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Spectra of Graphs

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Spectra of Graphs

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Preface

Algebraic graph theory is the branch of mathematics that studies graphs by using algebraic properties of associated matrices. In particular, *spectral graph theory* studies the relation between graph properties and the spectrum of the adjacency matrix or Laplace matrix. And the theory of *association schemes* and *coherent configurations* studies the algebra generated by associated matrices.

Spectral graph theory is a useful subject. The founders of Google computed the Perron-Frobenius eigenvector of the web graph and became billionaires. The second-largest eigenvalue of a graph gives information about expansion and randomness properties. The smallest eigenvalue gives information about independence number and chromatic number. Interlacing gives information about substructures. The fact that eigenvalue multiplicities must be integral provides strong restrictions. And the spectrum provides a useful invariant.

This book gives the standard elementary material on spectra in Chapter 1. Important applications of graph spectra involve the largest, second-largest, or smallest eigenvalue, or interlacing, topics that are discussed in Chapters 3 and 4. Afterwards, special topics such as trees, groups and graphs, Euclidean representations, and strongly regular graphs are discussed. Strongly related to strongly regular graphs are regular two-graphs, and Chapter 10 mainly discusses Seidel's work on sets of equiangular lines. Strongly regular graphs form the first nontrivial case of (symmetric) association schemes, and Chapter 11 gives a very brief introduction to this topic and Delsarte's linear programming bound. Chapter 12 very briefly mentions the main facts on distance-regular graphs, including some major developments that have occurred since the monograph [54] was written (proof of the Bannai-Ito conjecture, construction by Van Dam and Koolen of the twisted Grassmann graphs, determination of the connectivity of distance-regular graphs). Instead of working over $\mathbb{R}$, one can work over $\mathbb{F}_p$ or $\mathbb{Z}$ and obtain more detailed information. Chapter 13 considers p-ranks and Smith normal forms. Finally, Chapters 14 and 15 return to the real spectrum and consider when a graph is determined by its spectrum and when it has only few eigenvalues.

In Spring 2006, both authors gave a series of lectures at IPM, the Institute for Studies in Theoretical Physics and Mathematics, in Tehran. The lecture notes were combined and published as an IPM report. Those notes grew into the present text, of which the on-line version still is called `ipm.pdf`. We aim at researchers, teachers, and graduate students interested in graph spectra. The reader is assumed to be familiar with basic linear algebra and eigenvalues, but we did include a chapter on some more advanced topics in linear algebra, such as the Perron-Frobenius theorem and eigenvalue interlacing. The exercises at the end of the chapters vary from easy but interesting applications of the treated theory to little excursions into related topics.

This book shows the influence of Seidel. For other books on spectral graph theory, see CHUNG [93], CVETKOVIĆ, DOOB & SACHS [115], and CVETKOVIĆ, ROWLINSON & SIMIĆ [120]. For more algebraic graph theory, see BIGGS [30], GODSIL [172], and GODSIL & ROYLE [177]. For association schemes and distance-regular graphs, see BANNAI & ITO [21] and BROUWER, COHEN & NEUMAIER [54].

Amsterdam *Andries Brouwer*
December 2010 *Willem Haemers*

Contents

Chapter 1
Graph Spectrum

This chapter presents some simple results on graph spectra. We assume the reader is familiar with elementary linear algebra and graph theory. Throughout, J will denote the all-1 matrix, and $\mathbf{1}$ is the all-1 vector.

1.1 Matrices associated to a graph

Let Γ be a graph without multiple edges. The *adjacency matrix* of Γ is the 0-1 matrix A indexed by the vertex set $V\Gamma$ of Γ, where $A_{xy} = 1$ when there is an edge from x to y in Γ and $A_{xy} = 0$ otherwise. Occasionally we consider multigraphs (possibly with loops), in which case A_{xy} equals the number of edges from x to y.

Let Γ be an undirected graph without loops. The (vertex-edge) *incidence matrix* of Γ is the 0-1 matrix M, with rows indexed by the vertices and columns indexed by the edges, where $M_{xe} = 1$ when vertex x is an endpoint of edge e.

Let Γ be a directed graph without loops. The *directed incidence matrix* of Γ is the matrix N, with rows indexed by the vertices and columns by the edges, where $N_{xe} = -1, 1, 0$ when x is the head of e, the tail of e, or not on e, respectively.

