Kendall Atkinson
Weimin Han

TEXTS IN APPLIED MATHEMATICS 39

Theoretical Numerical Analysis

A Functional Analysis Framework

Third Edition

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A Functional Analysis Framework

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Dedicated to

Daisy and Clyde Atkinson
Hazel and Wray Fleming

and

Daqing Han, Suzhen Qin
Huidi Tang, Elizabeth and Michael

Series Preface

Mathematics is playing an ever more important role in the physical and biological sciences, provoking a blurring of boundaries between scientific disciplines and a resurgence of interest in the modern as well as the classical techniques of applied mathematics. This renewal of interest, both in research and teaching, has led to the establishment of the series: *Texts in Applied Mathematics* (*TAM*).

The development of new courses is a natural consequence of a high level of excitement on the research frontier as newer techniques, such as numerical and symbolic computer systems, dynamical systems, and chaos, mix with and reinforce the traditional methods of applied mathematics. Thus, the purpose of this textbook series is to meet the current and future needs of these advances and to encourage the teaching of new courses.

TAM will publish textbooks suitable for use in advanced undergraduate and beginning graduate courses, and will complement the *Applied Mathematical Sciences* (*AMS*) series, which will focus on advanced textbooks and research-level monographs.

Pasadena, California — J.E. Marsden
Providence, Rhode Island — L. Sirovich
College Park, Maryland — S.S. Antman

Preface

This textbook has grown out of a course which we teach periodically at the University of Iowa. We have beginning graduate students in mathematics who wish to work in numerical analysis from a theoretical perspective, and they need a background in those "tools of the trade" which we cover in this text. In the past, such students would ordinarily begin with a one-year course in *real and complex analysis*, followed by a one or two semester course in *functional analysis* and possibly a graduate level course in *ordinary differential equations, partial differential equations*, or *integral equations*. We still expect our students to take most of these standard courses. The course based on this book allows these students to move more rapidly into a research program.

The textbook covers basic results of functional analysis, approximation theory, Fourier analysis and wavelets, calculus and iteration methods for nonlinear equations, finite difference methods, Sobolev spaces and weak formulations of boundary value problems, finite element methods, elliptic variational inequalities and their numerical solution, numerical methods for solving integral equations of the second kind, boundary integral equations for planar regions with a smooth boundary curve, and multivariable polynomial approximations. The presentation of each topic is meant to be an introduction with a certain degree of depth. Comprehensive references on a particular topic are listed at the end of each chapter for further reading and study. For this third edition, we add a chapter on multivariable polynomial approximation and we revise numerous sections from the second edition to varying degrees. A good number of new exercises are included.

The material in the text is presented in a mixed manner. Some topics are treated with complete rigour, whereas others are simply presented without proof and perhaps illustrated (e.g. the principle of uniform boundedness). We have chosen to avoid introducing a formalized framework for *Lebesgue measure and integration* and also for *distribution theory*. Instead we use standard results on the completion of normed spaces and the unique extension of densely defined bounded linear operators. This permits us to introduce the Lebesgue spaces formally and without their concrete realization using measure theory. We describe some of the standard material on measure theory and distribution theory in an intuitive manner, believing this is sufficient for much of the subsequent mathematical development. In addition, we give a number of deeper results without proof, citing the existing literature. Examples of this are the *open mapping theorem, Hahn-Banach theorem, the principle of uniform boundedness*, and a number of the results on *Sobolev spaces*.

The choice of topics has been shaped by our research program and interests at the University of Iowa. These topics are important elsewhere, and we believe this text will be useful to students at other universities as well.

The book is divided into chapters, sections, and subsections as appropriate. Mathematical relations (equalities and inequalities) are numbered by chapter, section and their order of occurrence. For example, (1.2.3) is the third numbered mathematical relation in Section 1.2 of Chapter 1. Definitions, examples, theorems, lemmas, propositions, corollaries and remarks are numbered consecutively within each section, by chapter and section. For example, in Section 1.1, Definition 1.1.1 is followed by an example labeled as Example 1.1.2.

We give exercises at the end of most sections. The exercises are numbered consecutively by chapter and section. At the end of each chapter, we provide some short discussions of the literature, including recommendations for additional reading.

During the preparation of the book, we received helpful suggestions from numerous colleagues and friends. We particularly thank P.G. Ciarlet, William A. Kirk, Wenbin Liu, and David Stewart for the first edition, B. Bialecki, R. Glowinski, and A.J. Meir for the second edition, and Yuan Xu for the third edition. It is a pleasure to acknowledge the skillful editorial assistance from the Series Editor, Achi Dosanjh.

Contents

1

Linear Spaces

Linear (or vector) spaces are the standard setting for studying and solving a large proportion of the problems in differential and integral equations, approximation theory, optimization theory, and other topics in applied mathematics. In this chapter, we gather together some concepts and results concerning various aspects of linear spaces, especially some of the more important linear spaces such as Banach spaces, Hilbert spaces, and certain function spaces that are used frequently in this work and in applied mathematics generally.

1.1 Linear spaces

A linear space is a set of elements equipped with two binary operations, called vector addition and scalar multiplication, in such a way that the operations behave linearly.

