapping, the *dilatation* is

$$K_f(z) := K_\mu(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|}. \tag{1.5.2}$$

Note that $K_f < \infty$ a.e. if and only if $|\mu(z)| < 1$ a.e., and that $K_f \in L^\infty$ if and only if $||\mu||_\infty < 1$.

1.6 Ellipticity of the Beltrami Equation

In the classical case (1) in Sect. 1.4, the system (B) is *uniformly* or, in a different terminology, *strongly elliptic*, i.e.,

$$\varDelta = \alpha\gamma - \beta^2 \geq \varDelta_0 > 0, \tag{1.6.1}$$

and in the relaxed classical case (2) in Sect. 1.5, (B) is *elliptic*, i.e.,

$$\varDelta = \alpha\gamma - \beta^2 \geq 0. \tag{1.6.2}$$

Chapter 2
Preliminaries

2.1 BMO Functions in $\mathbb{C}$

The class BMO was introduced by John and Nirenberg in the paper [122] and soon became an important concept in harmonic analysis, partial differential equations, and related areas; see, e.g., [21, 24, 84, 103, 200] and [239].

A real-valued function u in a domain D in $\mathbb{C}$ is said to be of *bounded mean oscillation* in D, $u \in \mathrm{BMO}(D)$, if $u \in L^1_{\mathrm{loc}}(D)$, and

$$\|u\|_* := \sup_B \frac{1}{|B|} \int_B |u(z) - u_B| \, \mathrm{d}m(z) < \infty, \tag{2.1.1}$$

where the supremum is taken over all discs B in D, $\mathrm{d}m(z)$ corresponds to the Lebesgue measure in $\mathbb{C}$, $|B|$ is the Lebesgue measure of B, and

$$u_B = \frac{1}{|B|} \int_B u(z) \, \mathrm{d}m(z).$$

We write $u \in \mathrm{BMO}_{\mathrm{loc}}(D)$ if $u \in \mathrm{BMO}(U)$ for every relatively compact subdomain U of D (we also write BMO or $\mathrm{BMO}_{\mathrm{loc}}$ if it is clear from the context what D is).

A function φ in BMO is said to have *vanishing mean oscillation* (abbreviated as $\varphi \in VMO$), if the supremum in (2.1.1) taken over all disks B in D with $|B| < \varepsilon$ converges to 0 as $\varepsilon \to 0$. VMO was introduced by Sarason in [227]. A large number of papers are devoted to the existence, uniqueness and properties of solutions for various kinds of differential equations and, in particular, of elliptic type with coefficients of the class VMO; see, e.g., [61, 119, 166, 184, 190].

If $u \in \mathrm{BMO}$ and c is a constant, then $u + c \in \mathrm{BMO}$ and $\|u\|_* = \|u + c\|_*$. Obviously $L^\infty \subset \mathrm{BMO}$.

John and Nirenberg [122] established the following fundamental fact (see also [103]):

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Lemma 2.1. *If u is a nonconstant function in* BMO(D)*, then*

$$|\{z \in B : |u(z) - u_B| > t\}| \le a\mathrm{e}^{-\frac{b}{\|u\|_*}\cdot t} \cdot |B| \tag{2.1.2}$$

for every disc B in D and all $t > 0$, where a and b are absolute positive constants which do not depend on B and u. Conversely, if $u \in L^1_{\text{loc}}$ and if for every disc B in D and for all $t > 0$

$$|\{z \in B : |u(z) - u_B| > t\}| < a\mathrm{e}^{-bt}|B| \tag{2.1.3}$$

for some positive constants a and b, then $u \in$ BMO(D).

We will need the following lemma which follows from Lemma 2.1:

Lemma 2.2. *If u is a nonconstant function in* BMO(D)*, then*

$$|\{z \in B : |u(z)| > \tau\}| \le A\mathrm{e}^{-\beta\tau} \cdot |B| \tag{2.1.4}$$

for every disc B in D and all $\tau > |u_B|$, where

$$\beta = b/\|u\|_* \quad \textit{and} \quad A = a\mathrm{e}^{b|u_B|/\|u\|_*}, \tag{2.1.5}$$

and the constants a and b are as in Lemma 2.1.