Let Γ be an undirected graph without loops. The *Laplace matrix* of Γ is the matrix L indexed by the vertex set of Γ, with zero row sums, where $L_{xy} = -A_{xy}$ for $x \neq y$. If D is the diagonal matrix, indexed by the vertex set of Γ such that D_{xx} is the degree (valency) of x, then $L = D - A$. The matrix $Q = D + A$ is called the *signless Laplace matrix* of Γ.

An important property of the Laplace matrix L and the signless Laplace matrix Q is that they are positive semidefinite. Indeed, one has $Q = MM^\top$ and $L = NN^\top$ if M is the incidence matrix of Γ and N the directed incidence matrix of the directed graph obtained by orienting the edges of Γ in an arbitrary way. It follows that for any vector u one has $u^\top Lu = \sum_{xy}(u_x - u_y)^2$ and $u^\top Qu = \sum_{xy}(u_x + u_y)^2$, where the sum is over the edges of Γ.

A.E. Brouwer and W.H. Haemers, *Spectra of Graphs*, Universitext,
DOI 10.1007/978-1-4614-1939-6_1,

1.2 The spectrum of a graph

The (ordinary) *spectrum* of a finite graph Γ is by definition the spectrum of the adjacency matrix A, that is, its set of eigenvalues together with their multiplicities. The *Laplace spectrum* of a finite undirected graph without loops is the spectrum of the Laplace matrix L.

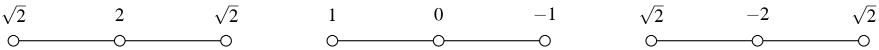
The rows and columns of a matrix of order n are numbered from 1 to n, while A is indexed by the vertices of Γ, so that writing down A requires one to assign some numbering to the vertices. However, the spectrum of the matrix obtained does not depend on the numbering chosen. It is the spectrum of the linear transformation A on the vector space K^X of maps from X into K, where X is the vertex set and K is some field such as $\mathbb{R}$ or $\mathbb{C}$.

The *characteristic polynomial* of Γ is that of A, that is, the polynomial p_A defined by $p_A(\theta) = \det(\theta I - A)$.

Example Let Γ be the path P_3 with three vertices and two edges. Assigning some arbitrary order to the three vertices of Γ, we find that the adjacency matrix A becomes one of

$$\begin{bmatrix} 0&1&1\\ 1&0&0\\ 1&0&0 \end{bmatrix} \text{ or } \begin{bmatrix} 0&1&0\\ 1&0&1\\ 0&1&0 \end{bmatrix} \text{ or } \begin{bmatrix} 0&0&1\\ 0&0&1\\ 1&1&0 \end{bmatrix}.$$

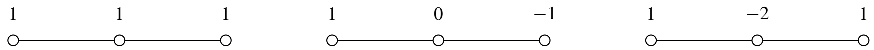
The characteristic polynomial is $p_A(\theta) = \theta^3 - 2\theta$. The spectrum is $\sqrt{2}$, 0, $-\sqrt{2}$. The eigenvectors are:



Here, for an eigenvector u, we write u_x as a label at the vertex x. One has $Au = \theta u$ if and only if $\sum_{y \leftarrow x} u_y = \theta u_x$ for all x. The Laplace matrix L of this graph is one of

$$\begin{bmatrix} 2&-1&-1\\ -1&1&0\\ -1&0&1 \end{bmatrix} \text{ or } \begin{bmatrix} 1&-1&0\\ -1&2&-1\\ 0&-1&1 \end{bmatrix} \text{ or } \begin{bmatrix} 1&0&-1\\ 0&1&-1\\ -1&-1&2 \end{bmatrix}.$$

Its eigenvalues are 0, 1 and 3. The Laplace eigenvectors are:



One has $Lu = \theta u$ if and only if $\sum_{y \sim x} u_y = (d_x - \theta) u_x$ for all x, where d_x is the degree of the vertex x.

Example Let Γ be the directed triangle with adjacency matrix

$$A = \begin{bmatrix} 0&1&0\\ 0&0&1\\ 1&0&0 \end{bmatrix}.$$

Then A has characteristic polynomial $p_A(\theta) = \theta^3 - 1$ and spectrum 1, ω, ω^2, where ω is a primitive cube root of unity.

Example Let Γ be the directed graph with two vertices and a single directed edge. Then $A = \begin{bmatrix} 0 1 \\ 0 0 \end{bmatrix}$ with $p_A(\theta) = \theta^2$, so A has the eigenvalue 0 with geometric multiplicity (that is, the dimension of the corresponding eigenspace) equal to 1 and algebraic multiplicity (that is, its multiplicity as a root of the polynomial p_A) equal to 2.

