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Cooperative Control Design

A Systematic, Passivity-Based Approach

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To Xiaoqing
H.B.

To Stephanie
M.A.

To Debbie
J.T.W.

Preface

Recent advances in sensing, communication and computation technologies have enabled a group of agents, such as robots, to communicate or sense their relative information and to perform tasks in a collaborative fashion. The past few years witnessed rapidly-growing research in cooperative control technology. Applications range from space interferometry sensing to environmental monitoring, to distributed computing, and distributed target surveillance and tracking. However, the analytical techniques used in cooperative control algorithms have been disparate, and a unified theory has been wanting.

In this book, we present a passivity-based framework that allows a systematic design of scalable and decentralized cooperative control laws. This framework makes explicit the passivity properties used implicitly in existing results and simplifies the design and analysis of a complex network of agents by exploiting the network structure and inherent passivity properties of agent dynamics. As we demonstrate in the book, this passivity-based framework can be easily tailored to handle classes of cooperative control problems. Each of these problems has important applications in practice. For example, formation control and coordinated path following allow vehicles to maintain a tight relative configuration, thereby achieving effective group sensing or drag force reduction. Attitude coordination ensures that multiple spacecraft keep a precise relative attitude, which is important in space interferometry missions. Cooperative load transport enables robots to move an object around meanwhile exerting desired internal forces on the object.

To demonstrate the design flexibility offered by the passivity-based framework, we feature various adaptive redesigns. Adaptivity is important in multi-agent systems, as it implies that the agents can adjust their control according to changing conditions. A particularly useful scenario is when a group leader has the group mission plan while the remaining agents have only partial information of the plan. As the leader autonomously modifies the mission plan, it is essential that the rest of the group be able to adapt. We develop adaptive designs with which the agents recover the leader's mission plan. Another scenario that requires adaptivity is when uncertainty exists in agents' controls, such as wind and viscous damping. In this case, the agents should adapt their control laws to simultaneously accommodate this uncertainty and achieve the required task. We illustrate this scenario with adaptive designs for agreement of multiple Euler-Lagrange systems.

The intention of this book is to summarize in a coherent manner the authors' recent research in passivity-based cooperative control of multi-agent systems. Related passivity approaches are applicable to other interconnected systems, such as data communication networks, biomolecular systems, building heating, ventilation, and air conditioning (HVAC) systems, and power networks, though these applications are outside the scope of this book. The organization of this book is as follows:
Chapter 1 introduces cooperative control and presents the necessary background in graph theory and passivity.
Chapter 2 presents the passivity-based approach to cooperative control and applies it to the agreement problem. We illustrate a position-based formation control problem that can be transformed into the agreement problem. We also study a distance-based formation control under the same passivity-based framework and compare it with the position-based formation control.
Chapters 3 and 4 consider the situation where only one agent possesses the group reference velocity information and develop adaptive designs that recover such information for the other agents. Chapter 3 adopts an internal model approach while Chapter 4 parameterizes the reference velocity with time-varying basis functions. These two adaptive schemes illustrate the design flexibility offered by the passivity-based approach.
Chapter 5 investigates attitude agreement of multiple rigid bodies. We adapt the results in Chapters 2 and 4 to a similar passivity-based framework that addresses the agreement of agents in $SO(3)$.
Chapter 6 studies agreement problem of multiple Euler-Lagrange systems. In particular, we consider the case where uncertainty exists in agents' dynamics. We present two adaptive designs that ensure the agreement in the presence of uncertainty.
Chapter 7 presents a synchronized path following problem, where the agents achieve tracking of a desired formation by synchronizing path parameters. The synchronization of the path parameters is solved by the passivity-based agreement designs in Chapter 2.
Chapter 8 studies cooperatively transporting a common load by multiple robots. We formulate this problem in a similar fashion to the position-based formation control in Chapter 2 and validate it with experiments.
Chapter 9 provides an investigation of robustness properties for a second order linear cooperative control system.
The Appendix includes technical proofs and tools utilized in this book.

