

Universitext

UTX

Andries E. Brouwer
Willem H. Haemers

Spectra of Graphs

 Springer

Universitext

Universitext

Series Editors:

Sheldon Axler
San Francisco State University

Vincenzo Capasso
Università degli Studi di Milano

Carles Casacuberta
Universitat de Barcelona

Angus J. MacIntyre
Queen Mary, University of London

Kenneth Ribet
University of California, Berkeley

Claude Sabbah
CNRS, École Polytechnique

Endre Süli
University of Oxford

Wojbor A. Woźczyński
Case Western Reserve University

Universitext is a series of textbooks that presents material from a wide variety of mathematical disciplines at master's level and beyond. The books, often well class-tested by their author, may have an informal, personal even experimental approach to their subject matter. Some of the most successful and established books in the series have evolved through several editions, always following the evolution of teaching curricula, to very polished texts.

Thus as research topics trickle down into graduate-level teaching, first textbooks written for new, cutting-edge courses may make their way into *Universitext*.

For further volumes:
<http://www.springer.com/series/223>

Andries E. Brouwer • Willem H. Haemers

Spectra of Graphs

 Springer

Andries E. Brouwer
Department of Mathematics
Eindhoven University of Technology
Eindhoven
The Netherlands

Willem H. Haemers
Department of Econometrics
and Operations Research
Tilburg University
Tilburg
The Netherlands

ISSN 0172-5939 e-ISSN 2191-6675
ISBN 978-1-4614-1938-9 e-ISBN 978-1-4614-1939-6
DOI 10.1007/978-1-4614-1939-6
Springer New York Dordrecht Heidelberg London

Library of Congress Control Number: 2011944191

Mathematics Subject Classification (2010): 05Exx, 05Bxx, 05Cxx, 05Dxx, 15Axx, 51Exx, 94Bxx, 94Cxx

© Andries E. Brouwer and Willem H. Haemers 2012

All rights reserved. This work may not be translated or copied in whole or in part without the written permission of the publisher (Springer Science+Business Media, LLC, 233 Spring Street, New York, NY 10013, USA), except for brief excerpts in connection with reviews or scholarly analysis. Use in connection with any form of information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed is forbidden.

The use in this publication of trade names, trademarks, service marks, and similar terms, even if they are not identified as such, is not to be taken as an expression of opinion as to whether or not they are subject to proprietary rights.

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

Algebraic graph theory is the branch of mathematics that studies graphs by using algebraic properties of associated matrices. In particular, *spectral graph theory* studies the relation between graph properties and the spectrum of the adjacency matrix or Laplace matrix. And the theory of *association schemes* and *coherent configurations* studies the algebra generated by associated matrices.

Spectral graph theory is a useful subject. The founders of Google computed the Perron-Frobenius eigenvector of the web graph and became billionaires. The second-largest eigenvalue of a graph gives information about expansion and randomness properties. The smallest eigenvalue gives information about independence number and chromatic number. Interlacing gives information about substructures. The fact that eigenvalue multiplicities must be integral provides strong restrictions. And the spectrum provides a useful invariant.

This book gives the standard elementary material on spectra in Chapter 1. Important applications of graph spectra involve the largest, second-largest, or smallest eigenvalue, or interlacing, topics that are discussed in Chapters 3 and 4. Afterwards, special topics such as trees, groups and graphs, Euclidean representations, and strongly regular graphs are discussed. Strongly related to strongly regular graphs are regular two-graphs, and Chapter 10 mainly discusses Seidel's work on sets of equiangular lines. Strongly regular graphs form the first nontrivial case of (symmetric) association schemes, and Chapter 11 gives a very brief introduction to this topic and Delsarte's linear programming bound. Chapter 12 very briefly mentions the main facts on distance-regular graphs, including some major developments that have occurred since the monograph [54] was written (proof of the Bannai-Ito conjecture, construction by Van Dam and Koolen of the twisted Grassmann graphs, determination of the connectivity of distance-regular graphs). Instead of working over \mathbb{R} , one can work over \mathbb{F}_p or \mathbb{Z} and obtain more detailed information. Chapter 13 considers p -ranks and Smith normal forms. Finally, Chapters 14 and 15 return to the real spectrum and consider when a graph is determined by its spectrum and when it has only few eigenvalues.

In Spring 2006, both authors gave a series of lectures at IPM, the Institute for Studies in Theoretical Physics and Mathematics, in Tehran. The lecture notes were

combined and published as an IPM report. Those notes grew into the present text, of which the on-line version still is called `ipm.pdf`. We aim at researchers, teachers, and graduate students interested in graph spectra. The reader is assumed to be familiar with basic linear algebra and eigenvalues, but we did include a chapter on some more advanced topics in linear algebra, such as the Perron-Frobenius theorem and eigenvalue interlacing. The exercises at the end of the chapters vary from easy but interesting applications of the treated theory to little excursions into related topics.

This book shows the influence of Seidel. For other books on spectral graph theory, see CHUNG [93], CVETKOVIĆ, DOOB & SACHS [115], and CVETKOVIĆ, ROWLINSON & SIMIĆ [120]. For more algebraic graph theory, see BIGGS [30], GODSIL [172], and GODSIL & ROYLE [177]. For association schemes and distance-regular graphs, see BANNAI & ITO [21] and BROUWER, COHEN & NEUMAIER [54].