1.2.1 Characteristic polynomial

Let Γ be a directed graph on n vertices. For any directed subgraph C of Γ that is a union of directed cycles, let $c(C)$ be its number of cycles. Then the characteristic polynomial $p_A(t) = \det(tI - A)$ of Γ can be expanded as $\sum c_i t^{n-i}$, where $c_i = \sum_C (-1)^{c(C)}$, with C running over all regular directed subgraphs with in- and outdegree 1 on i vertices.

(Indeed, this is just a reformulation of the definition of the determinant as $\det M = \sum_\sigma \operatorname{sgn}(\sigma) M_{1\sigma(1)} \cdots M_{n\sigma(n)}$. Note that when the permutation σ with $n-i$ fixed points is written as a product of nonidentity cycles, its sign is $(-1)^e$, where e is the number of even cycles in this product. Since the number of odd nonidentity cycles is congruent to i (mod 2), we have $\operatorname{sgn}(\sigma) = (-1)^{i+c(\sigma)}$.)

For example, the directed triangle has $c_0 = 1$, $c_3 = -1$. Directed edges that do not occur in directed cycles do not influence the (ordinary) spectrum.

The same description of $p_A(t)$ holds for undirected graphs (with each edge viewed as a pair of opposite directed edges).

Since $\frac{d}{dt} \det(tI - A) = \sum_x \det(tI - A_x)$ where A_x is the submatrix of A obtained by deleting row and column x, it follows that $p'_A(t)$ is the sum of the characteristic polynomials of all single-vertex-deleted subgraphs of Γ.

1.3 The spectrum of an undirected graph

Suppose Γ is undirected and simple with n vertices. Since A is real and symmetric, all its eigenvalues are real. Also, for each eigenvalue θ, its algebraic multiplicity coincides with its geometric multiplicity, so that we may omit the adjective and just speak about "multiplicity". Conjugate algebraic integers have the same multiplicity. Since A has zero diagonal, its trace $\operatorname{tr} A$, and hence the sum of the eigenvalues, is zero.

Similarly, L is real and symmetric, so that the Laplace spectrum is real. Moreover, L is positive semidefinite and singular, so we may denote the eigenvalues by

$\mu_1, \dots, \mu_n$, where $0 = \mu_1 \le \mu_2 \le \dots \le \mu_n$. The sum of these eigenvalues is $\operatorname{tr} L$, which is twice the number of edges of Γ.

Finally, also Q has real spectrum and nonnegative eigenvalues (but is not necessarily singular). We have $\operatorname{tr} Q = \operatorname{tr} L$.

1.3.1 Regular graphs

A graph Γ is called *regular* of degree (or valency) k when every vertex has precisely k neighbors. So, Γ is regular of degree k precisely when its adjacency matrix A has row sums k, i.e., when $A\mathbf{1} = k\mathbf{1}$ (or $AJ = kJ$).

If Γ is regular of degree k, then for every eigenvalue θ we have $|\theta| \le k$. (One way to see this is by observing that if $|t| > k$ then the matrix $tI - A$ is strictly diagonally dominant, and hence nonsingular, so that t is not an eigenvalue of A.)

If Γ is regular of degree k, then $L = kI - A$. It follows that if Γ has ordinary eigenvalues $k = \theta_1 \ge \dots \ge \theta_n$ and Laplace eigenvalues $0 = \mu_1 \le \mu_2 \le \dots \le \mu_n$, then $\theta_i = k - \mu_i$ for $i = 1, \dots, n$. The eigenvalues of $Q = kI + A$ are $2k, k + \theta_2, \dots, k + \theta_n$.

1.3.2 Complements

The *complement* $\overline{\Gamma}$ of Γ is the graph with the same vertex set as Γ, where two distinct vertices are adjacent whenever they are nonadjacent in Γ. So, if Γ has adjacency matrix A, then $\overline{\Gamma}$ has adjacency matrix $\overline{A} = J - I - A$ and Laplace matrix $\overline{L} = nI - J - L$.

Because eigenvectors of L are also eigenvectors of J, the eigenvalues of $\overline{L}$ are $0, n - \mu_n, \dots, n - \mu_2$. (In particular, $\mu_n \le n$.)

If Γ is regular we have a similar result for the ordinary eigenvalues: if Γ is k-regular with eigenvalues $\theta_1 \ge \dots \ge \theta_n$, then the eigenvalues of the complement are $n - k - 1$, $-1 - \theta_n$, $\dots$, $-1 - \theta_2$.