Definition 1.1.1 *Let V be a set of objects, to be called* vectors*; and let $\mathbb{K}$ be a set of scalars, either $\mathbb{R}$, the set of real numbers, or $\mathbb{C}$, the set of complex numbers. Assume there are two operations: $(u, v) \mapsto u + v \in V$ and $(\alpha, v) \mapsto \alpha v \in V$, called* addition *and* scalar multiplication *respectively, defined for any $u, v \in V$ and any $\alpha \in \mathbb{K}$. These operations are to satisfy the following rules.*

1. *$u + v = v + u$ for any $u, v \in V$ (commutative law);*
2. *$(u + v) + w = u + (v + w)$ for any $u, v, w \in V$ (associative law);*

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3. *there is an element* $0 \in V$ *such that* $0+v = v$ *for any* $v \in V$ *(existence of the zero element);*

4. *for any* $v \in V$*, there is an element* $-v \in V$ *such that* $v + (-v) = 0$ *(existence of negative elements);*

5. $1v = v$ *for any* $v \in V$*;*

6. $\alpha(\beta v) = (\alpha\beta)v$ *for any* $v \in V$*, any* $\alpha, \beta \in \mathbb{K}$ *(associative law for scalar multiplication);*

7. $\alpha(u + v) = \alpha u + \alpha v$ *and* $(\alpha + \beta)v = \alpha v + \beta v$ *for any* $u, v \in V$*, and any* $\alpha, \beta \in \mathbb{K}$ *(distributive laws).*

Then V *is called a* linear space*, or a* vector space*.*

When $\mathbb{K}$ is the set of the real numbers, V is a real linear space; and when $\mathbb{K}$ is the set of the complex numbers, V becomes a complex linear space. In this work, most of the time we only deal with real linear spaces. So when we say V is a linear space, the reader should usually assume V is a real linear space, unless explicitly stated otherwise.

Some remarks are in order concerning the definition of a linear space. From the commutative law and the associative law, we observe that to add several elements, the order of summation does not matter, and it does not cause any ambiguity to write expressions such as $u + v + w$ or $\sum_{i=1}^{n} u_i$. By using the commutative law and the associative law, it is not difficult to verify that the zero element and the negative element $(-v)$ of a given element $v \in V$ are unique, and they can be equivalently defined through the relations $v + 0 = v$ for any $v \in V$, and $(-v) + v = 0$. Below, we write $u - v$ for $u + (-v)$. This defines the subtraction of two vectors. Sometimes, we will also refer to a vector as a point.

Example 1.1.2 (a) The set $\mathbb{R}$ of the real numbers is a real linear space when the addition and scalar multiplication are the usual addition and multiplication. Similarly, the set $\mathbb{C}$ of the complex numbers is a complex linear space.

(b) Let d be a positive integer. The letter d is used generally in this work for the spatial dimension. The set of all vectors with d real components, with the usual vector addition and scalar multiplication, forms a linear space $\mathbb{R}^d$. A typical element in $\mathbb{R}^d$ can be expressed as $\boldsymbol{x} = (x_1, \dots, x_d)^T$, where $x_1, \dots, x_d \in \mathbb{R}$. Similarly, $\mathbb{C}^d$ is a complex linear space.

(c) Let $\Omega \subset \mathbb{R}^d$ be an open set of $\mathbb{R}^d$. In this work, the symbol Ω always stands for an open subset of $\mathbb{R}^d$. The set of all the continuous functions on Ω forms a linear space $C(\Omega)$, under the usual addition and scalar multiplication of functions: For $f, g \in C(\Omega)$, the function $f + g$ defined by

$$(f + g)(\boldsymbol{x}) = f(\boldsymbol{x}) + g(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega,$$

belongs to $C(\Omega)$, as does the scalar multiplication function αf defined through

$$(\alpha f)(\boldsymbol{x}) = \alpha\, f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega.$$

Similarly, $C(\overline{\Omega})$ denotes the space of continuous functions on the closed set $\overline{\Omega}$. Clearly, any $C(\overline{\Omega})$ function is continuous on Ω, and thus can be viewed as a $C(\Omega)$ function. Conversely, if $f \in C(\Omega)$ is uniformly continuous on Ω and Ω is bounded, then f can be continuously extended to $\partial\Omega$, the boundary of Ω, and the extended function belongs to $C(\overline{\Omega})$. Recall that f defined on Ω is uniformly continuous if for any $\varepsilon > 0$, there exists a $\delta = \delta(f, \varepsilon) > 0$ such that

$$|f(\boldsymbol{x}) - f(\boldsymbol{y})| < \varepsilon$$

whenever $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ with $\|\boldsymbol{x} - \boldsymbol{y}\| < \delta$. Note that a $C(\Omega)$ function can behave badly near $\partial\Omega$; consider for example $f(x) = \sin(1/x)$, $0 < x < 1$, for x near 0.

(d) A related function space is $C(D)$, containing all functions $f : D \to \mathbb{K}$ which are continuous on a general set $D \subset \mathbb{R}^d$. The arbitrary set D can be an open or closed set in $\mathbb{R}^d$, or perhaps neither; and it can be a lower dimensional set such as a portion of the boundary of an open set in $\mathbb{R}^d$. When D is a closed and bounded subset of $\mathbb{R}^d$, a function from the space $C(D)$ is necessarily bounded.

(e) For any non-negative integer m, we may define the space $C^m(\Omega)$ as the set of all the functions, which together with their derivatives of orders up to m are continuous on Ω. We may also define $C^m(\overline{\Omega})$ to be the space of all the functions, which together with their derivatives of orders up to m are continuous on $\overline{\Omega}$. These function spaces are discussed at length in Section 1.4.