Proof. For $t > 0$, let $\tau = t + |u_B|$, $D_1 = \{z \in B : |u(z)| > \tau\}$, and $D_2 = \{z \in B : |u(z) - u_B| > t\}$. Then, by the triangle inequality, $D_1 \subset D_2$, and hence, by (2.1.2),

$$|D_1| \le |D_2| \le a\mathrm{e}^{b|u_B|/\|u\|_*} \cdot \mathrm{e}^{-\tau b/\|u\|_*} \cdot |B|,$$

which implies (2.1.4) with A and β as in (2.1.5). □

Proposition 2.1. BMO $\subset L^p_{\text{loc}}$ *for all $p \in [1,\infty)$.*

Proof. Let $u \in$ BMO. Then by (2.1.4),

$$\int_B |u(z)|^p \mathrm{d}m(z) \le |B|\{|u_B|^p + A\int_{|u_B|}^{\infty} t^p \mathrm{e}^{-\beta t}\mathrm{d}t\} < \infty. \qquad \square$$

Remark 2.1. Given a domain D, $D \subset \mathbb{C}$, there is a nonnegative real-valued function u in D such that $u(z) \le Q(z)$ a.e. for some $Q(z)$ in BMO(D) and $u \notin$ BMO(D). For $D = \mathbb{C}$, one can take, for instance, $Q(x,y) = 1 + |\log|y||$ and $u(x,y) = Q(x,y)$ if $y > 0$ and $u(x,y) = 1$ if $y \le 0$.

2.2 BMO Functions in $\overline{\mathbb{C}}$

Later on, we will need also several facts about BMO functions on $\overline{\mathbb{C}}$ and their relations to BMO functions on $\mathbb{C}$.

We identify $\overline{\mathbb{C}}$ with the unit sphere

$$S^2 = \{x \in R^3 : |x| = 1\}$$

and the functions on $\overline{\mathbb{C}}$ with the functions on S^2. This is done with the aid of the stereographic projection P of S^2 onto $\overline{\mathbb{C}}$ which is given for $(x_1, x_2, x_3) \in S^2 \setminus (0,0,1)$ by

$$z = P(x_1, x_2, x_3) = \frac{x_1 + \mathrm{i}x_2}{1 - x_3}.$$

A real-valued measurable function u in a domain $D \subset \overline{\mathbb{C}}$ is said to be in $\mathrm{BMO}(D)$, if u is locally integrable with respect to the spherical area and

$$\|u\|_{*\sigma} = \sup_B \frac{1}{\sigma(B)} \int_B |u - u_B| \,\mathrm{d}\sigma < \infty, \tag{2.2.1}$$

where the supremum is taken over all spherical discs B in D. Here, $\sigma(B)$ denotes the spherical area of B, $\mathrm{d}\sigma = 4\mathrm{d}x_1\mathrm{d}x_2/(1 + x_1^2 + x_2^2)^2$, and

$$u_B = \frac{1}{\sigma(B)} \int_B u \,\mathrm{d}\sigma. \tag{2.2.2}$$

The following two lemmas enable one to decide whether a function u in a domain $D \subset \overline{\mathbb{C}}$ belongs to $\mathrm{BMO}(D)$ (in the spherical sense) by considering the restriction u_0 of u to $D_0 = D \setminus \{\infty\}$ (see [200], p. 7):

Lemma 2.3. $u \in \mathrm{BMO}(D)$ *if* $u_0 \in \mathrm{BMO}(D_0)$. *Furthermore,*

$$c^{-1}\|u_0\|_* \le \|u\|_{*\sigma} \le c\|u_0\|_*, \tag{2.2.3}$$

where c is an absolute constant.

The following lemma is a consequence of Lemmas 2.2 and 2.3:

Lemma 2.4. *If either* $u \in \mathrm{BMO}(D)$ *or* $u_0 \in \mathrm{BMO}(D_0)$, *then for* $\tau > \gamma$,

$$\sigma\{z \in B : |u(z)| > \tau\} \le \alpha \mathrm{e}^{-\beta\tau} \tag{2.2.4}$$

for every spherical disc B in D where the constants α, β, *and* γ *depend on B as well as on the function u.*

Proof. If $\overline{B} \in \mathbb{C}$, then by Lemma 2.3, we have (2.1.4), and since $\sigma(E) \le 4|E|$ for every measurable set $E \subset \mathbb{C}$, (2.2.4) follows. If $\infty \in \overline{B}$, then for suitable rotation R of S^2, ∞ is exterior to $\overline{B'}$, $B' = R(B)$, and the assertion follows by Lemma 2.3 and the validity of (2.2.4) with B' and $\hat{u} = u \circ R^{-1}$ instead of B and u. Now, in view of the invariance of the spherical area with respect to rotations, by Lemmas 2.2 and 2.3, we have again (2.2.4). □