1.3.3 Walks

From the spectrum one can read off the number of closed walks of a given length.

Proposition 1.3.1 *Let h be a nonnegative integer. Then $(A^h)_{xy}$ is the number of walks of length h from x to y. In particular, $(A^2)_{xx}$ is the degree of the vertex x, and* $\operatorname{tr} A^2$ *equals twice the number of edges of Γ; similarly,* $\operatorname{tr} A^3$ *is six times the number of triangles in Γ.*

1.3.4 Diameter

We saw that all eigenvalues of a single directed edge are zero. For undirected graphs this does not happen.

Proposition 1.3.2 *Let Γ be an undirected graph. All its eigenvalues are zero if and only if Γ has no edges. The same holds for the Laplace eigenvalues and the signless Laplace eigenvalues.*

More generally, we find a lower bound for the diameter:

Proposition 1.3.3 *Let Γ be a connected graph with diameter d. Then Γ has at least $d+1$ distinct eigenvalues, at least $d+1$ distinct Laplace eigenvalues, and at least $d+1$ distinct signless Laplace eigenvalues.*

Proof Let M be any nonnegative symmetric matrix with rows and columns indexed by $V\Gamma$ and such that for distinct vertices x,y we have $M_{xy} > 0$ if and only if $x \sim y$. Let the distinct eigenvalues of M be $\theta_1,\dots,\theta_t$. Then $(M-\theta_1 I)\cdots(M-\theta_t I) = 0$, so that M^t is a linear combination of $I, M, \dots, M^{t-1}$. But if $d(x,y) = t$ for two vertices x,y of Γ, then $(M^i)_{xy} = 0$ for $0 \le i \le t-1$ and $(M^t)_{xy} > 0$, a contradiction. Hence $t > d$. This applies to $M = A$, to $M = nI - L$, and to $M = Q$, where A is the adjacency matrix, L is the Laplace matrix, and Q is the signless Laplace matrix of Γ. □

Distance-regular graphs, discussed in Chapter 12, have equality here. For an upper bound on the diameter, see §4.7.

1.3.5 Spanning trees

From the Laplace spectrum of a graph one can determine the number of spanning trees (which will be nonzero only if the graph is connected).

Proposition 1.3.4 *Let Γ be an undirected (multi)graph with at least one vertex, and Laplace matrix L with eigenvalues $0 = \mu_1 \le \mu_2 \le \ldots \le \mu_n$. Let ℓ_{xy} be the (x,y)-cofactor of L. Then the number N of spanning trees of Γ equals*

$$N = \ell_{xy} = \det(L + \frac{1}{n^2}J) = \frac{1}{n}\mu_2\cdots\mu_n \quad \textit{for any} \quad x,y \in V\Gamma.$$

(The (i,j)-*cofactor* of a matrix M is by definition $(-1)^{i+j}\det M(i,j)$, where $M(i,j)$ is the matrix obtained from M by deleting row i and column j. Note that ℓ_{xy} does not depend on an ordering of the vertices of Γ.)

Proof Let L^S, for $S \subseteq V\Gamma$, denote the matrix obtained from L by deleting the rows and columns indexed by S, so that $\ell_{xx} = \det L^{\{x\}}$. The equality $N = \ell_{xx}$ follows by induction on n, and for fixed $n > 1$ on the number of edges incident with x. Indeed, if $n = 1$ then $\ell_{xx} = 1$. Otherwise, if x has degree 0, then $\ell_{xx} = 0$ since $L^{\{x\}}$ has zero row sums. Finally, if xy is an edge, then deleting this edge from Γ diminishes ℓ_{xx} by $\det L^{\{x,y\}}$, which by induction is the number of spanning trees of Γ with edge xy contracted, which is the number of spanning trees containing the edge xy. This shows $N = \ell_{xx}$.

Now $\det(tI - L) = t\prod_{i=2}^{n}(t-\mu_i)$ and $(-1)^{n-1}\mu_2\cdots\mu_n$ is the coefficient of t, that is, is $\frac{d}{dt}\det(tI-L)|_{t=0}$. But $\frac{d}{dt}\det(tI-L) = \sum_x \det(tI - L^{\{x\}})$, so $\mu_2\cdots\mu_n = \sum_x \ell_{xx} = nN$.