Amsterdam
December 2010

Andries Brouwer
Willem Haemers

Contents

Preface	v
1 Graph Spectrum	1
1.1 Matrices associated to a graph	1
1.2 The spectrum of a graph	2
1.2.1 Characteristic polynomial	3
1.3 The spectrum of an undirected graph	3
1.3.1 Regular graphs	4
1.3.2 Complements	4
1.3.3 Walks	4
1.3.4 Diameter	5
1.3.5 Spanning trees	5
1.3.6 Bipartite graphs	6
1.3.7 Connectedness	7
1.4 Spectrum of some graphs	8
1.4.1 The complete graph	8
1.4.2 The complete bipartite graph	8
1.4.3 The cycle	8
1.4.4 The path	9
1.4.5 Line graphs	9
1.4.6 Cartesian products	10
1.4.7 Kronecker products and bipartite double	10
1.4.8 Strong products	11
1.4.9 Cayley graphs	11
1.5 Decompositions	11
1.5.1 Decomposing K_{10} into Petersen graphs	12
1.5.2 Decomposing K_n into complete bipartite graphs	12
1.6 Automorphisms	12
1.7 Algebraic connectivity	13
1.8 Cosppectral graphs	14
1.8.1 The 4-cube	14

1.8.2	Seidel switching	15
1.8.3	Godsil-McKay switching	16
1.8.4	Reconstruction	16
1.9	Very small graphs	16
1.10	Exercises	17
2	Linear Algebra	21
2.1	Simultaneous diagonalization	21
2.2	Perron-Frobenius theory	22
2.3	Equitable partitions	24
2.3.1	Equitable and almost equitable partitions of graphs	25
2.4	The Rayleigh quotient	25
2.5	Interlacing	26
2.6	Schur's inequality	28
2.7	Schur complements	28
2.8	The Courant-Weyl inequalities	29
2.9	Gram matrices	29
2.10	Diagonally dominant matrices	30
2.10.1	Geršgorin circles	31
2.11	Projections	31
2.12	Exercises	32
3	Eigenvalues and Eigenvectors of Graphs	33
3.1	The largest eigenvalue	33
3.1.1	Graphs with largest eigenvalue at most 2	34
3.1.2	Subdividing an edge	35
3.1.3	The Kelmans operation	36
3.2	Interlacing	37
3.3	Regular graphs	37
3.4	Bipartite graphs	38
3.5	Cliques and cocliques	38
3.5.1	Using weighted adjacency matrices	39
3.6	Chromatic number	40
3.6.1	Using weighted adjacency matrices	42
3.6.2	Rank and chromatic number	42
3.7	Shannon capacity	42
3.7.1	Lovász's ϑ -function	44
3.7.2	The Haemers bound on the Shannon capacity	45
3.8	Classification of integral cubic graphs	46
3.8.1	A quotient of the hexagonal grid	47
3.8.2	Cubic graphs with loops	47
3.8.3	The classification	47
3.9	The largest Laplace eigenvalue	50
3.10	Laplace eigenvalues and degrees	51
3.11	The Grone-Merris conjecture	53

3.11.1	Threshold graphs	53
3.11.2	Proof of the Grone-Merris conjecture	53
3.12	The Laplacian for hypergraphs	56
3.12.1	Dominance order	58
3.13	Applications of eigenvectors	58
3.13.1	Ranking	59
3.13.2	Google PageRank	59
3.13.3	Cutting	60
3.13.4	Graph drawing	61
3.13.5	Clustering	61
3.13.6	Graph isomorphism	62
3.13.7	Searching an eigenspace	63
3.14	Stars and star complements	63
3.15	Exercises	64
4	The Second-Largest Eigenvalue	67
4.1	Bounds for the second-largest eigenvalue	67
4.2	Large regular subgraphs are connected	68
4.3	Randomness	68
4.4	Random walks	69
4.5	Expansion	70
4.6	Toughness and Hamiltonicity	71
4.6.1	The Petersen graph is not Hamiltonian	72
4.7	Diameter bound	72
4.8	Separation	73
4.8.1	Bandwidth	74
4.8.2	Perfect matchings	75
4.9	Block designs	77
4.10	Polarities	79
4.11	Exercises	80
5	Trees	83
5.1	Characteristic polynomials of trees	83
5.2	Eigenvectors and multiplicities	85
5.3	Sign patterns of eigenvectors of graphs	86
5.4	Sign patterns of eigenvectors of trees	87
5.5	The spectral center of a tree	88
5.6	Integral trees	89
5.7	Exercises	90
6	Groups and Graphs	93
6.1	$\Gamma(G, H, S)$	93
6.2	Spectrum	93
6.3	Non-Abelian Cayley graphs	94
6.4	Covers	95

6.5	Cayley sum graphs	97
6.5.1	(3,6)-fullerenes	97
6.6	Exercises	99
7	Topology	101
7.1	Embeddings	101
7.2	Minors	102
7.3	The Colin de Verdière invariant	102
7.4	The Van der Holst-Laurent-Schrijver invariant	103
8	Euclidean Representations	105
8.1	Examples	105
8.2	Euclidean representation	105
8.3	Root lattices	106
8.3.1	Examples	107
8.3.2	Root lattices	108
8.3.3	Classification	109
8.4	The Cameron-Goethals-Seidel-Shult theorem	111
8.5	Further applications	112
8.6	Exercises	113
9	Strongly Regular Graphs	115
9.1	Strongly regular graphs	115
9.1.1	Simple examples	115
9.1.2	The Paley graphs	116
9.1.3	Adjacency matrix	117
9.1.4	Imprimitive graphs	117
9.1.5	Parameters	118
9.1.6	The half case and cyclic strongly regular graphs	118
9.1.7	Strongly regular graphs without triangles	119
9.1.8	Further parameter restrictions	120
9.1.9	Strongly regular graphs from permutation groups	121
9.1.10	Strongly regular graphs from quasisymmetric designs	121
9.1.11	Symmetric 2-designs from strongly regular graphs	122
9.1.12	Latin square graphs	122
9.1.13	Partial geometries	124
9.2	Strongly regular graphs with eigenvalue -2	124
9.3	Connectivity	125
9.4	Cocliques and colorings	127
9.5	Automorphisms	129
9.6	Generalized quadrangles	129
9.6.1	Parameters	129
9.6.2	Constructions of generalized quadrangles	130
9.6.3	Strongly regular graphs from generalized quadrangles	131
9.6.4	Generalized quadrangles with lines of size 3	132