2.2.1 Removability of Isolated Singularities of BMO Functions

The following lemma holds for the BMO functions and cannot be extended to the $\mathrm{BMO}_{\mathrm{loc}}$ functions (see [200], p. 5):

Lemma 2.5. *Let E be a discrete set in a domain D, $D \subset \mathbb{C}$, and let u be a function in $\mathrm{BMO}(D \setminus E)$. Then any extension $\hat{u}$ of u on D is in $\mathrm{BMO}(D)$ and $\|u\|_* = \|\hat{u}\|_*$.*

2.2.2 BMO Functions, qc Mappings, qc Arc, and Symmetric Extensions

We say that a Jordan curve E in $\overline{\mathbb{C}}$ is a *K-quasicircle* if $E = f(\partial\Delta)$ for some K-quasiconformal map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$; see Sect. 3.1 below. A curve E is a *qc* curve if it is a subarc of a quasicircle. The following lemma is a special case of a theorem by Reimann on the characterization of quasiconformal maps in R^n, $n \geq 2$, in terms of the induced isomorphism on BMO (see [199], p. 266):

Lemma 2.6. *If f is a K-qc map of a domain D in $\mathbb{C}$ onto a domain D' and $u \in \mathrm{BMO}(D')$, then $v = u \circ f$ belongs to $\mathrm{BMO}(D)$, and*

$$\|v\|_* \leq c\|u\|_*$$

where c is a constant which depends only on K.

The next lemma can be found in [121].

Lemma 2.7. *Let D be a Jordan curve such that ∂D is a K-quasicircle, and let u be a $\mathrm{BMO}(D)$ function. Then u has an extension $\hat{u}$ on $\mathbb{C}$ which belongs to $\mathrm{BMO}(\mathbb{C})$ and*

$$\|\hat{u}\|_* \leq c\|u\|_*,$$

where c depends only on K.

The following lemmas, which concern symmetric extensions of BMO functions, will be needed in studying the reflection principle and boundary behavior of the so-called BMO-quasiconformal and BMO-quasiregular mappings in Chap. 5. It should be noted that these lemmas cannot be extended to $\mathrm{BMO}_{\mathrm{loc}}$ functions (see [200], p. 8).

Lemma 2.8. *If $u \in \mathrm{BMO}(\mathbb{D})$ and $\hat{u}$ is an extension of u on $\mathbb{C}$, which satisfies the symmetry condition*

$$\hat{u}(t) = \begin{cases} u(z) & \text{if } z \in \mathbb{D}, \\ u(z/|z|^2) & \text{if } z \in \mathbb{C}\setminus\mathbb{D}, \end{cases}$$

then $\hat{u} \in \mathrm{BMO}(\mathbb{C})$ and $\|u\|_* = \|\hat{u}\|_*$.

Similarly, one can prove the following lemma:

Lemma 2.9. *Let D be a domain in $\mathbb{C}$, E a free boundary arc in ∂D which is either a line segment or a circular arc, D^* a domain which is symmetric to D with respect to the corresponding line or circle such that $D \cap D^* = \emptyset$ and $\Omega = D \cup D^* \cup E$. If $u \in \mathrm{BMO}(D)$ and $\hat{u}$ is an extension of u on Ω which satisfies the symmetry condition*

$$\hat{u}(z) = \begin{cases} u(z) & \text{if } z \in D, \\ u(z^*) & \text{if } z \in D^*, \end{cases}$$

then $\hat{u} \in \mathrm{BMO}(\Omega)$ and $\|\hat{u}\|_* = \|u\|_*$.

