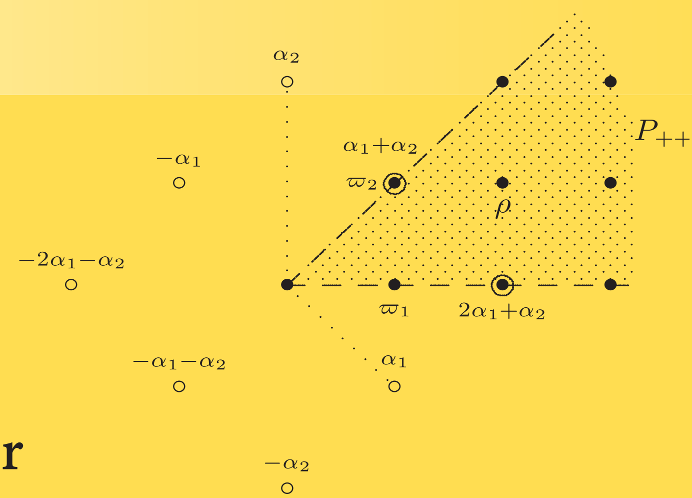
Graduate Texts in Mathematics

Roe Goodman · Nolan R. Wallach

Symmetry, Representations, and Invariants



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Contents

Preface

Symmetry, in the title of this book, should be understood as the geometry of Lie (and algebraic) group actions. The basic algebraic and analytic tools in the study of symmetry are representation and invariant theory. These three threads are precisely the topics of this book. The earlier chapters can be studied at several levels. An advanced undergraduate or beginning graduate student can learn the theory for the classical groups using only linear algebra, elementary abstract algebra, and advanced calculus, with further exploration of the key examples and concepts in the numerous exercises following each section. The more sophisticated reader can progress through the first ten chapters with occasional forward references to Chapter 11 for general results about algebraic groups. This allows great flexibility in the use of this book as a course text. The authors have used various chapters in a variety of courses; we suggest ways in which courses can be based on the book later in this preface. Finally, we have taken care to make the main theorems and applications meaningful for the reader who wishes to use the book as a reference to this vast subject.

The authors are gratified that their earlier text, *Representations and Invariants of the Classical Groups* [56], was well received. The present book has the same aim: an entry into the powerful techniques of Lie and algebraic group theory. The parts of the previous book that have withstood the authors' many revisions as they lectured from its material have been retained; these parts appear here after substantial rewriting and reorganization. The first four chapters are, in large part, newly written and offer a more direct and elementary approach to the subject. Several of the later parts of the book are also new. While we continue to look upon the classical groups as both fundamental in their own right and as important examples for the general theory, the results are now stated and proved in their natural generality. These changes justify the more accurate new title for the present book.

We have taken special care to make the book readable at many levels of detail. A reader desiring only the statement of a pertinent result can find it through the table of contents and index, and then read and study it through the examples of its use that are generally given. A more serious reader wishing to delve into a proof of the result can read in detail a more computational proof that uses special properties of the classical groups, or, perhaps in a second reading, the proof in the general case (with occasional forward references to results from later chapters). Usually, there is a third possibility of a proof using analytic methods. Some material in the earlier book, although important in its own right, has been eliminated or replaced. There are new proofs of some of the key results of the theory such as the theorem of the highest weight, the theorem on complete reducibility, the duality theorem, and the Weyl character formula. We hope that our new presentation will make these fundamental tools more accessible.

The last two chapters of the book develop, via a basic introduction to complex algebraic groups, what has come to be called *geometric invariant theory*. This includes the notion of quotient space and the representation-theoretic analysis of the regular functions on a space with an algebraic group action. A full description of the material covered in the book is given later in the preface.