- 9.7 The $(81, 20, 1, 6)$ strongly regular graph 132
 - 9.7.1 Descriptions 133
 - 9.7.2 Uniqueness 134
 - 9.7.3 Independence and chromatic numbers 135
 - 9.7.4 Second subconstituent 136
- 9.8 Strongly regular graphs and two-weight codes 136
 - 9.8.1 Codes, graphs, and projective sets 136
 - 9.8.2 The correspondence between linear codes and subsets of a projective space 137
 - 9.8.3 The correspondence between projective two-weight codes, subsets of a projective space with two intersection numbers, and affine strongly regular graphs 138
 - 9.8.4 Duality for affine strongly regular graphs 140
 - 9.8.5 Cyclotomy 141
- 9.9 Table of parameters for strongly regular graphs 143
 - 9.9.1 Comments 146
- 9.10 Exercises 148
- 10 Regular Two-graphs** 151
 - 10.1 Strong graphs 151
 - 10.2 Two-graphs 152
 - 10.3 Regular two-graphs 154
 - 10.3.1 Related strongly regular graphs 155
 - 10.3.2 The regular two-graph on 276 points 156
 - 10.3.3 Coherent subsets 156
 - 10.3.4 Completely regular two-graphs 157
 - 10.4 Conference matrices 158
 - 10.5 Hadamard matrices 159
 - 10.5.1 Constructions 160
 - 10.6 Equiangular lines 161
 - 10.6.1 Equiangular lines in \mathbb{R}^d and two-graphs 161
 - 10.6.2 Bounds on equiangular sets of lines in \mathbb{R}^d or \mathbb{C}^d 162
 - 10.6.3 Bounds on sets of lines with few angles and sets of vectors with few distances 163
- 11 Association Schemes** 165
 - 11.1 Definition 165
 - 11.2 The Bose-Mesner algebra 166
 - 11.3 The linear programming bound 168
 - 11.3.1 Equality 169
 - 11.3.2 The code-clique theorem 169
 - 11.3.3 Strengthened LP bounds 170
 - 11.4 The Krein parameters 170
 - 11.5 Automorphisms 172
 - 11.5.1 The Moore graph on 3250 vertices 172

11.6 P - and Q -polynomial association schemes 173

11.7 Exercises 175

12 Distance-Regular Graphs 177

12.1 Parameters 177

12.2 Spectrum 178

12.3 Primitivity 178

12.4 Examples 178

12.4.1 Hamming graphs 178

12.4.2 Johnson graphs 179

12.4.3 Grassmann graphs 180

12.4.4 Van Dam-Koolen graphs 180

12.5 Bannai-Ito conjecture 180

12.6 Connectedness 181

12.7 Growth 181

12.8 Degree of eigenvalues 181

12.9 Moore graphs and generalized polygons 182

12.10 Euclidean representations 183

12.11 Extremality 183

12.12 Exercises 185

13 p -ranks 187

13.1 Reduction mod p 187

13.2 The minimal polynomial 188

13.3 Bounds for the p -rank 188

13.4 Interesting primes p 189

13.5 Adding a multiple of J 190

13.6 Paley graphs 191

13.7 Strongly regular graphs 192

13.8 Smith normal form 194

13.8.1 Smith normal form and spectrum 195

13.9 Exercises 197

14 Spectral Characterizations 199

14.1 Generalized adjacency matrices 199

14.2 Constructing cospectral graphs 200

14.2.1 Trees 201

14.2.2 Partial linear spaces 202

14.2.3 GM switching 202

14.2.4 Sunada's method 204

14.3 Enumeration 204

14.3.1 Lower bounds 204

14.3.2 Computer results 205

14.4 DS graphs 206

14.4.1 Spectrum and structure 206

- 14.4.2 Some DS graphs 208
- 14.4.3 Line graphs 210
- 14.5 Distance-regular graphs 212
 - 14.5.1 Strongly regular DS graphs 213
 - 14.5.2 Distance-regularity from the spectrum 214
 - 14.5.3 Distance-regular DS graphs 215
- 14.6 The method of Wang and Xu 217
- 14.7 Exercises 219
- 15 Graphs with Few Eigenvalues 221**
 - 15.1 Regular graphs with four eigenvalues 221
 - 15.2 Three Laplace eigenvalues 223
 - 15.3 Other matrices with at most three eigenvalues 224
 - 15.3.1 Few Seidel eigenvalues 224
 - 15.3.2 Three adjacency eigenvalues 225
 - 15.3.3 Three signless Laplace eigenvalues 227
 - 15.4 Exercises 227
- References 229**
- Author Index 243**
- Subject Index 247**

Chapter 1

Graph Spectrum

This chapter presents some simple results on graph spectra. We assume the reader is familiar with elementary linear algebra and graph theory. Throughout, J will denote the all-1 matrix, and $\mathbf{1}$ is the all-1 vector.

1.1 Matrices associated to a graph

Let Γ be a graph without multiple edges. The *adjacency matrix* of Γ is the 0-1 matrix A indexed by the vertex set $V\Gamma$ of Γ , where $A_{xy} = 1$ when there is an edge from x to y in Γ and $A_{xy} = 0$ otherwise. Occasionally we consider multigraphs (possibly with loops), in which case A_{xy} equals the number of edges from x to y .

Let Γ be an undirected graph without loops. The (vertex-edge) *incidence matrix* of Γ is the 0-1 matrix M , with rows indexed by the vertices and columns indexed by the edges, where $M_{xe} = 1$ when vertex x is an endpoint of edge e .

Let Γ be a directed graph without loops. The *directed incidence matrix* of Γ is the matrix N , with rows indexed by the vertices and columns by the edges, where $N_{xe} = -1, 1, 0$ when x is the head of e , the tail of e , or not on e , respectively.