2.3 FMO Functions

Let D be a domain in the complex plane $\mathbb{C}$. Following [112] and [113], we say that a function $\varphi : D \to \mathbb{R}$ has *finite mean oscillation* at a point $z_0 \in D$ if

$$d_\varphi(z_0) = \overline{\lim_{\varepsilon \to 0}} \; ⨍_{B(z_0,\varepsilon)} |\varphi(z) - \overline{\varphi}_\varepsilon(z_0)| \, \mathrm{d}m(z) \; < \; \infty, \tag{2.3.1}$$

where

$$\overline{\varphi}_\varepsilon(z_0) = ⨍_{B(z_0,\varepsilon)} \varphi(z) \, \mathrm{d}m(z) \tag{2.3.2}$$

is the mean value of the function $\varphi(z)$ over the disk $B(z_0, \varepsilon)$. Condition (2.3.1) includes the assumption that φ is integrable in some neighborhood of the point z_0. We call $d_\varphi(z_0)$ the *dispersion* of the function φ at the point z_0. We say also that a function $\varphi : D \to \mathbb{R}$ is of *finite mean oscillation in D*, abbreviated as $\varphi \in$ FMO(D) or simply $\varphi \in FMO$, if φ has a finite dispersion at every point $z_* \in D$.

Remark 2.2. Note that if a function $\varphi : D \to \mathbb{R}$ is integrable over $B(z_0, \varepsilon_0) \subset D$, then

$$⨍_{B(z_0,\varepsilon)} |\varphi(z) - \overline{\varphi}_\varepsilon(z_0)| \, \mathrm{d}m(z) \; \leq \; 2 \cdot \overline{\varphi}_\varepsilon(z_0), \tag{2.3.3}$$

and the left-hand side in (2.3.3) is continuous in the parameter $\varepsilon \in (0, \varepsilon_0]$ by the absolute continuity of the indefinite integral. Thus, for every $\delta_0 \in (0, \varepsilon_0)$,

$$\sup_{\varepsilon \in [\delta_0, \varepsilon_0]} \; ⨍_{B(z_0,\varepsilon)} |\varphi(z) - \overline{\varphi}_\varepsilon(z_0)| \, \mathrm{d}m(z) \; < \infty. \tag{2.3.4}$$

If (2.3.1) holds, then

$$\sup_{\varepsilon \in (0, \varepsilon_0]} \; ⨍_{B(z_0,\varepsilon)} |\varphi(z) - \overline{\varphi}_\varepsilon(z_0)| \, \mathrm{d}m(z) \; < \infty. \tag{2.3.5}$$

The value of the left-hand side of (2.3.5) is called the *maximal dispersion* of the function φ in the disk $B(z_0,\varepsilon_0)$.

Proposition 2.2. *If, for some collection of numbers* $\varphi_\varepsilon \in \mathbb{R}$, $\varepsilon \in (0,\varepsilon_0]$,

$$\overline{\lim_{\varepsilon\to 0}} \;\; \fint_{B(z_0,\varepsilon)} |\varphi(z) - \varphi_\varepsilon| \, \mathrm{d}m(z) \; < \infty, \tag{2.3.6}$$

then φ *is of finite mean oscillation at* z_0.

Proof. Indeed, by the triangle inequality,

$$\fint_{B(z_0,\varepsilon)} |\varphi(z) - \overline{\varphi}_\varepsilon(z_0)| \, \mathrm{d}m(z) \; \le \; \fint_{B(z_0,\varepsilon)} |\varphi(z) - \varphi_\varepsilon| \, \mathrm{d}m(z) \; + \; |\varphi_\varepsilon - \overline{\varphi}_\varepsilon(z_0)|$$

$$\le \; 2 \cdot \fint_{B(z_0,\varepsilon)} |\varphi(z) - \varphi_\varepsilon| \, \mathrm{d}m(z)\,. \qquad \square$$

Choosing in Proposition 2.2, in particular, $\varphi_\varepsilon \equiv 0$, $\varepsilon \in (0,\varepsilon_0]$, we obtain the following:

Corollary 2.1. *If, for a point* $z_0 \in D$,

$$\overline{\lim_{\varepsilon\to 0}} \; \fint_{B(z_0,\varepsilon)} |\varphi(z)| \, \mathrm{d}m(z) \; < \; \infty, \tag{2.3.7}$$

then φ *has finite mean oscillation at* z_0.