Since the sum of the columns of L is zero, so that one column is minus the sum of the other columns, we have $\ell_{xx} = \ell_{xy}$ for any x,y. Finally, the eigenvalues of $L + \frac{1}{n^2}J$ are $\frac{1}{n}$ and $\mu_2, \ldots, \mu_n$, so $\det(L + \frac{1}{n^2}J) = \frac{1}{n}\mu_2\cdots\mu_n$. □

For example, the multigraph of valency k on two vertices has Laplace matrix $L = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$ so $\mu_1 = 0$, $\mu_2 = 2k$, and $N = \frac{1}{2}.2k = k$.

If we consider the complete graph K_n, then $\mu_2 = \ldots = \mu_n = n$, and therefore K_n has $N = n^{n-2}$ spanning trees. This formula is due to CAYLEY [85]. Proposition 1.3.4 is implicit in KIRCHHOFF [242] and known as the *matrix-tree theorem*. There is a "1-line proof" of the above result using the *Cauchy-Binet formula*.

Proposition 1.3.5 (Cauchy-Binet) *Let A and B be $m \times n$ matrices. Then*

$$\det AB^\top = \sum_S \det A_S \det B_S,$$

where the sum is over the $\binom{n}{m}$ m-subsets S of the set of columns, and A_S (B_S) is the square submatrix of order m of A (resp. B) with columns indexed by S.

Second proof of Proposition 1.3.4 (sketch) Let N_x be the directed incidence matrix of Γ with row x deleted. Then $l_{xx} = \det N_x N_x^\top$. Apply the Cauchy-Binet formula to get l_{xx} as a sum of squares of determinants of size $n-1$. These determinants vanish unless the set S of columns is the set of edges of a spanning tree, in which case the determinant is ± 1. □

1.3.6 Bipartite graphs

A graph Γ is called *bipartite* when its vertex set can be partitioned into two disjoint parts X_1, X_2 such that all edges of Γ meet both X_1 and X_2. The adjacency matrix of a bipartite graph has the form $A = \begin{bmatrix} 0 & B \\ B^\top & 0 \end{bmatrix}$. It follows that the spectrum of a bipartite graph is symmetric w.r.t. 0: if $\begin{bmatrix} u \\ v \end{bmatrix}$ is an eigenvector with eigenvalue θ, then $\begin{bmatrix} u \\ -v \end{bmatrix}$ is an eigenvector with eigenvalue $-\theta$. (The converse also holds, see Proposition 3.4.1.)

For the ranks one has $\mathrm{rk}A = 2\,\mathrm{rk}B$. If $n_i = |X_i|$ $(i = 1,2)$ and $n_1 \geq n_2$, then $\mathrm{rk}A \leq 2n_2$, so Γ has eigenvalue 0 with multiplicity at least $n_1 - n_2$.

One cannot, in general, recognize bipartiteness from the Laplace or signless Laplace spectrum. For example, $K_{1,3}$ and $K_1 + K_3$ have the same signless Laplace spectrum and only the former is bipartite. And Figure 14.4 gives an example of a bipartite and a nonbipartite graph with the same Laplace spectrum. However, by Proposition 1.3.10 below, a graph is bipartite precisely when its Laplace spectrum and signless Laplace spectrum coincide.

1.3.7 Connectedness

The spectrum of a disconnected graph is easily found from the spectra of its connected components:

Proposition 1.3.6 *Let Γ be a graph with connected components Γ_i $(1 \leq i \leq s)$. Then the spectrum of Γ is the union of the spectra of Γ_i (and multiplicities are added). The same holds for the Laplace spectrum and the signless Laplace spectrum.* □

Proposition 1.3.7 *The multiplicity of 0 as a Laplace eigenvalue of an undirected graph Γ equals the number of connected components of Γ.*

Proof We have to show that a connected graph has Laplace eigenvalue 0 with multiplicity 1. As we saw earlier, $L = NN^\top$, where N is the incidence matrix of an orientation of Γ. Now $Lu = 0$ is equivalent to $N^\top u = 0$ (since $0 = u^\top Lu = ||N^\top u||^2$), that is, for every edge the vector u takes the same value on both endpoints. Since Γ is connected, that means that u is constant. □

Proposition 1.3.8 *Let the undirected graph Γ be regular of valency k. Then k is the largest eigenvalue of Γ, and its multiplicity equals the number of connected components of Γ.*

Proof We have $L = kI - A$. □

One cannot see from the spectrum alone whether a (nonregular) graph is connected: both $K_{1,4}$ and $K_1 + C_4$ have spectrum 2^1, 0^3, $(-2)^1$ (we write multiplicities as exponents). And both $\hat{E}_6$ and $K_1 + C_6$ have spectrum 2^1, 1^2, 0, $(-1)^2$, $(-2)^1$.