Let Γ be an undirected graph without loops. The *Laplace matrix* of Γ is the matrix L indexed by the vertex set of Γ , with zero row sums, where $L_{xy} = -A_{xy}$ for $x \neq y$. If D is the diagonal matrix, indexed by the vertex set of Γ such that D_{xx} is the degree (valency) of x , then $L = D - A$. The matrix $Q = D + A$ is called the *signless Laplace matrix* of Γ .

An important property of the Laplace matrix L and the signless Laplace matrix Q is that they are positive semidefinite. Indeed, one has $Q = MM^T$ and $L = NN^T$ if M is the incidence matrix of Γ and N the directed incidence matrix of the directed graph obtained by orienting the edges of Γ in an arbitrary way. It follows that for any vector u one has $u^T L u = \sum_{xy} (u_x - u_y)^2$ and $u^T Q u = \sum_{xy} (u_x + u_y)^2$, where the sum is over the edges of Γ .

1.2 The spectrum of a graph

The (ordinary) *spectrum* of a finite graph Γ is by definition the spectrum of the adjacency matrix A , that is, its set of eigenvalues together with their multiplicities. The *Laplace spectrum* of a finite undirected graph without loops is the spectrum of the Laplace matrix L .

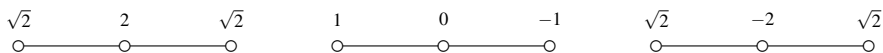
The rows and columns of a matrix of order n are numbered from 1 to n , while A is indexed by the vertices of Γ , so that writing down A requires one to assign some numbering to the vertices. However, the spectrum of the matrix obtained does not depend on the numbering chosen. It is the spectrum of the linear transformation A on the vector space K^X of maps from X into K , where X is the vertex set and K is some field such as \mathbb{R} or \mathbb{C} .

The *characteristic polynomial* of Γ is that of A , that is, the polynomial p_A defined by $p_A(\theta) = \det(\theta I - A)$.

Example Let Γ be the path P_3 with three vertices and two edges. Assigning some arbitrary order to the three vertices of Γ , we find that the adjacency matrix A becomes one of

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

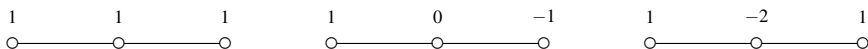
The characteristic polynomial is $p_A(\theta) = \theta^3 - 2\theta$. The spectrum is $\sqrt{2}, 0, -\sqrt{2}$. The eigenvectors are:



Here, for an eigenvector u , we write u_x as a label at the vertex x . One has $Au = \theta u$ if and only if $\sum_{y \leftarrow x} u_y = \theta u_x$ for all x . The Laplace matrix L of this graph is one of

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

Its eigenvalues are 0, 1 and 3. The Laplace eigenvectors are:



One has $Lu = \theta u$ if and only if $\sum_{y \sim x} u_y = (d_x - \theta)u_x$ for all x , where d_x is the degree of the vertex x .

Example Let Γ be the directed triangle with adjacency matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then A has characteristic polynomial $p_A(\theta) = \theta^3 - 1$ and spectrum $1, \omega, \omega^2$, where ω is a primitive cube root of unity.

Example Let Γ be the directed graph with two vertices and a single directed edge. Then $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ with $p_A(\theta) = \theta^2$, so A has the eigenvalue 0 with geometric multiplicity (that is, the dimension of the corresponding eigenspace) equal to 1 and algebraic multiplicity (that is, its multiplicity as a root of the polynomial p_A) equal to 2.

1.2.1 Characteristic polynomial

Let Γ be a directed graph on n vertices. For any directed subgraph C of Γ that is a union of directed cycles, let $c(C)$ be its number of cycles. Then the characteristic polynomial $p_A(t) = \det(tI - A)$ of Γ can be expanded as $\sum c_i t^{n-i}$, where $c_i = \sum_C (-1)^{c(C)}$, with C running over all regular directed subgraphs with in- and outdegree 1 on i vertices.

(Indeed, this is just a reformulation of the definition of the determinant as $\det M = \sum_{\sigma} \text{sgn}(\sigma) M_{1\sigma(1)} \cdots M_{n\sigma(n)}$. Note that when the permutation σ with $n - i$ fixed points is written as a product of nonidentity cycles, its sign is $(-1)^e$, where e is the number of even cycles in this product. Since the number of odd nonidentity cycles is congruent to $i \pmod{2}$, we have $\text{sgn}(\sigma) = (-1)^{i+c(\sigma)}$.)

For example, the directed triangle has $c_0 = 1$, $c_3 = -1$. Directed edges that do not occur in directed cycles do not influence the (ordinary) spectrum.

The same description of $p_A(t)$ holds for undirected graphs (with each edge viewed as a pair of opposite directed edges).

Since $\frac{d}{dt} \det(tI - A) = \sum_x \det(tI - A_x)$ where A_x is the submatrix of A obtained by deleting row and column x , it follows that $p'_A(t)$ is the sum of the characteristic polynomials of all single-vertex-deleted subgraphs of Γ .

1.3 The spectrum of an undirected graph

Suppose Γ is undirected and simple with n vertices. Since A is real and symmetric, all its eigenvalues are real. Also, for each eigenvalue θ , its algebraic multiplicity coincides with its geometric multiplicity, so that we may omit the adjective and just speak about “multiplicity”. Conjugate algebraic integers have the same multiplicity. Since A has zero diagonal, its trace $\text{tr}A$, and hence the sum of the eigenvalues, is zero.

Similarly, L is real and symmetric, so that the Laplace spectrum is real. Moreover, L is positive semidefinite and singular, so we may denote the eigenvalues by

μ_1, \dots, μ_n , where $0 = \mu_1 \leq \mu_2 \leq \dots \leq \mu_n$. The sum of these eigenvalues is $\text{tr}L$, which is twice the number of edges of Γ .

Finally, also Q has real spectrum and nonnegative eigenvalues (but is not necessarily singular). We have $\text{tr}Q = \text{tr}L$.