When our earlier text appeared there were few other introductions to the area. The most prominent included the fundamental text of Hermann Weyl, *The Classical Groups: Their Invariants and Representations* [164] and Chevalley's *The Theory of Lie groups I* [33], together with the more recent text *Lie Algebras* by Humphreys [76]. These remarkable volumes should be on the bookshelf of any serious student of the subject. In the interim, several other texts have appeared that cover, for the most part, the material in Chevalley's classic with extensions of his analytic group theory to Lie group theory and that also incorporate much of the material in Humphrey's text. Two books with a more substantial overlap but philosophically very different from ours are those by Knapp [86] and Procesi [123]. There is much for a student to learn from both of these books, which give an exposition of Weyl's methods in invariant theory that is different in emphasis from our book. We have developed the combinatorial aspects of the subject as consequences of the representations and invariants of the classical groups. In Hermann Weyl (and the book of Procesi) the opposite route is followed: the representations and invariants of the classical groups rest on a combinatorial determination of the representations of the symmetric group. Knapp's book is more oriented toward Lie group theory.

Organization

The logical organization of the book is illustrated in the chapter and section dependency chart at the end of the preface. A chapter or section listed in the chart depends on the chapters to which it is connected by a horizontal or rising line. This chart has a central spine; to the right are the more geometric aspects of the subject and on the left the more algebraic aspects. There are several intermediate terminal nodes in this chart (such as Sections 5.6 and 5.7, Chapter 6, and Chapters 9–10) that can serve as goals for courses or self study.

Chapter 1 gives an elementary approach to the classical groups, viewed either as Lie groups or algebraic groups, without using any deep results from differentiable manifold theory or algebraic geometry. Chapter 2 develops the basic structure of the classical groups and their Lie algebras, taking advantage of the defining representations. The complete reducibility of representations of $\mathfrak{sl}(2,\mathbb{C})$ is established by a variant of Cartan's original proof. The key Lie algebra results (Cartan subalgebras and root space decomposition) are then extended to arbitrary semisimple Lie algebras.

Chapter 3 is devoted to Cartan's highest-weight theory and the Weyl group. We give a new algebraic proof of complete reducibility for semisimple Lie algebras following an argument of V. Kac; the only tools needed are the complete reducibility for $\mathfrak{sl}(2,\mathbb{C})$ and the Casimir operator. The general treatment of associative algebras and their representations occurs in Chapter 4, where the key result is the general duality theorem for locally regular representations of a reductive algebraic group. The unifying role of the duality theorem is even more prominent throughout the book than it was in our previous book.

The machinery of Chapters 1–4 is then applied in Chapter 5 to obtain the principal results in classical representations and invariant theory: the first fundamental theorems for the classical groups and the application of invariant theory to representation theory via the duality theorem.

Chapters 6, on spinors, follows the corresponding chapter from our previous book, with some corrections and additional exercises. For the main result in Chapter 7—the Weyl character formula—we give a new algebraic group proof using the radial component of the Casimir operator (replacing the proof via Lie algebra cohomology in the previous book). This proof is a differential operator analogue of Weyl's original proof using compact real forms and the integration formula, which we also present in detail. The treatment of branching laws in Chapter 8 follows the same approach (due to Kostant) as in the previous book.

Chapters 9–10 apply all the machinery developed in previous chapters to analyze the tensor representations of the classical groups. In Chapter 9 we have added a discussion of the Littlewood–Richardson rule (including the role of the $\mathbf{GL}(n,\mathbb{C})$ branching law to reduce the proof to a well-known combinatorial construction). We have removed the partial harmonic decomposition of tensor space under orthogonal and symplectic groups that was treated in Chapter 10 of the previous book, and replaced it with a representation-theoretic treatment of the symmetry properties of curvature tensors for pseudo-Riemannian manifolds.

The general study of algebraic groups over $\mathbb{C}$ and homogeneous spaces begins in Chapter 11 (with the necessary background material from algebraic geometry in Appendix A). In Lie theory the examples are, in many cases, more difficult than the general theorems. As in our previous book, every new concept is detailed with its meaning for each of the classical groups. For example, in Chapter 11 every classical symmetric pair is described and a model is given for the corresponding affine variety, and in Chapter 12 the (complexified) Iwasawa decomposition is worked out explicitly. Also in Chapter 12 a proof of the celebrated Kostant–Rallis theorem for symmetric spaces is given and every implication for the invariant theory of classical groups is explained.