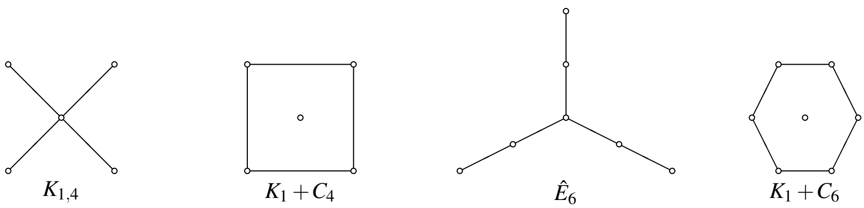


Fig. 1.1 Two pairs of cospectral graphs

Proposition 1.3.9 *The multiplicity of 0 as a signless Laplace eigenvalue of an undirected graph Γ equals the number of bipartite connected components of Γ.*

Proof Let M be the vertex-edge incidence matrix of Γ, so that $Q = MM^\top$. If $MM^\top u = 0$, then $M^\top u = 0$, so $u_x = -u_y$ for all edges xy, and the support of u is the union of a number of bipartite components of Γ. □

Proposition 1.3.10 *A graph Γ is bipartite if and only if the Laplace spectrum and the signless Laplace spectrum of Γ are equal.*

Proof If Γ is bipartite, the Laplace matrix L and the signless Laplace matrix Q are similar by a diagonal matrix D with diagonal entries ± 1 (that is, $Q = DLD^{-1}$). Therefore Q and L have the same spectrum. Conversely, if both spectra are the same, then by Propositions 1.3.7 and 1.3.9 the number of connected components equals the number of bipartite components. Hence Γ is bipartite. □

1.4 Spectrum of some graphs

In this section we discuss some special graphs and their spectra. All graphs in this section are finite, undirected, and simple. Observe that the all-1 matrix J of order n has rank 1, and that the all-1 vector $\mathbf{1}$ is an eigenvector with eigenvalue n, so the spectrum of J is $n^1,\ 0^{n-1}$. (Here and throughout, we write multiplicities as exponents where convenient and no confusion seems likely.)

1.4.1 The complete graph

Let Γ be the complete graph K_n on n vertices. Its adjacency matrix is $A = J - I$, and the spectrum is $(n-1)^1,\ (-1)^{n-1}$. The Laplace matrix is $nI - J$, which has spectrum $0^1,\ n^{n-1}$.

1.4.2 The complete bipartite graph

The spectrum of the complete bipartite graph $K_{m,n}$ is $\pm\sqrt{mn},\ 0^{m+n-2}$. The Laplace spectrum is $0^1,\ m^{n-1},\ n^{m-1},\ (m+n)^1$.

1.4.3 The cycle

Let Γ be the directed n-cycle D_n. Eigenvectors are $(1, \zeta, \zeta^2, \dots, \zeta^{n-1})^\top$, where $\zeta^n = 1$, and the corresponding eigenvalue is ζ. Thus, the spectrum consists precisely of the complex n-th roots of unity $e^{2\pi ij/n}$ $(j = 0, \dots, n-1)$.

Now consider the undirected n-cycle C_n. If B is the adjacency matrix of D_n, then $A = B + B^\top$ is the adjacency matrix of C_n. We find the same eigenvectors as before, with eigenvalues $\zeta + \zeta^{-1}$, so that the spectrum consists of the numbers $2\cos(2\pi j/n)$ $(j = 0,\ldots,n-1)$.

This graph is regular of valency 2, so the Laplace spectrum consists of the numbers $2 - 2\cos(2\pi j/n)$ $(j = 0,\ldots,n-1)$.

1.4.4 The path

Let Γ be the undirected path P_n with n vertices. The ordinary spectrum consists of the numbers $2\cos(\pi j/(n+1))$ $(j = 1,\ldots,n)$. The Laplace spectrum is $2 - 2\cos(\pi j/n)$ $(j = 0,\ldots,n-1)$.