Recall that a point $z_0 \in D$ is called a *Lebesgue point* of a function $\varphi : D \to \mathbb{R}$ if φ is integrable in a neighborhood of z_0 and

$$\lim_{\varepsilon\to 0} \;\; \fint_{B(z_0,\varepsilon)} |\varphi(z) - \varphi(z_0)| \, \mathrm{d}m(z) \; = \; 0. \tag{2.3.8}$$

It is known that, for every function $\varphi \in L^1(D)$, almost every point in D is a Lebesgue point. We thus have the following corollary:

Corollary 2.2. *Every function* $\varphi : D \to \mathbb{R}$, *which is locally integrable, has a finite mean oscillation at almost every point in D.*

Remark 2.3. Note that the function $\varphi(z) = \log(1/|z|)$ belongs to BMO in the unit disk Δ (see, e.g., [191], p. 5) and hence also to FMO. However, $\overline{\varphi}_\varepsilon(0) \to \infty$ as $\varepsilon \to 0$, showing that condition (2.3.7) is only sufficient but not necessary for a function φ to be of finite mean oscillation at z_0. Clearly, $\mathrm{BMO}(D) \subset \mathrm{BMO}_{\mathrm{loc}}(D) \subset \mathrm{FMO}(D)$ and $\mathrm{BMO}(D) \neq \mathrm{BMO}_{\mathrm{loc}}(D)$. Also, $\mathrm{BMO}_{\mathrm{loc}}(D) \neq \mathrm{FMO}(D)$ as is clear from the following examples.

2.3.1 Examples of Functions $\varphi \in FMO \setminus BMO_{loc}$

Set $z_n = 2^{-n},\ r_n = 2^{-pn^2},\ p > 1,\ c_n = 2^{2n^2}$, $D_n = \{z \in \mathbb{C} : |z - z_n| < r_n\}$, and

$$\varphi(z) = \sum_{n=1}^{\infty} c_n \chi_{D_n}(z),$$

where χ_E denotes the *characteristic function* of a set E, i.e.,

$$\chi_E(z) = \begin{cases} 1, & z \in E, \\ 0, & \text{otherwise.} \end{cases}$$

It is evident by Corollary 2.1 that $\varphi \in \mathrm{FMO}(\mathbb{C} \setminus \{0\})$.

To prove that $\varphi \in \mathrm{FMO}(0)$, fix N such that $(p-1)N > 1$, and set $\varepsilon = \varepsilon_N = z_N + r_N$. Then $\bigcup_{n \geq N} D_n \subset \mathbb{D}(\varepsilon) := \{z \in \mathbb{C} : |z| < \varepsilon\}$ and

$$\begin{aligned} \int_{\mathbb{D}(\varepsilon)} \varphi &= \sum_{n \geq N} \int_{D_n} \varphi = \pi \sum_{n \geq N} c_n r_n^2 \\ &= \sum_{n \geq N} 2^{2(1-p)n^2} < \sum_{n \geq N} 2^{2(1-p)n} \\ &< C \cdot [2^{(1-p)N}]^2 < 2C\varepsilon^2. \end{aligned}$$

Hence, $\varphi \in \mathrm{FMO}(0)$ and, consequently, $\varphi \in \mathrm{FMO}(\mathbb{C})$.

On the other hand,

$$\int_{\mathbb{D}(\varepsilon)} \varphi^p = \pi \sum_{n > N} c_n^p \cdot r_n^2 = \sum_{n > N} 1 = \infty.$$

Hence, $\varphi \notin L^p(\mathbb{D}(\varepsilon))$, and therefore, $\varphi \notin \mathrm{BMO}_{\mathrm{loc}}$ by Proposition 2.1.

We conclude this section by constructing functions $\varphi : \mathbb{C} \to \mathbb{R}$ of the class $C^\infty(\mathbb{C} \setminus \{0\})$ which belongs to FMO but not to L^p_{loc} for any $p > 1$ and hence not to $\mathrm{BMO}_{\mathrm{loc}}$. In the following example, $p = 1 + \delta$ with an arbitrarily small $\delta > 0$. Set

$$\varphi_0(z) = \begin{cases} \mathrm{e}^{\frac{1}{|z|^2 - 1}}, & \text{if } |z| < 1, \\ 0, & \text{if } |z| \geq 1. \end{cases} \tag{2.3.9}$$

Then φ_0 belongs to C_0^∞ and hence to $\mathrm{BMO}_{\mathrm{loc}}$. Consider the function

$$\varphi_\delta^*(z) = \begin{cases} \varphi_k(z), & \text{if } z \in B_k, \\ 0, & \text{if } z \in \mathbb{C} \setminus \bigcup B_k, \end{cases} \tag{2.3.10}$$

where $B_k = B(z_k, r_k),\ z_k = 2^{-k},\ r_k = 2^{-(1+\delta)k^2}$, $\delta > 0$, and

$$\varphi_k(z) = 2^{2k^2}\varphi_0\left(\frac{z - z_k}{r_k}\right), \quad z \in B_k,\ k = 2, 3, \ldots . \tag{2.3.11}$$

Then φ^*_δ is smooth in $\mathbb{C}\setminus\{0\}$ and thus belongs to $\mathrm{BMO_{loc}}(\mathbb{C}\setminus\{0\})$ and hence to $\mathrm{FMO}(\mathbb{C}\setminus\{0\})$.