(f) The space of continuous 2π-periodic functions is denoted by $C_p(2\pi)$. It is the set of all $f \in C(-\infty, \infty)$ for which

$$f(x + 2\pi) = f(x), \quad -\infty < x < \infty.$$

For an integer $k \geq 0$, the space $C_p^k(2\pi)$ denotes the set of all functions in $C_p(2\pi)$ which have k continuous derivatives on $(-\infty, \infty)$. We usually write $C_p^0(2\pi)$ as simply $C_p(2\pi)$. These spaces are used in connection with problems in which periodicity plays a major role. □

Definition 1.1.3 *A* subspace *W of the linear space V is a subset of V which is closed under the addition and scalar multiplication operations of V, i.e., for any $u, v \in W$ and any $\alpha \in \mathbb{K}$, we have $u + v \in W$ and $\alpha v \in W$.*

It can be verified that W itself, equipped with the addition and scalar multiplication operations of V, is a linear space.

Example 1.1.4 In the linear space $\mathbb{R}^3$,

$$W = \{\boldsymbol{x} = (x_1, x_2, 0)^T \mid x_1, x_2 \in \mathbb{R}\}$$

is a subspace, consisting of all the vectors on the x_1x_2-plane. In contrast,

$$\widehat{W} = \{\boldsymbol{x} = (x_1, x_2, 1)^T \mid x_1, x_2 \in \mathbb{R}\}$$

is not a subspace. Nevertheless, we observe that $\widehat{W}$ is a translation of the subspace W,

$$\widehat{W} = \boldsymbol{x}_0 + W$$

where $\boldsymbol{x}_0 = (0, 0, 1)^T$. The set $\widehat{W}$ is an example of an *affine* set. □

Given vectors $v_1, \ldots, v_n \in V$ and scalars $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$, we call

$$\sum_{i=1}^{n} \alpha_i v_i = \alpha_1 v_1 + \cdots + \alpha_n v_n$$

a *linear combination* of $v_1, \ldots, v_n$. It is meaningful to remove "redundant" vectors from the linear combination. Thus we introduce the concepts of linear dependence and independence.

Definition 1.1.5 *We say* $v_1, \ldots, v_n \in V$ *are* linearly dependent *if there are scalars* $\alpha_i \in \mathbb{K}$, $1 \le i \le n$, *with at least one* α_i *nonzero such that*

$$\sum_{i=1}^{n} \alpha_i v_i = 0. \tag{1.1.1}$$

We say $v_1, \ldots, v_n \in V$ *are* linearly independent *if they are not linearly dependent, in other words, if* (1.1.1) *implies* $\alpha_i = 0$ *for* $i = 1, 2, \ldots, n$.

We observe that $v_1, \ldots, v_n$ are linearly dependent if and only if at least one of the vectors can be expressed as a linear combination of the rest of the vectors. In particular, a set of vectors containing the zero element is always linearly dependent. Similarly, $v_1, \ldots, v_n$ are linearly independent if and only if none of the vectors can be expressed as a linear combination of the rest of the vectors; in other words, none of the vectors is "redundant".

Example 1.1.6 In $\mathbb{R}^d$, d vectors $\boldsymbol{x}^{(i)} = (x_1^{(i)}, \ldots, x_d^{(i)})^T$, $1 \le i \le d$, are linearly independent if and only if the determinant

$$\begin{vmatrix} x_1^{(1)} & \cdots & x_1^{(d)} \\ \vdots & \ddots & \vdots \\ x_d^{(1)} & \cdots & x_d^{(d)} \end{vmatrix}$$

is nonzero. This follows from a standard result in linear algebra. The condition (1.1.1) is equivalent to a homogeneous system of linear equations, and a standard result of linear algebra says that this system has $(0, \ldots, 0)^T$ as its only solution if and only if the above determinant is nonzero. □

Example 1.1.7 Within the space $C[0,1]$, the vectors $1, x, x^2, \ldots, x^n$ are linearly independent. This can be proved in several ways. Assuming

$$\sum_{j=0}^{n} \alpha_j x^j = 0, \quad 0 \le x \le 1,$$

we can form its first n derivatives. Setting $x = 0$ in this polynomial and its derivatives will lead to $\alpha_j = 0$ for $j = 0, 1, \ldots, n$. $\square$

Definition 1.1.8 *The* span *of $v_1, \ldots, v_n \in V$ is defined to be the set of all the linear combinations of these vectors:*

$$\operatorname{span}\{v_1, \ldots, v_n\} = \left\{ \sum_{i=1}^{n} \alpha_i v_i \;\middle|\; \alpha_i \in \mathbb{K},\ 1 \le i \le n \right\}.$$

Evidently, span $\{v_1, \ldots, v_n\}$ is a linear subspace of V. Most of the time, we apply this definition for the case where $v_1, \ldots, v_n$ are linearly independent.

Definition 1.1.9 *A linear space V is said to be* finite dimensional *if there exists a finite maximal set of independent vectors $\{v_1, \ldots, v_n\}$; i.e., the set $\{v_1, \ldots, v_n\}$ is linearly independent, but $\{v_1, \ldots, v_n, v_{n+1}\}$ is linearly dependent for any $v_{n+1} \in V$. The set $\{v_1, \ldots, v_n\}$ is called a* basis *of the space. If such a finite basis for V does not exist, then V is said to be* infinite dimensional.

We see that a basis is a set of independent vectors such that any vector in the space can be written as a linear combination of them. Obviously a basis is not unique, yet we have the following important result.

Theorem 1.1.10 *For a finite dimensional linear space, every basis for V contains exactly the same number of vectors. This number is called the* dimension *of the space, denoted by* $\dim V$.

A proof of this result can be found in most introductory textbooks on linear algebra; for example, see [6, Section 5.4].