The results in this book would not have been possible without the help of our colleagues and collaborators. We are indebted to Professors Petar Kokotović, Panagiotis Tsiotras, Derek Paley, Ming Cao, Arthur Sanderson, and A. Agung Julius for insightful comments on parts of this work. We also acknowledge Emrah Biyik, Ivar-André Ihle and Thor Fossen, whose joint work with the second author are presented in Section 4.5 and in Chapter 7. We would like to express our gratitude to Mehran Mesbahi, Wei Ren, Fumin Zhang and Sam Coogan for reviewing parts of the manuscript and for providing valuable feedback. We also thank our editors Steven Elliot and Andrew Leigh for their interest and efforts in publishing this book. The first author is grateful to Professors Kevin Lynch and Randy Freeman for offering him a precious opportunity to continue working on cooperative control as a research fellow at Northwestern University and for their constant guidance and support. He also thanks John Wason and Karen Chapin for help in experimental implementation, and Aranya Chakrabortty, Fabio Morbidi, Florian Dörfler, Chinpei Tang, and Tiejun Zhang for helpful discussions during the preparation of the book.

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Contents

Chapter 1
Introduction

1.1 What Is Cooperative Control?

Multiple robots or sensors have a number of advantages over a single agent, including robustness to failures of individual agents, reconfigurability, and the ability to perform challenging tasks such as environmental monitoring, target localization, that cannot be achieved by a single agent. A cooperative control system consists of a group of autonomous agents with sensing or communication capabilities, for example, robots with camera sensors and vehicles with communication devices. The goal of the group is to achieve prescribed agent and group behaviors using only local information available to each agent from sensing or communication devices. Such local information may include relative configuration and motion obtained from sensing or communication between agents, agent's sensor measurements, and so on. The relative sensing and communication dictates the architecture of information flow between agents. Thus, a cooperative system has four basic elements: group objective, agents, information topology, and control algorithms governing the motion of the agents.

Examples of group objectives in cooperative control include flocking, schooling, cohesion, guarding, escorting [112, 98, 131, 76, 51, 107], agreement [110, 75, 102, 63, 10, 103, 5], vehicle formation maintenance [124, 47, 36, 5], gradient climbing in an environmental field [8, 98, 27, 18], cooperative load transport [14, 145, 106, 129, 128], distributed estimation and optimal sensing [97, 99, 86, 153, 33], source seeking [151] and coverage control [34, 78]. Some of the cooperative control objectives involve only relative variables (e.g., relative positions) between agents while others depend on absolute variables (e.g., inertial positions) of agents. We illustrate below some examples of cooperative control.

Formation Control. A major focus in cooperative control is formation stability, where the group objective is to stabilize the relative distances/positions between the agents to prescribed desired values. Formation maintenance finds natural applications in coverage control, drag force reduction, and space interferometry. We take space interferometry as a motivating example. As shown in Fig. 1.1, space in-

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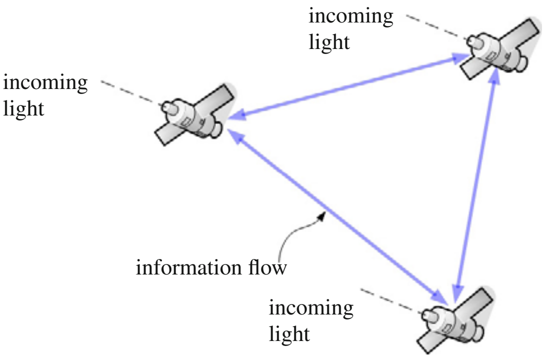

Fig. 1.1 Space interferometry using multiple spacecraft. Spacecraft must maintain precise relative formation and the same attitude towards the incoming light to generate an interferometry pattern. The information flow between spacecraft is setup by optical sensors, which measure relative positions between spacecraft.

terferometry uses multiple spacecraft pointing towards an object of interest. Each spacecraft collects light from the object. If the relative positions between spacecraft are maintained precisely, an interferometry pattern is generated and measurement of the magnitude and phase of this pattern can be obtained by coherently mixing the light from individual spacecraft [73, 118, 123]. Multiple such measurements allow reconstruction of the image of the object, which has a much finer resolution than any single spaceborne telescope achieves. In space, position information of individual spacecraft in the earth frame is imprecise or unavailable, whereas relative position between spacecraft can be measured precisely by optical sensors [74]. Thus, maintaining spacecraft formation in space may make use of only relative position information while maneuvering the formation to point towards objects of interest.