Now,

$$\int\limits_{B_k} \varphi_k(z)\ \mathrm{d}m(z) \ =\ 2^{-2\delta k^2}\int\limits_{\mathbb{C}} \varphi_0(z)\ \mathrm{d}m(z). \tag{2.3.12}$$

Hence,

$$\overline{\lim_{\varepsilon\to 0}}\ \fint_{B(\varepsilon)} \varphi^*_\delta(z)\ \mathrm{d}m(z)\ <\ \infty. \tag{2.3.13}$$

Thus, $\varphi \in \mathrm{FMO}$ by Corollary 2.1.

Indeed, by setting

$$K = K(\varepsilon) = \left[\log_2 \frac{1}{\varepsilon}\right] \le \log_2 \frac{1}{\varepsilon}, \tag{2.3.14}$$

where $[A]$ denotes the integral part of the number A, we have

$$J = \fint_{D(\varepsilon)} \varphi^*_\delta(z)\ \mathrm{d}m(z) \le I \cdot \sum_{k=K}^{\infty} 2^{-2\delta k^2}/\pi 2^{-2(K+1)}, \tag{2.3.15}$$

where $I = \int_{\mathbb{C}} \varphi(z)\ \mathrm{d}m(z)$. If $K\delta > 1$, i.e., $K > 1/\delta$, then

$$\sum_{k=K}^{\infty} 2^{-2\delta k^2} \le \sum_{k=K}^{\infty} 2^{-2k} = 2^{-2K}\sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^k = \frac{4}{3}\cdot 2^{-2K}, \tag{2.3.16}$$

i.e., $J \le 16I/3\pi$.

On the other hand,

$$\int\limits_{B_k} \varphi_k^{1+\delta}(z)\ \mathrm{d}m(z) = \int\limits_{\mathbb{C}} \varphi_0^{1+\delta}(z)\ \mathrm{d}m(z) \tag{2.3.17}$$

and hence $\varphi^*_\delta \notin L^{1+\delta}(U)$ for any neighborhood U of 0.

2.4 On Sobolev's Classes

We recall the necessary definitions and basic facts on L^p, $p \in [1,\infty]$, and the Sobolev spaces $W^{l,p}$ $l = 1,2,\ldots$. Given an open set U in $\mathbb{R}^n$ and a positive integer l, $C_0^l(U)$ denotes a collection of all functions $\varphi : U \to \mathbb{R}$ with compact support having all partial continuous derivatives of order at least l in U; $\varphi \in C_0^\infty(U)$ if $\varphi \in C_0^l(U)$ for all $l = 1,2,\ldots$. A vector $\alpha = (\alpha_1,\ldots,\alpha_n)$ with natural coordinates is called a *multi-index*. Every multi-index α is associated with the differential operator $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}$, where $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Now, let u and $v : U \to \mathbb{R}$ be locally integrable functions. The function v is called the *distributional derivative* $D^\alpha u$ of u if

$$\int_U u D^\alpha \varphi \,\mathrm{d}x = (-1)^{|\alpha|} \int_U v\varphi \,\mathrm{d}x \quad \forall\, \varphi \in C_0^\infty(U), \tag{2.4.1}$$

where the notation dx corresponds to the Lebesgue measure in $\mathbb{R}^n$.

The concept of the generalized derivative was introduced by Sobolev in [231]. The *Sobolev class* $W^{l,p}(U)$ consists of all functions $u : U \to \mathbb{R}$ in $L^p(U)$, $p \geq 1$, with generalized derivatives of order l summable of order p. A function $u : U \to \mathbb{R}$ belongs to $W^{l,p}_{\mathrm{loc}}(U)$ if $u \in W^{l,p}(U_*)$ for every open set U_* with compact closure $\overline{U_*} \subset U$. A similar notion is introduced for vector functions $f : U \to \mathbb{R}^m$ in the componentwise sense.