1.3.1 Regular graphs

A graph Γ is called *regular* of degree (or valency) k when every vertex has precisely k neighbors. So, Γ is regular of degree k precisely when its adjacency matrix A has row sums k , i.e., when $A\mathbf{1} = k\mathbf{1}$ (or $AJ = kJ$).

If Γ is regular of degree k , then for every eigenvalue θ we have $|\theta| \leq k$. (One way to see this is by observing that if $|t| > k$ then the matrix $tI - A$ is strictly diagonally dominant, and hence nonsingular, so that t is not an eigenvalue of A .)

If Γ is regular of degree k , then $L = kI - A$. It follows that if Γ has ordinary eigenvalues $k = \theta_1 \geq \dots \geq \theta_n$ and Laplace eigenvalues $0 = \mu_1 \leq \mu_2 \leq \dots \leq \mu_n$, then $\theta_i = k - \mu_i$ for $i = 1, \dots, n$. The eigenvalues of $Q = kI + A$ are $2k, k + \theta_2, \dots, k + \theta_n$.

1.3.2 Complements

The *complement* $\bar{\Gamma}$ of Γ is the graph with the same vertex set as Γ , where two distinct vertices are adjacent whenever they are nonadjacent in Γ . So, if Γ has adjacency matrix A , then $\bar{\Gamma}$ has adjacency matrix $\bar{A} = J - I - A$ and Laplace matrix $\bar{L} = nI - J - L$.

Because eigenvectors of L are also eigenvectors of J , the eigenvalues of \bar{L} are $0, n - \mu_n, \dots, n - \mu_2$. (In particular, $\mu_n \leq n$.)

If Γ is regular we have a similar result for the ordinary eigenvalues: if Γ is k -regular with eigenvalues $\theta_1 \geq \dots \geq \theta_n$, then the eigenvalues of the complement are $n - k - 1, -1 - \theta_n, \dots, -1 - \theta_2$.

1.3.3 Walks

From the spectrum one can read off the number of closed walks of a given length.

Proposition 1.3.1 *Let h be a nonnegative integer. Then $(A^h)_{xy}$ is the number of walks of length h from x to y . In particular, $(A^2)_{xx}$ is the degree of the vertex x , and $\text{tr}A^2$ equals twice the number of edges of Γ ; similarly, $\text{tr}A^3$ is six times the number of triangles in Γ .*

1.3.4 Diameter

We saw that all eigenvalues of a single directed edge are zero. For undirected graphs this does not happen.

Proposition 1.3.2 *Let Γ be an undirected graph. All its eigenvalues are zero if and only if Γ has no edges. The same holds for the Laplace eigenvalues and the signless Laplace eigenvalues.*

More generally, we find a lower bound for the diameter:

Proposition 1.3.3 *Let Γ be a connected graph with diameter d . Then Γ has at least $d + 1$ distinct eigenvalues, at least $d + 1$ distinct Laplace eigenvalues, and at least $d + 1$ distinct signless Laplace eigenvalues.*

Proof Let M be any nonnegative symmetric matrix with rows and columns indexed by $V\Gamma$ and such that for distinct vertices x, y we have $M_{xy} > 0$ if and only if $x \sim y$. Let the distinct eigenvalues of M be $\theta_1, \dots, \theta_t$. Then $(M - \theta_1 I) \cdots (M - \theta_t I) = 0$, so that M^t is a linear combination of I, M, \dots, M^{t-1} . But if $d(x, y) = t$ for two vertices x, y of Γ , then $(M^i)_{xy} = 0$ for $0 \leq i \leq t - 1$ and $(M^t)_{xy} > 0$, a contradiction. Hence $t > d$. This applies to $M = A$, to $M = nI - L$, and to $M = Q$, where A is the adjacency matrix, L is the Laplace matrix, and Q is the signless Laplace matrix of Γ . \square

Distance-regular graphs, discussed in Chapter 12, have equality here. For an upper bound on the diameter, see §4.7.

1.3.5 Spanning trees

From the Laplace spectrum of a graph one can determine the number of spanning trees (which will be nonzero only if the graph is connected).

Proposition 1.3.4 *Let Γ be an undirected (multi)graph with at least one vertex, and Laplace matrix L with eigenvalues $0 = \mu_1 \leq \mu_2 \leq \dots \leq \mu_n$. Let ℓ_{xy} be the (x, y) -cofactor of L . Then the number N of spanning trees of Γ equals*

$$N = \ell_{xy} = \det\left(L + \frac{1}{n^2}J\right) = \frac{1}{n}\mu_2 \cdots \mu_n \text{ for any } x, y \in V\Gamma.$$

(The (i, j) -cofactor of a matrix M is by definition $(-1)^{i+j} \det M(i, j)$, where $M(i, j)$ is the matrix obtained from M by deleting row i and column j . Note that ℓ_{xy} does not depend on an ordering of the vertices of Γ .)

Proof Let L^S , for $S \subseteq V\Gamma$, denote the matrix obtained from L by deleting the rows and columns indexed by S , so that $\ell_{xx} = \det L^{\{x\}}$. The equality $N = \ell_{xx}$ follows by induction on n , and for fixed $n > 1$ on the number of edges incident with x . Indeed, if $n = 1$ then $\ell_{xx} = 1$. Otherwise, if x has degree 0, then $\ell_{xx} = 0$ since $L^{\{x\}}$ has zero

row sums. Finally, if xy is an edge, then deleting this edge from Γ diminishes ℓ_{xx} by $\det L^{\{x,y\}}$, which by induction is the number of spanning trees of Γ with edge xy contracted, which is the number of spanning trees containing the edge xy . This shows $N = \ell_{xx}$.

Now $\det(tI - L) = t \prod_{i=2}^n (t - \mu_i)$ and $(-1)^{n-1} \mu_2 \cdots \mu_n$ is the coefficient of t , that is, is $\frac{d}{dt} \det(tI - L)|_{t=0}$. But $\frac{d}{dt} \det(tI - L) = \sum_x \det(tI - L^{\{x\}})$, so $\mu_2 \cdots \mu_n = \sum_x \ell_{xx} = nN$.