Example 1.1.11 The space $\mathbb{R}^d$ is d-dimensional. There are infinitely many possible choices for a basis of the space. A canonical basis for this space is $\{\boldsymbol{e}_i\}_{i=1}^{d}$, where $\boldsymbol{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)^T$ in which the single 1 is in component i. $\square$

Example 1.1.12 In the space $\mathbb{P}_n$ of the polynomials of degree less than or equal to n, $\{1, x, \ldots, x^n\}$ is a basis and we have $\dim(\mathbb{P}_n) = n + 1$. In the subspace

$$\mathbb{P}_{n,0} = \{p \in \mathbb{P}_n \mid p(0) = p(1) = 0\},$$

a basis is given by the functions $x\,(1-x)$, $x^2(1-x)$, $\ldots$, $x^{n-1}(1-x)$. We observe that

$$\dim(\mathbb{P}_{n,0}) = \dim(\mathbb{P}_n) - 2.$$

The difference 2 in the dimensions reflects the two zero value conditions at 0 and 1 in the definition of $\mathbb{P}_{n,0}$. $\square$

We now introduce the concept of a linear function.

Definition 1.1.13 *Let L be a function from one linear space V to another linear space W. We say L is a linear function if*
(a) *for all $u, v \in V$,*

$$L(u+v) = L(u) + L(v);$$

(b) *for all $v \in V$ and all $\alpha \in \mathbb{K}$,*

$$L(\alpha v) = \alpha L(v).$$

For such a linear function, we often write $L(v)$ for Lv.

This definition is extended and discussed extensively in Chapter 2. Other common names are *linear mapping*, *linear operator*, and *linear transformation.*

Definition 1.1.14 *Two linear spaces U and V are said to be* isomorphic, *if there is a linear bijective (i.e., one-to-one and onto) function $\ell : U \to V$.*

Many properties of a linear space U hold for any other linear space V that is isomorphic to U; and then the explicit contents of the space do not matter in the analysis of these properties. This usually proves to be convenient. One such example is that if U and V are isomorphic and are finite dimensional, then their dimensions are equal, a basis of V can be obtained from that of U by applying the mapping ℓ, and a basis of U can be obtained from that of V by applying the inverse mapping of ℓ.

Example 1.1.15 The set $\mathbb{P}_k$ of all polynomials of degree less than or equal to k is a subspace of continuous function space $C[0,1]$. An element in the space $\mathbb{P}_k$ has the form $a_0 + a_1 x + \cdots + a_k x^k$. The mapping $\ell : a_0 + a_1 x + \cdots + a_k x^k \mapsto (a_0, a_1, \ldots, a_k)^T$ is bijective from $\mathbb{P}_k$ to $\mathbb{R}^{k+1}$. Thus, $\mathbb{P}_k$ is isomorphic to $\mathbb{R}^{k+1}$. $\square$

Definition 1.1.16 *Let U and V be two linear spaces. The Cartesian product of the spaces, $W = U \times V$, is defined by*

$$W = \{w = (u,v) \mid u \in U,\ v \in V\}$$

endowed with componentwise addition and scalar multiplication

$$(u_1, v_1) + (u_2, v_2) = (u_1 + u_2, v_1 + v_2) \quad \forall\, (u_1, v_1), (u_2, v_2) \in W,$$
$$\alpha\,(u,v) = (\alpha\, u, \alpha\, v) \quad \forall\, (u,v) \in W,\ \forall\, \alpha \in \mathbb{K}.$$

It is easy to verify that W is a linear space. The definition can be extended in a straightforward way for the Cartesian product of any finite number of linear spaces.

Example 1.1.17 The real plane can be viewed as the *Cartesian product* of two real lines: $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. In general,

$$\mathbb{R}^d = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{d \text{ times}}. \qquad \square$$

Exercise 1.1.1 Show that the set of all continuous solutions of the differential equation $u''(x) + u(x) = 0$ is a finite-dimensional linear space. Is the set of all continuous solutions of $u''(x) + u(x) = 1$ a linear space?

Exercise 1.1.2 When is the set $\{v \in C[0,1] \mid v(0) = a\}$ a linear space?

Exercise 1.1.3 Show that in any linear space V, a set of vectors is always linearly dependent if one of the vectors is zero.

Exercise 1.1.4 Let $\{v_1, \ldots, v_n\}$ be a basis of an n-dimensional space V. Show that for any $v \in V$, there are scalars $\alpha_1, \ldots, \alpha_n$ such that

$$v = \sum_{i=1}^{n} \alpha_i v_i,$$

and the scalars $\alpha_1, \ldots, \alpha_n$ are uniquely determined by v.

Exercise 1.1.5 Assume U and V are finite dimensional linear spaces, and let $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_m\}$ be bases for them, respectively. Using these bases, create a basis for $W = U \times V$. Determine $\dim W$.

1.2 Normed spaces

The previous section is devoted to the algebraic structure of spaces. In this section, we turn to the topological structure of spaces. In numerical analysis, we need to frequently examine the closeness of a numerical solution to the exact solution. To answer the question quantitatively, we need to have a measure on the magnitude of the difference between the numerical solution and the exact solution. A norm of a vector in a linear space provides such a measure.

Definition 1.2.1 *Given a linear space V, a* norm $\|\cdot\|$ *is a function from V to $\mathbb{R}$ with the following properties.*

1. *$\|v\| \geq 0$ for any $v \in V$, and $\|v\| = 0$ if and only if $v = 0$;*

2. $\|\alpha v\| = |\alpha| \, \|v\|$ *for any* $v \in V$ *and* $\alpha \in \mathbb{K}$;
3. $\|u + v\| \le \|u\| + \|v\|$ *for any* $u, v \in V$.