A function $\omega : \mathbb{R}^n \to \mathbb{R}$ with a compact support in the unit ball $\mathbb{B}^n$ is called a *Sobolev averaging kernel* if ω is nonnegative and belongs to $C_0^\infty(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} \omega(x)\,\mathrm{d}x \;=\; 1. \tag{2.4.2}$$

The well-known example of such a function is $\omega(x) = \gamma\varphi(|x|^2 - \frac{1}{4})$, where $\varphi(t) = \mathrm{e}^{1/t}$ for $t < 0$ and $\varphi(t) \equiv 0$ for $t \geq 0$, and the constant γ is chosen so that (2.4.2) holds. Later on, we use only ω depending on $|x|$.

Let U be a nonempty bounded open subset of $\mathbb{R}^n$ and $f \in L^1(U)$. Extending f by zero outside of U, we set

$$f_h = \omega_h * f = \int_{|y|\leq 1} f(x+hy)\,\omega(y)\,\mathrm{d}y = \frac{1}{h^n}\int_U f(z)\,\omega\left(\frac{z-x}{h}\right)\mathrm{d}z, \tag{2.4.3}$$

where $f_h = \omega_h * f$, $\omega_h(y) = \omega(y/h)$, $h > 0$, is called the *Sobolev mean functions* for f. It is known that $f_h \in C_0^\infty(\mathbb{R}^n)$, $\|f_h\|_p \leq \|f\|_p$ for every $f \in L^p(U)$, $p \in [1,\infty]$, and $f_h \to f$ in $L^p(U)$ for every $f \in L^p(U)$, $p \in [1,\infty)$; see, e.g., 1.2.1 in [170]. It is clear that if f has a compact support in U, then f_h also has a compact support in U for small enough h.

A sequence $\varphi_k \in L^1(U)$ is called *weakly fundamental* if

$$\lim_{k_1,k_2\to\infty} \int\limits_U \Phi(x)\left(\varphi_{k_1}(x) - \varphi_{k_2}(x)\right)\, \mathrm{d}x = 0 \quad \forall\, \Phi \in L^\infty(U). \tag{2.4.4}$$

It is well known that the space $L^1(U)$ is *weakly complete*, i.e., every weakly fundamental sequence $\varphi_k \in L^1(U)$ *weakly converges* in $L^1(U)$, i.e., there is a function in $\varphi \in L^1(U)$ such that

$$\lim_{k\to\infty} \int\limits_U \Phi(x)\ \varphi_k(x)\ \mathrm{d}x = \int\limits_U \Phi(x)\ \varphi(x)\ \mathrm{d}x \qquad \forall\, \Phi \in L^\infty(U); \tag{2.4.5}$$

see, e.g., Theorem IV.8.6 in [73]:

Recall also the following statement (see, e.g., Theorem 1.2.5 in [110]).

Proposition 2.3. *Let f and $g \in L^1_{\mathrm{loc}}(U)$. If*

$$\int f\,\varphi\ \mathrm{d}x = \int g\,\varphi\ \mathrm{d}x \quad \forall\ \varphi \in C_0^\infty(U), \tag{2.4.6}$$

then $f = g$ a.e.

The following fact is known for the Sobolev classes $W^{1,p}(U)$, $p > 1$; see, e.g., Lemma III.3.5 in [202], also Theorem 4.6.1 in [79]. Note that this fact for $p = 2$ was known long ago in the plane for the so-called mappings with the bounded Dirichlet integral; see, e.g., Theorem 1 in [247].

Lemma 2.10. *Let U be a bounded open set in $\mathbb{R}^n$ and let $f_k : U \to \mathbb{R}$ be a sequence of functions of the class $W^{1,p}(U)$, $p > 1$. Suppose that the norm sequence $\|f_k\|_{1,p}$ is bounded and $f_k \to f$ as $k \to \infty$ in $L^1(U)$. Then $f \in W^{1,p}(U)$ and $\partial f_k/\partial x_j \to \partial f/\partial x_j$ as $k \to \infty$ weakly in $L^p(U)$.*

Here we apply instead of Lemma 2.10, which is not valid for $p = 1$, the following lemma; see, e.g., [222] or [220].