The ordinary spectrum follows by looking at C_{2n+2}. If $u(\zeta) = (1,\zeta,\zeta^2,\ldots,\zeta^{2n+1})^\top$ is an eigenvector of C_{2n+2}, where $\zeta^{2n+2} = 1$, then $u(\zeta)$ and $u(\zeta^{-1})$ have the same eigenvalue, $2\cos(\pi j/(n+1))$, and hence so has $u(\zeta) - u(\zeta^{-1})$. This latter vector has two zero coordinates distance $n+1$ apart and (for $\zeta \neq \pm 1$) induces an eigenvector on the two paths obtained by removing the two points where it is zero.

Eigenvectors of L with eigenvalue $2 - \zeta - \zeta^{-1}$ are $(1 + \zeta^{2n-1},\ldots,\zeta^j + \zeta^{2n-1-j},\ldots,\zeta^{n-1} + \zeta^n)$, where $\zeta^{2n} = 1$. One can check this directly, or view P_n as the result of folding C_{2n}, where the folding has no fixed vertices. An eigenvector of C_{2n} that is constant on the preimages of the folding yields an eigenvector of P_n with the same eigenvalue.

1.4.5 Line graphs

The *line graph* $L(\Gamma)$ of Γ is the graph with the edge set of Γ as vertex set, where two vertices are adjacent if the corresponding edges of Γ have an endpoint in common. If N is the incidence matrix of Γ, then $N^\top N - 2I$ is the adjacency matrix of $L(\Gamma)$. Since $N^\top N$ is positive semidefinite, the eigenvalues of a line graph are not smaller than -2. We have an explicit formula for the eigenvalues of $L(\Gamma)$ in terms of the signless Laplace eigenvalues of Γ.

Proposition 1.4.1 *Suppose Γ has m edges, and let $\rho_1 \geq \ldots \geq \rho_r$ be the positive signless Laplace eigenvalues of Γ. Then the eigenvalues of $L(\Gamma)$ are $\theta_i = \rho_i - 2$ for $i = 1,\ldots,r$, and $\theta_i = -2$ if $r < i \leq m$.*

Proof The signless Laplace matrix Q of Γ and the adjacency matrix B of $L(\Gamma)$ satisfy $Q = NN^\top$ and $B + 2I = N^\top N$. Because $NN^\top$ and $N^\top N$ have the same nonzero eigenvalues (multiplicities included), the result follows. □

Example Since the path P_n has line graph P_{n-1} and is bipartite, the Laplace and the signless Laplace eigenvalues of P_n are $2 + 2\cos\frac{\pi i}{n}$, $i = 1,\ldots,n$.

Corollary 1.4.2 *If Γ is a k-regular graph ($k \geq 2$) with n vertices, $e = kn/2$ edges, and eigenvalues θ_i ($i = 1,\ldots,n$), then $L(\Gamma)$ is $(2k-2)$-regular with eigenvalues $\theta_i + k - 2$ ($i = 1,\ldots,n$) and $e - n$ times -2.* □

The line graph of the complete graph K_n ($n \geq 2$) is known as the *triangular graph* $T(n)$. It has spectrum $2(n-2)^1$, $(n-4)^{n-1}$, $(-2)^{n(n-3)/2}$. The line graph of the regular complete bipartite graph $K_{m,m}$ ($m \geq 2$) is known as the *lattice graph* $L_2(m)$. It has spectrum $2(m-1)^1$, $(m-2)^{2m-2}$, $(-2)^{(m-1)^2}$. These two families of graphs, and their complements, are examples of strongly regular graphs, which will be the subject of Chapter 9. The complement of $T(5)$ is the famous *Petersen graph*. It has spectrum $3^1\ 1^5\ (-2)^4$.

1.4.6 Cartesian products

Given graphs Γ and Δ with vertex sets V and W, respectively, their *Cartesian product* $\Gamma \Box \Delta$ is the graph with vertex set $V \times W$, where $(v,w) \sim (v',w')$ when either $v = v'$ and $w \sim w'$ or $w = w'$ and $v \sim v'$. For the adjacency matrices we have $A_{\Gamma \Box \Delta} = A_\Gamma \otimes I + I \otimes A_\Delta$.

If u and v are eigenvectors for Γ and Δ with ordinary or Laplace eigenvalues θ and η, respectively, then the vector w defined by $w_{(x,y)} = u_x v_y$ is an eigenvector of $\Gamma \Box \Delta$ with ordinary or Laplace eigenvalue $\theta + \eta$.

For example, $L_2(m) = K_m \Box K_m$.