The space V *equipped with the norm* $\|\cdot\|$, $(V, \|\cdot\|)$, *is called a* normed linear space *or a* normed space. *We usually say* V *is a normed space when the definition of the norm is clear from the context.*

Some remarks are in order on the definition of a norm. The three axioms in the definition mimic the principal properties of the notion of the ordinary length of a vector in $\mathbb{R}^2$ or $\mathbb{R}^3$. The first axiom says the norm of any vector must be non-negative, and the only vector with zero norm is zero. The second axiom is usually called *positive homogeneity*. The third axiom is also called the *triangle inequality*, which is a direct extension of the triangle inequality on the plane: The length of any side of a triangle is bounded by the sum of the lengths of the other two sides. With the definition of a norm, we can use the quantity $\|u - v\|$ as a measure for the distance between u and v.

Definition 1.2.2 *Given a linear space* V, *a* semi-norm $|\cdot|$ *is a function from* V *to* $\mathbb{R}$ *with the properties of a norm except that* $|v| = 0$ *does not necessarily imply* $v = 0$.

One place in this work where the notion of a semi-norm plays an important role is in estimating the error of polynomial interpolation.

Example 1.2.3 (a) For $\boldsymbol{x} = (x_1, \dots, x_d)^T$, the formula

$$\|\boldsymbol{x}\|_2 = \left(\sum_{i=1}^{d} x_i^2 \right)^{1/2} \tag{1.2.1}$$

defines a norm in the space $\mathbb{R}^d$ (Exercise 1.2.6), called the *Euclidean norm*, which is the usual norm for the space $\mathbb{R}^d$. When $d = 1$, the norm coincides with the absolute value: $\|x\|_2 = |x|$ for $x \in \mathbb{R}$.
(b) More generally, for $1 \le p \le \infty$, the formulas

$$\|\boldsymbol{x}\|_p = \left(\sum_{i=1}^{d} |x_i|^p \right)^{1/p} \quad \text{for } 1 \le p < \infty, \tag{1.2.2}$$

$$\|\boldsymbol{x}\|_\infty = \max_{1 \le i \le d} |x_i| \tag{1.2.3}$$

define norms in the space $\mathbb{R}^d$ (see Exercise 1.2.6 for $p = 1, 2, \infty$, and Exercise 1.5.7 for other values of p). The norm $\|\cdot\|_p$ is called the *p-norm*, and $\|\cdot\|_\infty$ is called the *maximum* or *infinity norm*. It can be shown that

$$\|\boldsymbol{x}\|_\infty = \lim_{p \to \infty} \|\boldsymbol{x}\|_p$$

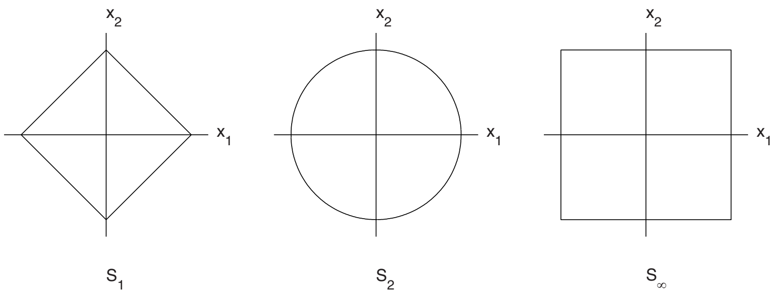

FIGURE 1.1. The unit circle $S_p = \{\boldsymbol{x} \in \mathbb{R}^2 \mid \|\boldsymbol{x}\|_p = 1\}$ for $p = 1, 2, \infty$

either directly or by using the inequality (1.2.6) given below. Again, when $d = 1$, all these norms coincide with the absolute value: $\|x\|_p = |x|$, $x \in \mathbb{R}$. Over $\mathbb{R}^d$, the most commonly used norms are $\|\cdot\|_p$, $p = 1, 2, \infty$. The unit circle in $\mathbb{R}^2$ for each of these norms is shown in Figure 1.2. □

Example 1.2.4 For $p \in [1, \infty]$, the space ℓ^p is defined as

$$\ell^p = \{v = (v_n)_{n \ge 1} \mid \|v\|_{\ell^p} < \infty\} \tag{1.2.4}$$

with the norm

$$\|v\|_{\ell^p} = \begin{cases} \left(\sum_{n=1}^{\infty} |v_n|^p \right)^{1/p} & \text{if } p < \infty, \\ \sup_{n \ge 1} |v_n| & \text{if } p = \infty. \end{cases}$$

Proof of the triangle inequality of the norm $\|\cdot\|_{\ell^p}$ is the content of Exercise 1.5.11. □

Example 1.2.5 (a) The standard norm for $C[a, b]$ is the *maximum norm*

$$\|f\|_\infty = \max_{a \le x \le b} |f(x)|, \quad f \in C[a, b].$$

This is also the norm for $C_p(2\pi)$ (with $a = 0$ and $b = 2\pi$), the space of continuous 2π-periodic functions introduced in Example 1.1.2 (f).
(b) For an integer $k > 0$, the standard norm for $C^k[a, b]$ is

$$\|f\|_{k,\infty} = \max_{0 \le j \le k} \|f^{(j)}\|_\infty, \quad f \in C^k[a, b].$$

This is also the standard norm for $C_p^k(2\pi)$. □

With the notion of a norm for V we can introduce a topology for V, and speak about *open* and *closed* sets in V.

Definition 1.2.6 *Let $(V, \|\cdot\|)$ be a normed space. Given $v_0 \in V$ and $r > 0$, the sets*

$$B(v_0, r) = \{v \in V \mid \|v - v_0\| < r\},$$
$$\overline{B}(v_0, r) = \{v \in V \mid \|v - v_0\| \leq r\}$$

are called the open and closed balls centered at v_0 with radius r. When $r = 1$ and $v_0 = 0$, we have unit balls.

