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Rainbow Connections of Graphs

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Preface

The concept of rainbow connection of a graph was first introduced by G. Chartrand, G.L. Johns, K.A. McKeon and P. Zhang in 2006. Let G be a nontrivial connected graph on which an edge-coloring $c: E(G) \to \{1, 2, \cdots, n\}$, $n \in \mathbb{N}$, is defined, where adjacent edges may be colored the same. A path is *rainbow* if no two edges of it are colored the same. An edge-colored graph G is *rainbow connected* if every two distinct vertices are connected by a rainbow path. An edge-coloring under which G is rainbow connected is called a *rainbow coloring*. Clearly, if a graph is rainbow connected, it must be connected. Conversely, every connected graph has a trivial edge-coloring that makes it rainbow connected by coloring edges with distinct colors. Thus, we define the *rainbow connection number* of a connected graph G, denoted by $rc(G)$, as the smallest number of colors that are needed in order to make G rainbow connected. A rainbow coloring using $rc(G)$ colors is called a *minimum rainbow coloring*. Obviously, the rainbow connection number can be viewed as a new kind of chromatic index.

The rainbow connection number is not only a natural combinatorial measure, but it also has applications to the secure transfer of classified information between agencies. In addition, the rainbow connection number can also be motivated by its interesting interpretation in the area of networking. Suppose that G represents a network (e.g., a cellular network). We wish to route messages between any two vertices in a pipeline, and require that each link on the route between the vertices (namely, each edge on the path) is assigned a distinct channel (e.g., a distinct frequency). Clearly, we want to minimize the number of distinct channels that we use in our network. This number is precisely $rc(G)$.

There is a vertex version of the rainbow connection, called the rainbow vertex-connection number $rvc(G)$, which was introduced by M. Krivelevich and R. Yuster. There are also the concepts of strong rainbow connection or rainbow diameter, the rainbow connectivity, and the rainbow index. For details, we refer to Sect. 1.4 of this book.

The rainbow connections of graphs are very new concepts. Recently, there has been great interest in these concepts and a lot of results have been published. The goal of this book is to bring together most of the results that deal with rainbow connections of graphs. We begin with an introductory chapter. In Chap. 2, we address the computing complexity of the rainbow connections. In general, it is NP-hard. Many upper bounds have been obtained in the literature, which appear in Chap. 3. In Chaps. 4 and 5, dense and sparse graphs and some graph classes are studied. Chapter 6 concerns graph products, such as the Cartesian product, the direct product, the strong product, and the composition or lexicographic product of graphs. Chapter 7 is on the rainbow connectivity, which actually includes the rainbow k-connectivity, the k-rainbow index, and the (k,ℓ)-rainbow index. In the final chapter, results of the vertex version, the rainbow vertex-connection number, are reported. In each chapter we list conjectures, open problems, or questions at appropriate places. We hope that this can motivate more young graph theorists and graduate students to do further study in this subject. We do not give proofs for all results. Instead, we only select some of them for which we give their proofs because we feel that these proofs employed some typical techniques, and these proof techniques are popular in the study of rainbow connections. New results are still appearing. There must be some or even many of them for which we have not realized their existence, and therefore have not included them in this book.

The readers of the book are expected to have some background in graph theory and some related knowledge in combinatorics, probability, algorithms, and complexity analysis. All relevant notions from graph theory are properly defined in Chap. 1, but also elsewhere where needed.

The anticipated readers of the book are mathematicians and students of mathematics, whose fields of interest are graph theory, combinatorial optimization as well as communication network design. Consequently, the present book will be found suitable for courses in these fields. The exposition of the details of the proofs of some main results will enable students to understand and eventually master a good part of graph theory and combinatorial optimization.

People working on communication networks may also be interested in some aspects of the book. They will find it useful for designing networks that can securely transfer classified information.

The material presented in this book was used in graph theory seminars, held three times at Nankai University, in 2009, 2010, and 2011. We thank all the members of our group for their help in the preparation of this book. Without their help, we would have not finished writing it in such a short period of time. We also thank the Natural Science Foundation of China (NSFC) for financial support to our research project on rainbow connections. Last, but not least, we are very grateful to the editor for algebraic combinatorics and graph theory of this new series of books of Springer Briefs, Professor Ping Zhang, for inviting us to write this book. Without her encouragement, this book may not exist.

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Contents

Chapter 1
Introduction

1.1 Basic Concepts

In this section, we want to collect most of the terminology and notations used in this monograph. For those not given here, they will be defined when needed.

All graphs considered in this book are finite, simple, and undirected. We follow the terminology and notations of [9] for all those not defined here. We use $V(G)$ and $E(G)$ to denote the set of vertices and the set of edges of G, respectively. For any subset X of $V(G)$, let $G[X]$ denote the subgraph induced by X, and $E[X]$ the edge set of $G[X]$; similarly, for any subset F of $E(G)$, let $G[F]$ denote the subgraph induced by F. Let $\mathscr{G}$ be a set of graphs. Then we denote $V(\mathscr{G}) = \bigcup_{G\in\mathscr{G}} V(G)$, and $E(\mathscr{G}) = \bigcup_{G\in\mathscr{G}} E(G)$, which is the union of all the graphs in $\mathscr{G}$.