Since the sum of the columns of L is zero, so that one column is minus the sum of the other columns, we have $\ell_{xx} = \ell_{xy}$ for any x, y . Finally, the eigenvalues of $L + \frac{1}{n^2}J$ are $\frac{1}{n}$ and μ_2, \dots, μ_n , so $\det(L + \frac{1}{n^2}J) = \frac{1}{n} \mu_2 \cdots \mu_n$. \square

For example, the multigraph of valency k on two vertices has Laplace matrix $L = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$ so $\mu_1 = 0$, $\mu_2 = 2k$, and $N = \frac{1}{2} \cdot 2k = k$.

If we consider the complete graph K_n , then $\mu_2 = \dots = \mu_n = n$, and therefore K_n has $N = n^{n-2}$ spanning trees. This formula is due to CAYLEY [85]. Proposition 1.3.4 is implicit in KIRCHHOFF [242] and known as the *matrix-tree theorem*. There is a “1-line proof” of the above result using the *Cauchy-Binet formula*.

Proposition 1.3.5 (Cauchy-Binet) *Let A and B be $m \times n$ matrices. Then*

$$\det AB^\top = \sum_S \det A_S \det B_S,$$

where the sum is over the $\binom{n}{m}$ m -subsets S of the set of columns, and A_S (B_S) is the square submatrix of order m of A (resp. B) with columns indexed by S .

Second proof of Proposition 1.3.4 (sketch) Let N_x be the directed incidence matrix of Γ with row x deleted. Then $\ell_{xx} = \det N_x N_x^\top$. Apply the Cauchy-Binet formula to get ℓ_{xx} as a sum of squares of determinants of size $n-1$. These determinants vanish unless the set S of columns is the set of edges of a spanning tree, in which case the determinant is ± 1 . \square

1.3.6 Bipartite graphs

A graph Γ is called *bipartite* when its vertex set can be partitioned into two disjoint parts X_1, X_2 such that all edges of Γ meet both X_1 and X_2 . The adjacency matrix of a bipartite graph has the form $A = \begin{bmatrix} 0 & B \\ B^\top & 0 \end{bmatrix}$. It follows that the spectrum of a bipartite graph is symmetric w.r.t. 0: if $\begin{bmatrix} u \\ v \end{bmatrix}$ is an eigenvector with eigenvalue θ , then $\begin{bmatrix} u \\ -v \end{bmatrix}$ is an eigenvector with eigenvalue $-\theta$. (The converse also holds, see Proposition 3.4.1.)

For the ranks one has $\text{rk} A = 2 \text{rk} B$. If $n_i = |X_i|$ ($i = 1, 2$) and $n_1 \geq n_2$, then $\text{rk} A \leq 2n_2$, so Γ has eigenvalue 0 with multiplicity at least $n_1 - n_2$.

One cannot, in general, recognize bipartiteness from the Laplace or signless Laplace spectrum. For example, $K_{1,3}$ and $K_1 + K_3$ have the same signless Laplace

spectrum and only the former is bipartite. And Figure 14.4 gives an example of a bipartite and a nonbipartite graph with the same Laplace spectrum. However, by Proposition 1.3.10 below, a graph is bipartite precisely when its Laplace spectrum and signless Laplace spectrum coincide.

1.3.7 Connectedness

The spectrum of a disconnected graph is easily found from the spectra of its connected components:

Proposition 1.3.6 *Let Γ be a graph with connected components Γ_i ($1 \leq i \leq s$). Then the spectrum of Γ is the union of the spectra of Γ_i (and multiplicities are added). The same holds for the Laplace spectrum and the signless Laplace spectrum.* \square

Proposition 1.3.7 *The multiplicity of 0 as a Laplace eigenvalue of an undirected graph Γ equals the number of connected components of Γ .*

Proof We have to show that a connected graph has Laplace eigenvalue 0 with multiplicity 1. As we saw earlier, $L = NN^T$, where N is the incidence matrix of an orientation of Γ . Now $Lu = 0$ is equivalent to $N^T u = 0$ (since $0 = u^T Lu = \|N^T u\|^2$), that is, for every edge the vector u takes the same value on both endpoints. Since Γ is connected, that means that u is constant. \square

Proposition 1.3.8 *Let the undirected graph Γ be regular of valency k . Then k is the largest eigenvalue of Γ , and its multiplicity equals the number of connected components of Γ .*

Proof We have $L = kI - A$. \square

One cannot see from the spectrum alone whether a (nonregular) graph is connected: both $K_{1,4}$ and $K_1 + C_4$ have spectrum $2^1, 0^3, (-2)^1$ (we write multiplicities as exponents). And both \hat{E}_6 and $K_1 + C_6$ have spectrum $2^1, 1^2, 0, (-1)^2, (-2)^1$.

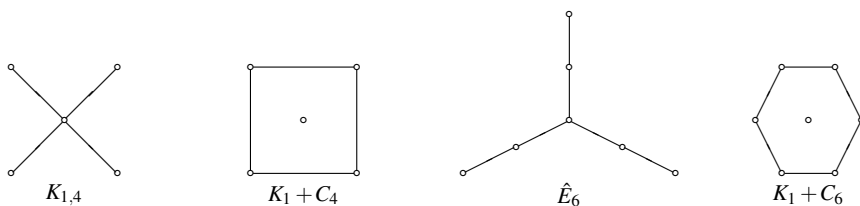


Fig. 1.1 Two pairs of cospectral graphs

Proposition 1.3.9 *The multiplicity of 0 as a signless Laplace eigenvalue of an undirected graph Γ equals the number of bipartite connected components of Γ .*

Proof Let M be the vertex-edge incidence matrix of Γ , so that $Q = MM^\top$. If $MM^\top u = 0$, then $M^\top u = 0$, so $u_x = -u_y$ for all edges xy , and the support of u is the union of a number of bipartite components of Γ . \square