Definition 1.2.7 *Let $A \subset V$ be a set in a normed linear space. The set A is* open *if for every $v \in A$, there is an $r > 0$ such that $B(v, r) \subset A$. The set A is* closed *in V if its complement $V \backslash A$ is open in V.*

1.2.1 Convergence

With the notion of a norm at our disposal, we can define the important concept of convergence.

Definition 1.2.8 *Let V be a normed space with the norm $\|\cdot\|$. A sequence $\{u_n\} \subset V$ is* convergent *to $u \in V$ if*

$$\lim_{n\to\infty} \|u_n - u\| = 0.$$

We say that u is the limit of the sequence $\{u_n\}$, and write $u_n \to u$ as $n \to \infty$, or $\lim_{n\to\infty} u_n = u$.

It can be verified that any sequence can have at most one limit.

Definition 1.2.9 *A function $f : V \to \mathbb{R}$ is said to be* continuous *at $u \in V$ if for any sequence $\{u_n\}$ with $u_n \to u$, we have $f(u_n) \to f(u)$ as $n \to \infty$. The function f is said to be continuous on V if it is continuous at every $u \in V$.*

Proposition 1.2.10 *The norm function $\|\cdot\|$ is continuous.*

Proof. We need to show that if $u_n \to u$, then $\|u_n\| \to \|u\|$. This follows from the *backward triangle inequality* (Exercise 1.2.1)

$$|\,\|u\| - \|v\|\,| \leq \|u - v\| \quad \forall\, u, v \in V, \tag{1.2.5}$$

derived from the triangle inequality. $\square$

Example 1.2.11 Consider the space $V = C[0,1]$. Let $x_0 \in [0,1]$. We define the function

$$\ell_{x_0}(v) = v(x_0), \quad v \in V.$$

Assume $v_n \to v$ in V as $n \to \infty$. Then

$$|\ell_{x_0}(v_n) - \ell_{x_0}(v)| \le \|v_n - v\|_V \to 0 \quad \text{as } n \to \infty.$$

Hence, the point value function ℓ_{x_0} is continuous on $C[0,1]$. □

We have seen that on a linear space various norms can be defined. Different norms give different measures of size for a given vector in the space. Consequently, different norms may give rise to different forms of convergence.

Definition 1.2.12 *We say two norms $\|\cdot\|_{(1)}$ and $\|\cdot\|_{(2)}$ are equivalent if there exist positive constants c_1, c_2 such that*

$$c_1\|v\|_{(1)} \le \|v\|_{(2)} \le c_2\|v\|_{(1)} \quad \forall\, v \in V.$$

With two such equivalent norms, a sequence $\{u_n\}$ converges in one norm if and only if it converges in the other norm:

$$\lim_{n\to\infty} \|u_n - u\|_{(1)} = 0 \quad \Longleftrightarrow \quad \lim_{n\to\infty} \|u_n - u\|_{(2)} = 0.$$

Conversely, if each sequence converging with respect to one norm also converges with respect to the other norm, then the two norms are equivalent; proof of this statement is left as Exercise 1.2.15.

Example 1.2.13 For the norms (1.2.2)–(1.2.3) on $\mathbb{R}^d$, it is straightforward to show

$$\|\boldsymbol{x}\|_\infty \le \|\boldsymbol{x}\|_p \le d^{1/p}\|\boldsymbol{x}\|_\infty \quad \forall\, \boldsymbol{x} \in \mathbb{R}^d. \tag{1.2.6}$$

So all the norms $\|\boldsymbol{x}\|_p$, $1 \le p \le \infty$, on $\mathbb{R}^d$ are equivalent. □

More generally, we have the following well-known result. For a proof, see [15, p. 483].

Theorem 1.2.14 *Over a finite dimensional space, any two norms are equivalent.*

Thus, on a finite dimensional space, different norms lead to the same convergence notion. Over an infinite dimensional space, however, such a statement is no longer valid.

Example 1.2.15 Let V be the space of all continuous functions on $[0,1]$. For $u \in V$, in analogy with Example 1.2.3, we may define the following norms

$$\|v\|_p = \left[\int_0^1 |v(x)|^p dx\right]^{1/p}, \quad 1 \le p < \infty, \tag{1.2.7}$$

$$\|v\|_\infty = \sup_{0 \le x \le 1} |v(x)|. \tag{1.2.8}$$

Now consider a sequence of functions $\{u_n\} \subset V$, defined by

$$u_n(x) = \begin{cases} 1 - nx, & 0 \le x \le \dfrac{1}{n}, \\ 0, & \dfrac{1}{n} < x \le 1. \end{cases}$$

It is easy to show that

$$\|u_n\|_p = [n(p+1)]^{-1/p}, \quad 1 \le p < \infty.$$

Thus we see that the sequence $\{u_n\}$ converges to $u = 0$ in the norm $\|\cdot\|_p$, $1 \le p < \infty$. On the other hand,

$$\|u_n\|_\infty = 1, \quad n \ge 1,$$

so $\{u_n\}$ does not converge to $u = 0$ in the norm $\|\cdot\|_\infty$. □

As we have seen in the last example, in an infinite dimensional space, some norms are not equivalent. Convergence defined by one norm can be stronger than that by another.

Example 1.2.16 Consider again the space of all continuous functions on $[0,1]$, and the family of norms $\|\cdot\|_p$, $1 \le p < \infty$, and $\|\cdot\|_\infty$. We have, for any $p \in [1, \infty)$,

$$\|v\|_p \le \|v\|_\infty \quad \forall\, v \in V.$$

Therefore, convergence in $\|\cdot\|_\infty$ implies convergence in $\|\cdot\|_p$, $1 \le p < \infty$, but not conversely (see Example 1.2.15). Convergence in $\|\cdot\|_\infty$ is usually called *uniform convergence.* □

With the notion of convergence, we can define the concept of an infinite series in a normed space.