For example, the *hypercube* 2^n, also called Q_n, is the Cartesian product of n factors K_2. The spectrum of K_2 is $1, -1$, and hence the spectrum of 2^n consists of the numbers $n - 2i$ with multiplicity $\binom{n}{i}$ ($i = 0,1,\ldots,n$).

1.4.7 Kronecker products and bipartite double

Given graphs Γ and Δ with vertex sets V and W, respectively, their *Kronecker product* (or *direct product*, or *conjunction*) $\Gamma \otimes \Delta$ is the graph with vertex set $V \times W$, where $(v,w) \sim (v',w')$ when $v \sim v'$ and $w \sim w'$. The adjacency matrix of $\Gamma \otimes \Delta$ is the Kronecker product of the adjacency matrices of Γ and Δ.

If u and v are eigenvectors for Γ and Δ with eigenvalues θ and η, respectively, then the vector $w = u \otimes v$ (with $w_{(x,y)} = u_x v_y$) is an eigenvector of $\Gamma \otimes \Delta$ with eigenvalue $\theta\eta$. Thus, the spectrum of $\Gamma \otimes \Delta$ consists of the products of the eigenvalues of Γ and Δ.

Given a graph Γ, its *bipartite double* is the graph $\Gamma \otimes K_2$ (with for each vertex x of Γ two vertices x' and x'', and for each edge xy of Γ two edges $x'y''$ and $x''y'$). If Γ is bipartite, its double is just the union of two disjoint copies. If Γ is connected and not bipartite, then its double is connected and bipartite. If Γ has spectrum Φ, then $\Gamma \otimes K_2$ has spectrum $\Phi \cup -\Phi$.

The notation $\Gamma \times \Delta$ is used in the literature both for the Cartesian product and for the Kronecker product of two graphs. We avoid it here.

1.4.8 Strong products

Given graphs Γ and Δ with vertex sets V and W, respectively, their *strong product* $\Gamma \boxtimes \Delta$ is the graph with vertex set $V \times W$, where two distinct vertices (v,w) and (v',w') are adjacent whenever v and v' are equal or adjacent in Γ, and w and w' are equal or adjacent in Δ. If A_Γ and A_Δ are the adjacency matrices of Γ and Δ, then $((A_\Gamma + I) \otimes (A_\Delta + I)) - I$ is the adjacency matrix of $\Gamma \boxtimes \Delta$. It follows that the eigenvalues of $\Gamma \boxtimes \Delta$ are the numbers $(\theta + 1)(\eta + 1) - 1$, where θ and η run through the eigenvalues of Γ and Δ, respectively.

Note that the edge set of the strong product of Γ and Δ is the union of the edge sets of the Cartesian product and the Kronecker product of Γ and Δ.

For example, $K_{m+n} = K_m \boxtimes K_n$.

1.4.9 Cayley graphs

Let G be an Abelian group and $S \subseteq G$. The *Cayley graph* on G with difference set S is the (directed) graph Γ with vertex set G and edge set $E = \{(x,y) \mid y - x \in S\}$. Now Γ is regular with in- and outvalency $|S|$. The graph Γ will be undirected when $S = -S$.

It is easy to compute the spectrum of finite Cayley graphs (on an Abelian group). Let χ be a character of G, that is, a map $\chi : G \to \mathbb{C}^*$ such that $\chi(x+y) = \chi(x)\chi(y)$. Then $\sum_{y \sim x} \chi(y) = (\sum_{s \in S} \chi(s))\chi(x)$, so the vector $(\chi(x))_{x \in G}$ is a right eigenvector of the adjacency matrix A of Γ with eigenvalue $\chi(S) := \sum_{s \in S} \chi(s)$. The $n = |G|$ distinct characters give independent eigenvectors, so one obtains the entire spectrum in this way.

For example, the directed pentagon (with in- and outvalency 1) is a Cayley graph for $G = \mathbb{Z}_5$ and $S = \{1\}$. The characters of G are the maps $i \mapsto \zeta^i$ for some fixed fifth root of unity ζ. Hence the directed pentagon has spectrum $\{\zeta \mid \zeta^5 = 1\}$.

The undirected pentagon (with valency 2) is the Cayley graph for $G = \mathbb{Z}_5$ and $S = \{-1, 1\}$. The spectrum of the pentagon becomes $\{\zeta + \zeta^{-1} \mid \zeta^5 = 1\}$, that is, consists of 2 and $\frac{1}{2}(-1 \pm \sqrt{5})$ (both with multiplicity 2).

1.5 Decompositions

Here we present two nontrivial applications of linear algebra to graph decompositions.