Proposition 1.3.10 *A graph Γ is bipartite if and only if the Laplace spectrum and the signless Laplace spectrum of Γ are equal.*

Proof If Γ is bipartite, the Laplace matrix L and the signless Laplace matrix Q are similar by a diagonal matrix D with diagonal entries ± 1 (that is, $Q = DLD^{-1}$). Therefore Q and L have the same spectrum. Conversely, if both spectra are the same, then by Propositions 1.3.7 and 1.3.9 the number of connected components equals the number of bipartite components. Hence Γ is bipartite. \square

1.4 Spectrum of some graphs

In this section we discuss some special graphs and their spectra. All graphs in this section are finite, undirected, and simple. Observe that the all-1 matrix J of order n has rank 1, and that the all-1 vector $\mathbf{1}$ is an eigenvector with eigenvalue n , so the spectrum of J is $n^1, 0^{n-1}$. (Here and throughout, we write multiplicities as exponents where convenient and no confusion seems likely.)

1.4.1 The complete graph

Let Γ be the complete graph K_n on n vertices. Its adjacency matrix is $A = J - I$, and the spectrum is $(n-1)^1, (-1)^{n-1}$. The Laplace matrix is $nI - J$, which has spectrum $0^1, n^{n-1}$.

1.4.2 The complete bipartite graph

The spectrum of the complete bipartite graph $K_{m,n}$ is $\pm\sqrt{mn}, 0^{m+n-2}$. The Laplace spectrum is $0^1, m^{n-1}, n^{m-1}, (m+n)^1$.

1.4.3 The cycle

Let Γ be the directed n -cycle D_n . Eigenvectors are $(1, \zeta, \zeta^2, \dots, \zeta^{n-1})^\top$, where $\zeta^n = 1$, and the corresponding eigenvalue is ζ . Thus, the spectrum consists precisely of the complex n -th roots of unity $e^{2\pi i j/n}$ ($j = 0, \dots, n-1$).

Now consider the undirected n -cycle C_n . If B is the adjacency matrix of D_n , then $A = B + B^\top$ is the adjacency matrix of C_n . We find the same eigenvectors as before, with eigenvalues $\zeta + \zeta^{-1}$, so that the spectrum consists of the numbers $2 \cos(2\pi j/n)$ ($j = 0, \dots, n-1$).

This graph is regular of valency 2, so the Laplace spectrum consists of the numbers $2 - 2 \cos(2\pi j/n)$ ($j = 0, \dots, n-1$).

1.4.4 The path

Let Γ be the undirected path P_n with n vertices. The ordinary spectrum consists of the numbers $2 \cos(\pi j/(n+1))$ ($j = 1, \dots, n$). The Laplace spectrum is $2 - 2 \cos(\pi j/n)$ ($j = 0, \dots, n-1$).

The ordinary spectrum follows by looking at C_{2n+2} . If $u(\zeta) = (1, \zeta, \zeta^2, \dots, \zeta^{2n+1})^\top$ is an eigenvector of C_{2n+2} , where $\zeta^{2n+2} = 1$, then $u(\zeta)$ and $u(\zeta^{-1})$ have the same eigenvalue, $2 \cos(\pi j/(n+1))$, and hence so has $u(\zeta) - u(\zeta^{-1})$. This latter vector has two zero coordinates distance $n+1$ apart and (for $\zeta \neq \pm 1$) induces an eigenvector on the two paths obtained by removing the two points where it is zero.

Eigenvectors of L with eigenvalue $2 - \zeta - \zeta^{-1}$ are $(1 + \zeta^{2n-1}, \dots, \zeta^j + \zeta^{2n-1-j}, \dots, \zeta^{n-1} + \zeta^n)$, where $\zeta^{2n} = 1$. One can check this directly, or view P_n as the result of folding C_{2n} , where the folding has no fixed vertices. An eigenvector of C_{2n} that is constant on the preimages of the folding yields an eigenvector of P_n with the same eigenvalue.

1.4.5 Line graphs

The *line graph* $L(\Gamma)$ of Γ is the graph with the edge set of Γ as vertex set, where two vertices are adjacent if the corresponding edges of Γ have an endpoint in common. If N is the incidence matrix of Γ , then $N^\top N - 2I$ is the adjacency matrix of $L(\Gamma)$. Since $N^\top N$ is positive semidefinite, the eigenvalues of a line graph are not smaller than -2 . We have an explicit formula for the eigenvalues of $L(\Gamma)$ in terms of the signless Laplace eigenvalues of Γ .

Proposition 1.4.1 *Suppose Γ has m edges, and let $\rho_1 \geq \dots \geq \rho_r$ be the positive signless Laplace eigenvalues of Γ . Then the eigenvalues of $L(\Gamma)$ are $\theta_i = \rho_i - 2$ for $i = 1, \dots, r$, and $\theta_i = -2$ if $r < i \leq m$.*

Proof The signless Laplace matrix Q of Γ and the adjacency matrix B of $L(\Gamma)$ satisfy $Q = NN^\top$ and $B + 2I = N^\top N$. Because NN^\top and $N^\top N$ have the same nonzero eigenvalues (multiplicities included), the result follows. \square

Example Since the path P_n has line graph P_{n-1} and is bipartite, the Laplace and the signless Laplace eigenvalues of P_n are $2 + 2 \cos \frac{\pi i}{n}$, $i = 1, \dots, n$.

Corollary 1.4.2 *If Γ is a k -regular graph ($k \geq 2$) with n vertices, $e = kn/2$ edges, and eigenvalues θ_i ($i = 1, \dots, n$), then $L(\Gamma)$ is $(2k - 2)$ -regular with eigenvalues $\theta_i + k - 2$ ($i = 1, \dots, n$) and $e - n$ times -2 . \square*

The line graph of the complete graph K_n ($n \geq 2$) is known as the *triangular graph* $T(n)$. It has spectrum $2(n - 2)^1, (n - 4)^{n-1}, (-2)^{n(n-3)/2}$. The line graph of the regular complete bipartite graph $K_{m,m}$ ($m \geq 2$) is known as the *lattice graph* $L_2(m)$. It has spectrum $2(m - 1)^1, (m - 2)^{2m-2}, (-2)^{(m-1)^2}$. These two families of graphs, and their complements, are examples of strongly regular graphs, which will be the subject of Chapter 9. The complement of $T(5)$ is the famous *Petersen graph*. It has spectrum $3^1, 1^5, (-2)^4$.