Lemma 2.11. *Let U be a bounded open set in $\mathbb{R}^n$ and let $f_k : U \to \mathbb{R}$ be a sequence of functions of the class $W^{1,1}(U)$. Suppose that $f_k \to f$ as $k \to \infty$ weakly in $L^1(U)$, $\partial f_k/\partial x_j$, $k = 1,2,\ldots$, $j = 1,2,\ldots,n$ are uniformly bounded in $L^1(U)$ and their indefinite integrals are absolutely equicontinuous. Then $f \in W^{1,1}(U)$ and $\partial f_k/\partial x_j \to \partial f/\partial x_j$ as $k \to \infty$ weakly in $L^1(U)$.*

The proof of Lemma 2.11 is based on Proposition 2.3 and the above mentioned criterion of weak convergence in the space L^1.

Remark 2.4. The weak convergence $f_k \to f$ in $L^1(U)$ implies that

$$\sup_k \|f_k\|_1 \ < \ \infty;$$

see, e.g., IV.8.7 in [73]. The latter together with

$$\sup_k \|\partial f_k/\partial x_j\|_1 < \infty,$$

$j = 1,2,\ldots,n$, implies that $f_k \to f$ by the norm in L^q for every $1 \le q < n/(n-1)$, the limit function f belongs to $BV(U)$, the class of functions of bounded variation, but, generally speaking, not to the class $W^{1,1}(U)$; see, e.g., Remark in 4.6 and Theorem 5.2.1 in [79]. Thus, the additional condition of Lemma 2.11 on absolute equicontinuity of the indefinite integrals of $\partial f_k/\partial x_j$ is essential; cf. also Remark to Theorem I.2.4 in [197].

Proof of Lemma 2.11. It is known that the space L^1 is weakly complete; see Theorem IV.8.6 in [73]. Thus, it suffices to prove that the sequences $\partial f_k/\partial x_j$ are weakly fundamental in $L^1(U)$.

Indeed, by the definition of generalized derivatives, we have that

$$\int_U \varphi(x)\,\frac{\partial f_k}{\partial x_j}\,\mathrm{d}x \;=\; -\int_U f_k(x)\,\frac{\partial \varphi}{\partial x_j}\,\mathrm{d}x \quad \forall\, \varphi \in C_0^\infty(U). \tag{2.4.7}$$

Note that the integrals on the right-hand side in (2.4.7) are bounded linear functionals in $L^1(U)$ and the sequence f_k is weakly fundamental in $L^1(U)$ because $f_k \to f$ weakly in $L^1(U)$. Hence, in particular,

$$\int_U \varphi(x)\,\left(\frac{\partial f_{k_1}}{\partial x_j} - \frac{\partial f_{k_2}}{\partial x_j}\right)\mathrm{d}x \to 0 \quad \forall\, \varphi \in C_0^\infty(U)$$

as k_1 and $k_2 \to \infty$.

Now, let $\Phi \in L^\infty(U)$. Then $\|\Phi_h\|_\infty \le \|\Phi\|_\infty$ and $\Phi_h \to \Phi$ in the norm of $L^1(U)$ for its Sobolev mean functions Φ_h, and hence, $\Phi_h \to \Phi$ in measure as $h \to 0$. Set $\varphi_m = \Phi_{h_m}$, where $\Phi_{h_m} \to \Phi$ a.e. as $m \to \infty$. Considering restrictions of Φ to compacta in U, we may assume that $\varphi_m \in C_0^\infty(U)$. By the Egoroff theorem, $\varphi_m \to \Phi$ uniformly on a set $S \subset U$ such that $|U \setminus S| < \delta$, where $\delta > 0$ can be arbitrary small; see, e.g., III.6.12 in [73]. Given $\varepsilon > 0$, we have that

$$\left|\int_S (\Phi(x) - \varphi_m(x))\left(\frac{\partial f_{k_1}}{\partial x_j} - \frac{\partial f_{k_2}}{\partial x_j}\right)\mathrm{d}x\right|$$
$$\le 2\cdot \max_{x\in S} |\,\Phi(x) - \varphi_m(x)\,| \cdot \sup_{k=1,2,\ldots} \int_U \left|\frac{\partial f_k}{\partial x_j}\right| \mathrm{d}x \le \frac{\varepsilon}{3}$$

for all large enough m. Choosing one such m, we have that

$$\left|\int_U \varphi_m(x)\,\left(\frac{\partial f_{k_1}}{\partial x_j} - \frac{\partial f_{k_2}}{\partial x_j}\right)\,\mathrm{d}x\right| \;\le\; \frac{\varepsilon}{3}$$