Definition 1.2.17 *Let $\{v_n\}_{n=1}^\infty$ be a sequence in a normed space V. Define the partial sums $s_n = \sum_{i=1}^n v_i$, $n = 1, 2, \cdots$. If $s_n \to s$ in V, then we say the series $\sum_{i=1}^\infty v_i$ converges, and write*

$$\sum_{i=1}^\infty v_i = \lim_{n \to \infty} s_n = s.$$

Definition 1.2.18 *Let $V_1 \subset V_2$ be two subsets in a normed space V. We say the set V_1 is dense in V_2 if for any $u \in V_2$ and any $\varepsilon > 0$, there is a $v \in V_1$ such that $\|v - u\| < \varepsilon$.*

Example 1.2.19 Let $p \in [1, \infty)$ and $\Omega \subset \mathbb{R}^d$ be an open bounded set. Then the subspace $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$. The subspace of all the polynomials is also dense in $L^p(\Omega)$. □

We now extend the definition of a basis to an infinite dimensional normed space.

Definition 1.2.20 *Suppose V is an infinite dimensional normed space.*
(a) *We say that V has a countably-infinite basis if there is a sequence $\{v_i\}_{i\ge1} \subset V$ for which the following is valid: For each $v \in V$, we can find scalars $\{\alpha_{n,i}\}_{i=1}^n$, $n = 1, 2, \ldots$, such that*

$$\left\| v - \sum_{i=1}^{n} \alpha_{n,i} v_i \right\| \to 0 \quad \text{as } n \to \infty.$$

The space V is also said to be separable*. The sequence $\{v_i\}_{i\ge1}$ is called a basis if any finite subset of the sequence is linearly independent.*
(b) *We say that V has a* Schauder basis $\{v_n\}_{n\ge1}$ *if for each $v \in V$, it is possible to write*

$$v = \sum_{n=1}^{\infty} \alpha_n v_n$$

as a convergent series in V for a unique choice of scalars $\{\alpha_n\}_{n\ge1}$.

We see that the normed space V is separable if it has a countable dense subset. From Example 1.2.19, we conclude that for $p \in [1, \infty)$, $L^p(\Omega)$ is separable since the set of all the polynomials with rational coefficients is countable and is dense in $L^p(\Omega)$.

From the uniqueness requirement for a Schauder basis, we deduce that $\{v_n\}$ must be independent. A normed space having a Schauder basis can be shown to be separable. However, the converse is not true; see [77] for an example of a separable Banach space that does not have a Schauder basis. In the space ℓ^2, $\{\boldsymbol{e}_j = (0, \cdots, 0, 1_j, 0, \cdots)\}_{j=1}^\infty$ forms a Schauder basis since any $\boldsymbol{x} = (x_1, x_2, \cdots) \in \ell^2$ can be uniquely written as $\boldsymbol{x} = \sum_{j=1}^\infty x_j \boldsymbol{e}_j$. It can be proved that the set $\{1, \cos nx, \sin nx\}_{n=1}^\infty$ forms a Schauder basis in $L^p(-\pi, \pi)$ for $p \in (1, \infty)$; see the discussion in Section 4.1.

1.2.2 Banach spaces

The concept of a normed space is usually too general, and special attention is given to a particular type of normed space called a *Banach space*.

Definition 1.2.21 *Let V be a normed space. A sequence $\{u_n\} \subset V$ is called a* Cauchy sequence *if*

$$\lim_{m,n\to\infty} \|u_m - u_n\| = 0.$$

Obviously, a convergent sequence is a Cauchy sequence. In other words, being a Cauchy sequence is a necessary condition for a sequence to converge. Note that in showing convergence with Definition 1.2.8, one has to know the limit, and this is not convenient in many circumstances. On the contrary, it is usually relatively easier to determine if a given sequence is a Cauchy sequence. So it is natural to ask if a Cauchy sequence is convergent. In the finite dimensional space $\mathbb{R}^d$, any Cauchy sequence is convergent. However, in a general infinite dimensional space, a Cauchy sequence may fail to converge, as is demonstrated in the next example.

Example 1.2.22 Let $\Omega \subset \mathbb{R}^d$ be a bounded open set. For $v \in C(\overline{\Omega})$ and $1 \le p < \infty$, define the p-norm

$$\|v\|_p = \left[\int_\Omega |v(\boldsymbol{x})|^p dx\right]^{1/p}. \tag{1.2.9}$$

Here, $\boldsymbol{x} = (x_1, \dots, x_d)^T$ and $dx = dx_1 dx_2 \cdots dx_d$. In addition, define the ∞-*norm* or *maximum norm*

$$\|v\|_\infty = \max_{\boldsymbol{x}\in\overline{\Omega}} |v(\boldsymbol{x})|.$$

The space $C(\overline{\Omega})$ with $\|\cdot\|_\infty$ is a Banach space, since the uniform limit of continuous functions is itself continuous.