1.4.6 Cartesian products

Given graphs Γ and Δ with vertex sets V and W , respectively, their *Cartesian product* $\Gamma \square \Delta$ is the graph with vertex set $V \times W$, where $(v, w) \sim (v', w')$ when either $v = v'$ and $w \sim w'$ or $w = w'$ and $v \sim v'$. For the adjacency matrices we have $A_{\Gamma \square \Delta} = A_{\Gamma} \otimes I + I \otimes A_{\Delta}$.

If u and v are eigenvectors for Γ and Δ with ordinary or Laplace eigenvalues θ and η , respectively, then the vector w defined by $w_{(x,y)} = u_x v_y$ is an eigenvector of $\Gamma \square \Delta$ with ordinary or Laplace eigenvalue $\theta + \eta$.

For example, $L_2(m) = K_m \square K_m$.

For example, the *hypercube* 2^n , also called Q_n , is the Cartesian product of n factors K_2 . The spectrum of K_2 is $1, -1$, and hence the spectrum of 2^n consists of the numbers $n - 2i$ with multiplicity $\binom{n}{i}$ ($i = 0, 1, \dots, n$).

1.4.7 Kronecker products and bipartite double

Given graphs Γ and Δ with vertex sets V and W , respectively, their *Kronecker product* (or *direct product*, or *conjunction*) $\Gamma \otimes \Delta$ is the graph with vertex set $V \times W$, where $(v, w) \sim (v', w')$ when $v \sim v'$ and $w \sim w'$. The adjacency matrix of $\Gamma \otimes \Delta$ is the Kronecker product of the adjacency matrices of Γ and Δ .

If u and v are eigenvectors for Γ and Δ with eigenvalues θ and η , respectively, then the vector $w = u \otimes v$ (with $w_{(x,y)} = u_x v_y$) is an eigenvector of $\Gamma \otimes \Delta$ with eigenvalue $\theta\eta$. Thus, the spectrum of $\Gamma \otimes \Delta$ consists of the products of the eigenvalues of Γ and Δ .

Given a graph Γ , its *bipartite double* is the graph $\Gamma \otimes K_2$ (with for each vertex x of Γ two vertices x' and x'' , and for each edge xy of Γ two edges $x'y''$ and $x''y'$). If Γ is bipartite, its double is just the union of two disjoint copies. If Γ is connected and not bipartite, then its double is connected and bipartite. If Γ has spectrum Φ , then $\Gamma \otimes K_2$ has spectrum $\Phi \cup -\Phi$.

The notation $\Gamma \times \Delta$ is used in the literature both for the Cartesian product and for the Kronecker product of two graphs. We avoid it here.

1.4.8 Strong products

Given graphs Γ and Δ with vertex sets V and W , respectively, their *strong product* $\Gamma \boxtimes \Delta$ is the graph with vertex set $V \times W$, where two distinct vertices (v, w) and (v', w') are adjacent whenever v and v' are equal or adjacent in Γ , and w and w' are equal or adjacent in Δ . If A_Γ and A_Δ are the adjacency matrices of Γ and Δ , then $((A_\Gamma + I) \otimes (A_\Delta + I)) - I$ is the adjacency matrix of $\Gamma \boxtimes \Delta$. It follows that the eigenvalues of $\Gamma \boxtimes \Delta$ are the numbers $(\theta + 1)(\eta + 1) - 1$, where θ and η run through the eigenvalues of Γ and Δ , respectively.

Note that the edge set of the strong product of Γ and Δ is the union of the edge sets of the Cartesian product and the Kronecker product of Γ and Δ .

For example, $K_{m+n} = K_m \boxtimes K_n$.

1.4.9 Cayley graphs

Let G be an Abelian group and $S \subseteq G$. The *Cayley graph* on G with difference set S is the (directed) graph Γ with vertex set G and edge set $E = \{(x, y) \mid y - x \in S\}$. Now Γ is regular with in- and outvalency $|S|$. The graph Γ will be undirected when $S = -S$.

It is easy to compute the spectrum of finite Cayley graphs (on an Abelian group). Let χ be a character of G , that is, a map $\chi : G \rightarrow \mathbb{C}^*$ such that $\chi(x+y) = \chi(x)\chi(y)$. Then $\sum_{y \sim x} \chi(y) = (\sum_{s \in S} \chi(s))\chi(x)$, so the vector $(\chi(x))_{x \in G}$ is a right eigenvector of the adjacency matrix A of Γ with eigenvalue $\chi(S) := \sum_{s \in S} \chi(s)$. The $n = |G|$ distinct characters give independent eigenvectors, so one obtains the entire spectrum in this way.

For example, the directed pentagon (with in- and outvalency 1) is a Cayley graph for $G = \mathbb{Z}_5$ and $S = \{1\}$. The characters of G are the maps $i \mapsto \zeta^i$ for some fixed fifth root of unity ζ . Hence the directed pentagon has spectrum $\{\zeta \mid \zeta^5 = 1\}$.

The undirected pentagon (with valency 2) is the Cayley graph for $G = \mathbb{Z}_5$ and $S = \{-1, 1\}$. The spectrum of the pentagon becomes $\{\zeta + \zeta^{-1} \mid \zeta^5 = 1\}$, that is, consists of 2 and $\frac{1}{2}(-1 \pm \sqrt{5})$ (both with multiplicity 2).

1.5 Decompositions

Here we present two nontrivial applications of linear algebra to graph decompositions.