Agreement. In the agreement problem, the group objective is the convergence of distributed variables of interest (agents' positions or headings, phase of oscillations, etc.) to a common value. Given different contexts, the agreement problem is also called *consensus*, or *synchronization*. As shown in Fig. 1.1, space interferometry requires spacecraft to align their attitudes with each other, which is an agreement problem. Agreement problem also has potential applications in schooling and flocking in distributed robotics and biological systems [100, 112, 127, 24], distributed estimation and sensor fusion [99, 104], fire surveillance [25] and distributed computing [148, 140, 16], among others.

Optimal Sensing. The group objective for optimal sensing is to optimally place the agents' positions so that certain meaningful utility functions are maximized. Examples of utility functions include probability of detecting a target [34, 78] and information obtained from a sensor network [86, 33]. In this case, the utility functions usually depend on the absolute positions of the agents.

Most cooperative control problems concern coordinated motion of agents in different scenarios. Therefore, agent dynamics become important in achieving different group objectives. Small mobile sensors or robots can be controlled by directly manipulating their velocities. Such agents are commonly modeled by a first order kinematic model. Depending on the group objective, the agent model can be a simple integrator or a set of integrators subject to nonholonomic constraints, such as a unicycle model. If the group objective is to control the position of the sensor to improve sensing capability, it may suffice to model the sensors as massless points with single integrator kinematics. When the velocities of agents are not directly manipulatable or the masses of the agents are significant, double integrator agent dynamics are more appropriate. In numerous applications, the attitudes of the agents play an important role, which means that agent models in Euler-Lagrangian form or in Hamiltonian form may be required.

To achieve the group objective, each agent may need information from other agents. If agent i has access to agent j's information, the information of agent j flows to agent i and agent j is a neighbor of agent i. The abstract structure of the information flows in the group is then represented as a graph, where each agent is a node and the information flows between them are represented as links [1]. In many applications, the information flow between agents is achieved by direct sensing or communication. For example, to control the relative distances between the agents, the agents obtain their relative distance information by sensing or by communicating their inertial positions. In some applications, the information flow is realized through a physical medium. For example, consider a group of agents transporting a payload. By interacting with the payload, the agents can obtain the relative information between them without explicit communication. We will illustrate such an example in Chapter 8.

One of the main challenges in cooperative control design is to achieve prescribed group objectives by *distributed* feedback laws. The distributed laws make use of information available only to individual agents. Such information includes the information flow from neighbors, and sensor measurements from agent itself. Take the agreement problem as an example. When modeled as a first order integrator, the agent can aggregate the differences between its own state and its neighbors' and take that aggregated value as a feedback control. In this case, the control algorithm is distributed since it only employs information from neighboring agents. If the control algorithm and the agent model are both linear, stability can be analyzed by examining the eigenvalues of the closed-loop system matrix with the help of algebraic graph theory. This approach leads to simple stability criteria for the agreement problem, e.g., [63, 111, 102, 103, 47, 76, 87].

However, for some applications of cooperative control, only nonlinear algorithms can achieve the objective. Consider the following formation control problem: The group objective is to stabilize relative distances (the Euclidean norms of relative positions) between agents to desired values. In this case, the desired equilibria are spheres, which are compact sets containing more than one point. When each agent is modeled as a linear system, such as a double integrator, there is no linear feedback law globally stabilizing the desired equilibria. This is because a linear agent model with linear feedback results in a linear system, whose equilibria can simply

[1] A brief introduction to graph theory will be presented later in this chapter.

be a point or a subspace. Thus, only nonlinear feedback laws may solve this formation control problem. Indeed, most of the formation control algorithms have been proposed in the form of nonlinear artificial attraction and repulsion forces between neighboring agents. The design and analysis of such rules make use of graph theory and potential function methods.

1.2 What Is in This Book?

For different cooperative control problems, there are different control design methods. In this book, we introduce a *unifying* passivity-based framework for cooperative control problems. Under this passivity-based framework, we develop robust, adaptive, and scalable design techniques that address a broad class of cooperative control problems, including the formation control and the agreement problem discussed above.

This framework makes explicit the passivity properties used implicitly in the Lyapunov analysis of several earlier results, including [131, 98, 102], and simplifies the design and analysis of a complex network of agents by exploiting the network structure and inherent passivity properties of agent dynamics. With this simplification, the passivity approach further overcomes the simplifying assumptions of existing designs and offers numerous advantages, including:

1. Admissibility of complex and heterogenous agent dynamics: Unlike some of the existing cooperative control literature where the agent is modeled as a point robot, the passivity approach allows high order and nonlinear dynamics, including Lagrangian and Hamiltonian systems. As illustrated in Chapter 5, attitude coordination among multiple rigid bodies can be studied under this passivity framework. In this case, the agent dynamics are in the Hamiltonian form. Chapter 6 discusses the agreement of multiple Lagrangian systems. The passivity approach is further applicable to *heterogenous* systems in which the agent dynamics and parameters, such as masses, dampings, vary across the group.