The space $C(\overline{\Omega})$ with the norm $\|\cdot\|_p$, $1 \le p < \infty$, is not a Banach space. To illustrate this, we consider the space $C[0,1]$ and a sequence in $C[0,1]$ defined as follows:

$$u_n(x) = \begin{cases} 0, & 0 \le x \le 1/2 - 1/(2n), \\ n\,x - (n-1)/2, & 1/2 - 1/(2n) \le x \le 1/2 + 1/(2n), \\ 1, & 1/2 + 1/(2n) \le x \le 1. \end{cases}$$

Let

$$u(x) = \begin{cases} 0, & 0 \le x < 1/2, \\ 1, & 1/2 < x \le 1. \end{cases}$$

Then $\|u_n - u\|_p \to 0$ as $n \to \infty$, i.e., the sequence $\{u_n\}$ converges to u in the norm $\|\cdot\|_p$. But obviously no matter how we define $u(1/2)$, the limit function u is not continuous. □

Although a Cauchy sequence is not necessarily convergent, it does converge if it has a convergent subsequence.

Proposition 1.2.23 *If a Cauchy sequence contains a convergent subsequence, then the entire sequence converges to the same limit.*

Proof. Let $\{u_n\}$ be a Cauchy sequence in a normed space V, with a subsequence $\{u_{n_j}\}$ converging to $u \in V$. Then for any $\varepsilon > 0$, there exist positive integers n_0 and j_0 such that

$$\|u_m - u_n\| \le \frac{\varepsilon}{2} \quad \forall\, m, n \ge n_0,$$
$$\|u_{n_j} - u\| \le \frac{\varepsilon}{2} \quad \forall\, j \ge j_0.$$

Let $N = \max\{n_0, n_{j_0}\}$. Then

$$\|u_n - u\| \le \|u_n - u_N\| + \|u_N - u\| \le \varepsilon \quad \forall\, n \ge N.$$

Therefore, $u_n \to u$ as $n \to \infty$. □

Definition 1.2.24 *A normed space is said to be* complete *if every Cauchy sequence from the space converges to an element in the space. A complete normed space is called a* Banach space*.*

Example of Banach spaces include $C([a,b])$ and $L^p(a,b)$, $1 \le p \le \infty$, with their standard norms.

1.2.3 Completion of normed spaces

It is important to be able to deal with function spaces using a norm of our choice, as such a norm is often important or convenient in the formulation of a problem or in the analysis of a numerical method. The following theorem allows us to do this. A proof is discussed in [135, p. 84].

Theorem 1.2.25 *Let V be a normed space. Then there is a complete normed space W with the following properties:*
(a) *There is a subspace $\widehat{V} \subset W$ and a bijective (one-to-one and onto) linear function $\mathcal{I} : V \to \widehat{V}$ with*

$$\|\mathcal{I}v\|_W = \|v\|_V \quad \forall\, v \in V.$$

The function $\mathcal{I}$ is called an isometric isomorphism *of the spaces V and $\widehat{V}$.*
(b) *The subspace $\widehat{V}$ is dense in W, i.e., for any $w \in W$, there is a sequence $\{\widehat{v}_n\} \subset \widehat{V}$ such that*

$$\|w - \widehat{v}_n\|_W \to 0 \quad \text{as } n \to \infty.$$

The space W is called the completion *of V, and W is unique up to an isometric isomorphism.*

The spaces V and $\widehat{V}$ are generally identified, meaning no distinction is made between them. However, we also consider cases where it is important to note the distinction. An important example of the theorem is to let V be the rational numbers and W be the real numbers $\mathbb{R}$. One way in which $\mathbb{R}$ can be defined is as a set of equivalence classes of Cauchy sequences of rational numbers, and $\widehat{V}$ can be identified with those equivalence classes of Cauchy sequences whose limit is a rational number. A proof of the above theorem can be made by mimicking this commonly used construction of the real numbers from the rational numbers.

Theorem 1.2.25 guarantees the existence of a unique abstract completion of an arbitrary normed vector space. However, it is often possible, and indeed desirable, to give a more concrete definition of the completion of a given normed space; much of the subject of *real analysis* is concerned with this topic. In particular, the subject of *Lebesgue measure and integration* deals with the completion of $C(\overline{\Omega})$ under the norms of (1.2.9), $\|\cdot\|_p$ for $1 \le p < \infty$. A complete development of Lebesgue measure and integration is given in any standard textbook on real analysis; for example, see Royden [198] or Rudin [199]. We do not introduce formally and rigorously the concepts of *measurable set* and *measurable function*. Rather we think of measure theory intuitively as described in the following paragraphs. Our rationale for this is that the details of Lebesgue measure and integration can often be bypassed in most of the material we present in this text.

Measurable subsets of $\mathbb{R}$ include the standard open and closed intervals with which we are familiar. Multi-variable extensions of intervals to $\mathbb{R}^d$ are also measurable, together with countable unions and intersections of them. In particular, open sets and closed sets are measurable. Intuitively, the measure of a set $D \subset \mathbb{R}^d$ is its "length", "area", "volume", or suitable generalization; and we denote the *measure* of D by $\operatorname{meas}(D)$. For a formal discussion of measurable set, see Royden [198] or Rudin [199].

To introduce the concept of measurable function, we begin by defining a step function. A function v on a measurable set D is a *step function* if D can be decomposed into a finite number of pairwise disjoint measurable subsets $D_1, \dots, D_k$ with $v(\boldsymbol{x})$ constant over each D_j. We say a function v on D is a *measurable function* if it is the pointwise limit of a sequence of step functions. This includes, for example, all continuous functions on D.

For each such measurable set D_j, we define a *characteristic function*

$$\chi_j(\boldsymbol{x}) = \begin{cases} 1, & \boldsymbol{x} \in D_j, \\ 0, & \boldsymbol{x} \notin D_j. \end{cases}$$

A general step function over the decomposition $D_1, \dots, D_k$ of D can then be written as

$$v(\boldsymbol{x}) = \sum_{j=1}^{k} \alpha_j \chi_j(\boldsymbol{x}), \quad \boldsymbol{x} \in D \tag{1.2.10}$$