A *clique* of a graph G is defined as a complete subgraph of G, and a *maximal clique* is a clique that is not contained in any larger clique of G. For a set S, $|S|$ denotes the cardinality of S. An edge in a connected graph is called a *bridge* if its removal disconnects the graph. A graph with no bridges is called a *bridgeless* graph. A path on n vertices is denoted by P_n, whose length is $n-1$ and denoted by $\ell(P_n)$. A vertex is called *pendant* if its degree is 1. We call a path of G with length k a *pendant k-length path* if one of its end vertex has degree 1 and all inner vertices have degree 2 in G. By definition, a pendant k-length path contains a pendant ℓ-length path ($1 \le \ell \le k$). A pendant 1-length path is a *pendant edge*. We denote by C_n a cycle on n vertices. For $n \ge 3$, the *wheel* W_n is constructed by joining a new vertex to every vertex of C_n. Let $K_{s,t}$ be a complete bipartite graph whose sizes of two parts are s,t, respectively. The *line graph* of a graph G is the graph $L(G)$ (or $L^1(G)$) whose vertex set $V(L(G)) = E(G)$ and two vertices e_1, e_2 of $L(G)$ are adjacent if and only if they are adjacent in G. The *iterated line graph* of a graph G, denoted by $L^2(G)$, is the line graph of the graph $L(G)$. More generally, the *k-iterated line graph* $L^k(G)$ is the line graph of $L^{k-1}(G)$ $(k \ge 2)$. An *intersection graph* of a family $\mathscr{F}$ of sets is a graph whose vertices can be mapped to the sets in $\mathscr{F}$ such that there is an edge between two vertices in the graph if and only if the corresponding two sets in $\mathscr{F}$ have a nonempty intersection. An *interval graph* is an intersection

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graph of intervals on the real line. A *unit interval graph* is an intersection graph of unit length intervals on the real line. A *circular arc graph* is an intersection graph of arcs on a circle. An independent triple of vertices x, y, z in a graph G is an *asteroidal triple* (AT) if between every pair of vertices in the triple, there is a path that does not contain any neighbor of the third. A graph without ATs is called an AT-*free graph* [25]. A graph G is a *threshold graph* if there exists a weight function $w : V(G) \to \mathbb{R}$ and a real constant t such that two vertices $u, v \in V(G)$ are adjacent if and only if $w(u) + w(v) \geq t$. A bipartite graph $G(A,B)$ is called a *chain graph* if the vertices of A can be ordered as $A = (a_1, a_2, \cdots, a_k)$ such that $N(a_1) \subseteq N(a_2) \subseteq \cdots \subseteq N(a_k)$ [96].

Let Γ be a group [88], and let a be an element of Γ. We use $\langle a \rangle$ to denote the cyclic subgroup of Γ generated by a. The number of elements of $\langle a \rangle$ is called the *order* of a, denoted by $|a|$. A pair of elements a and b in a group commutes if $ab = ba$. A group is *Abelian* if every pair of its elements commutes. A *Cayley graph* of Γ with respect to S is the graph $C(\Gamma, S)$ with vertex set Γ in which two vertices x and y are adjacent if and only if $xy^{-1} \in S$ (or equivalently, $yx^{-1} \in S$), where $S \subseteq \Gamma \setminus \{1\}$ is closed under taking inverse.

A k-regular graph G of order v is said to be *strongly regular* and denoted by $SRG(v, k, \lambda, \mu)$ if there are integers λ and μ such that every two adjacent vertices have λ common neighbors and every two nonadjacent vertices have μ common neighbors.

Let G be a connected graph. The *distance* between two vertices u and v in G, denoted by $d(u,v)$, is the length of a shortest path between them in G. The *eccentricity* of a vertex v is $ecc(v) := \max_{x \in V(G)} d(v,x)$. The *diameter* of G is $diam(G) := \max_{x \in V(G)} ecc(x)$. The *radius* of G is $rad(G) := \min_{x \in V(G)} ecc(x)$. Distance between a vertex v and a set $S \subseteq V(G)$ is $d(v,S) := \min_{x \in S} d(v,x)$. The k-*step open neighborhood* of a set $S \subseteq V(G)$ is $N^k(S) := \{x \in V(G) | d(x,S) = k\}$, $k \in \{0, 1, 2, \cdots\}$. A set $D \subseteq V(G)$ is called a k-*step dominating set* of G if every vertex in G is at a distance at most k from D. Further, if D induces a connected subgraph of G, it is called a *connected* k-*step dominating set* of G. The cardinality of a minimum connected k-step dominating set in G is called its *connected* k-*step domination number*, denoted by $\gamma_c^k(G)$. We call a two-step dominating set k-*strong* [55] if every vertex that is not dominated by it has at least k neighbors that are dominated by it. In [13], Chandran, Das, Rajendraprasad, and Varma made two new definitions which will be useful in the sequel. A dominating set D in a graph G is called a *two-way dominating set* if every pendant vertex of G is included in D. In addition, if $G[D]$ is connected, we call D a *connected two-way dominating set*. A (connected) two-step dominating set D of vertices in a graph G is called a *(connected) two-way two-step dominating set* if (1) every pendant vertex of G is included in D and (2) every vertex in $N^2(D)$ has at least two neighbors in $N^1(D)$. Note that if $\delta(G) \geq 2$, then every (connected) dominating set in G is a (connected) two-way dominating set.

Let F be a subgraph of a graph G. An *ear* of F in G is a nontrivial path in G whose ends are in F but whose internal vertices are not. A *nested sequence* of graphs is a sequence $(G_0, G_1, \cdots, G_k)$ of graphs such that $G_i \subset G_{i+1}, 0 \leq i < k$. An *ear*