2. Design flexibility, robustness and adaptivity: The passivity approach abstracts the common core of several multi-agent coordination problems, such as formation stabilization, group agreement, and attitude coordination. Because passivity involves only input-output variables, it has inherent robustness to unknown model parameters. Since passivity is closely related to Lyapunov stability, this passivity approach lends itself to systematic adaptive designs that enhance robustness of cooperative systems. Such design flexibility and adaptivity will be demonstrated in this book by the adaptive designs in Chapters 3, 4 and 6.

3. Modularity and scalability: The passivity framework yields decentralized controllers which allow the agents to make decisions based on relative information with respect to their neighbors, such as relative distance. A key advantage of the passivity-based design is its *modularity*, which means that the control laws do not rely on the knowledge of number of other agents, the communication structure of the network, or any other global network parameters.

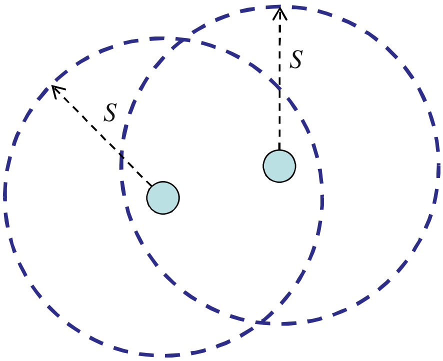

Fig. 1.2 If the sensing/communication ranges for both robots are chosen to be the same and one agent is within the sensing/communication range of the other agent, the information flow between them is symmetric. $S > 0$ denotes the sensing or communication radius.

Our major assumptions for this passivity-based approach are:

Bidirectional Information Topology: Control algorithms with bidirectional information topology tend to have inherent stability properties as we explicate with the help of passivity arguments in this book. Although directional information topology can render stability for first order agents [103, 109], it may lead to instability for high order agents, as we illustrate in Example 2.3 in Chapter 2.

Bidirectional information topology also appears naturally in a number of cooperative control applications. For example, as shown in Fig. 1.2, the information topology of agents with the same sensing range can be modeled as bidirectional. In the load transport problem studied in Chapter 8, the agents exert force on the payload and receive reaction forces from the payload. The exerted force and the reaction force contain implicitly the relative motion information between the agents and the payload. Thus, the information flows are bidirectional.

Static Information Topology: Our design assumes that the information topology remains unchanged. This is not a restrictive assumption since in most practical situations, the information topology remains static for a certain period of time. If that period of time is long enough, by standard dwell time arguments [91, 56], the closed-loop system remains stable. Note that for first order linear consensus protocols, such as those studied in [103, 63], robustness to arbitrary switching topology has been justified. However, for higher order systems, it is well known that switching may lead to instability [80, 81]. We will show that for first order protocols, the passivity-based framework can handle a broad class of switching topology whereas for higher order cooperative systems, topology switching improperly may result in instability.

1.3 Notation and Definition

• We denote by $\mathbb{R}$ and by $\mathbb{C}$ the set of real numbers and complex numbers, respectively. The notation $\mathbb{R}_{\geq 0}$ denotes the set of all real nonnegative numbers. The real part and the imaginary part of a complex number $x \in \mathbb{C}$ are given by $Re[x]$ and $Im[x]$, respectively.
• All the vectors in this book are column vectors. The set of p by 1 real vectors is denoted by $\mathbb{R}^p$ while the set of p by q real matrices is denoted by $\mathbb{R}^{p\times q}$. The transpose of a matrix $A \in \mathbb{R}^{p\times q}$ is given by $A^T \in \mathbb{R}^{q\times p}$.
•$\mathscr{N}(A)$ and $\mathscr{R}(A)$ are the null space (kernel) and the range space of a matrix A, respectively. I_p and $\mathbf{0}_p$ denote the $p \times p$ identity and zero matrices, respectively. The $p \times q$ zero matrix is denoted by $\mathbf{0}_{p\times q}$. Likewise, 1_N and 0_N denote the $N \times 1$ vector with each entry of 1 and 0, respectively. Without confusion, we will also use 0 to denote a vector of zeros with a compatible dimension.
• The Kronecker product of matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ is defined as

$$A \otimes B := \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \in \mathbb{R}^{mp\times nq}, \tag{1.1}$$

and satisfies the properties

$$(A \otimes B)^T = A^T \otimes B^T \tag{1.2}$$

$$(A \otimes I_p)(C \otimes I_p) = (AC) \otimes I_p \tag{1.3}$$

where A and C are assumed to be compatible for multiplication.
• The maximum and minimum eigenvalues of a symmetric matrix A are denoted by $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$, respectively.
• For a vector $x \in \mathbb{R}^p$, $|x|$ denotes its 2-norm, that is $|x| = \sqrt{x^T x}$.
• The norm of a matrix A is defined as its induced norm $||A|| = \sqrt{\lambda_{\max}(A^T A)}$.
• We use the notation $\mathrm{diag}\{K_1, K_2, \cdots, K_n\}$ to denote the block diagonal matrix

$$\begin{pmatrix} K_1 & \mathbf{0}_{p\times q} & \cdots & \mathbf{0}_{p\times q} \\ \mathbf{0}_{p\times q} & K_2 & \cdots & \mathbf{0}_{p\times q} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{p\times q} & \mathbf{0}_{p\times q} & \cdots & K_n \end{pmatrix} \tag{1.4}$$

where $K_i \in \mathbb{R}^{p\times q}$, $i = 1, \cdots, n$.
• The notation $K = K^T > 0$ means that K is a symmetric positive definite matrix while $k > 0$ implies k is a positive scalar.
• Given a vector $v \in \mathbb{R}^3$, the cross product $v\times$ is a linear operator, and can be represented in a coordinate frame as left-multiplication by the skew-symmetric matrix:

$$\widehat{v} = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} \tag{1.5}$$

where $(v_1,\ v_2,\ v_3)$ are the components of v in the given coordinate frame. The inverse operation of cross product is given by $^\vee$, that is

$$(\widehat{v})^\vee = v. \tag{1.6}$$

• For the coordinate frame representation of a vector, the leading superscript indicates the reference frame while the subscript i denotes the agent i. The superscript d means the desired value. As an illustration, ${}^j v_i^d$ means the desired velocity of the ith agent in the jth frame.
• A function is said to be C^k if its partial derivatives exist and are continuous up to order k.
• Given a C^2 function $P : \mathbb{R}^p \to \mathbb{R}$ we denote by ∇P its gradient vector, and by $\nabla^2 P$ its Hessian matrix.
• A function $\alpha : [0,a) \to \mathbb{R}_{\geq 0}$ is of class K if it is continuous, strictly increasing and satisfies $\alpha(0) = 0$. It is said to belong to class K_∞ if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class KL if, for each fixed s, the function $\beta(r,s)$ belongs to class K with respect to r and, for each fixed r, the function $\beta(r,s)$ is decreasing with respect to s and $\beta(r,s) \to 0$ as $s \to \infty$. An example of class KL functions is shown in Fig. 1.3.

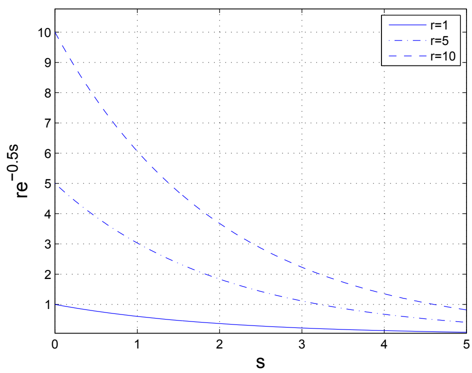

Fig. 1.3 The function $\beta(r,s) = re^{-0.5s}$ is of class KL because for fixed r, $re^{-0.5s}$ is decreasing and converges to zero as s converges to ∞ and for fixed s, $re^{-0.5s}$ is monotonically increasing with respect to r.

• The system $\dot{x} = f(x,u)$ is said to be Input-to-State Stable (ISS) [125, 126] if there exist functions $\beta \in KL$, $\rho \in K$ such that for any initial state $x(t_0)$ and any bounded input $u(t)$, the solution $x(t)$ exists for all $t \geq 0$ and satisfies