This book can serve for several different courses. An introductory one-term course in Lie groups, algebraic groups, and representation theory with emphasis on the classical groups can be based on Chapters 1–3 (with reference to Appendix D as needed). Chapters 1–3 and 11 (with reference to Appendix A as needed) can be the core of a one-term introductory course on algebraic groups in characteristic zero. For students who have already had an introductory course in Lie algebras and Lie groups, Chapters 3 and 4 together with Chapters 6–10 contain ample material for a second course emphasizing representations, character formulas, and their applications. An alternative (more advanced) second-term course emphasizing the geometric side of the subject can be based on topics from Chapters 3, 4, 11, and 12. A year-long course on representations and classical invariant theory along the lines of Weyl's book would follow Chapters 1–5, 7, 9, and 10. The exercises have been revised and many new ones added (there are now more than 350, most with several parts and detailed hints for solution). Although none of the exercises are used in the proofs of the results in the book, we consider them an essential part of courses based on this book. Working through a significant number of the exercises helps a student learn the general concepts, fine structure, and applications of representation and invariant theory.

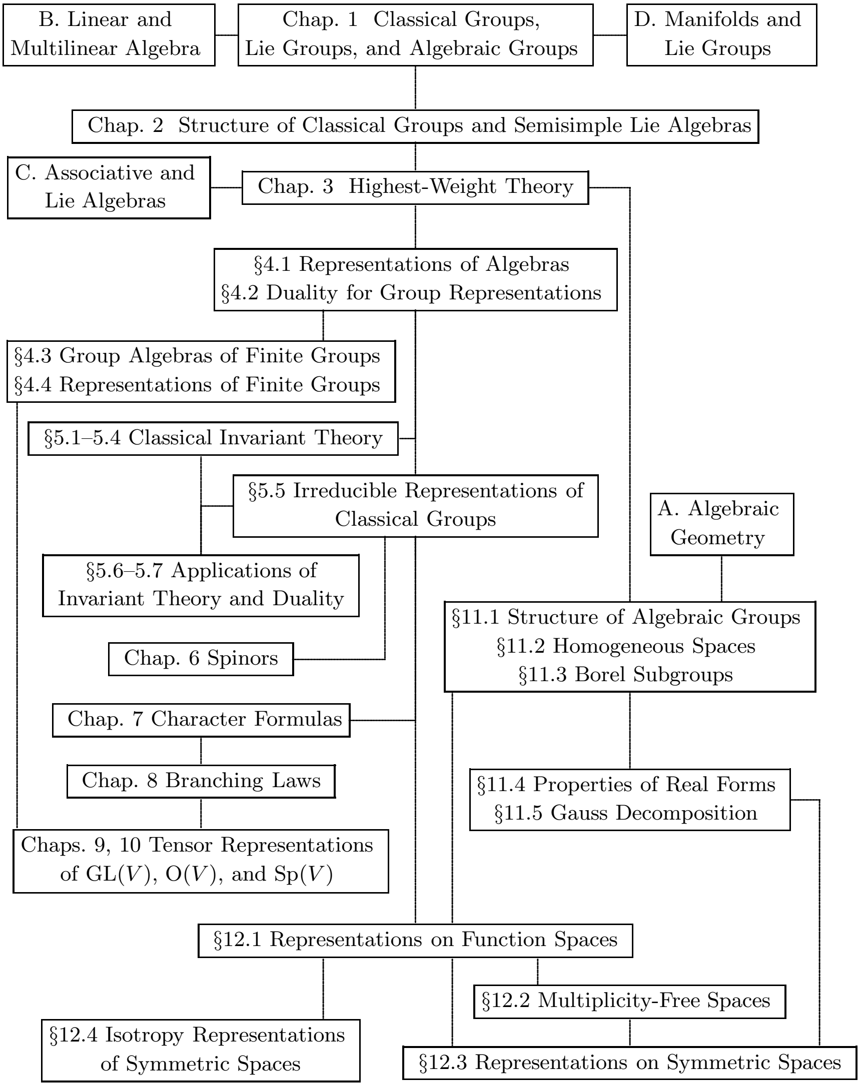
Acknowledgments

In the end-of-chapter notes we have attempted to give credits for the results in the book and some idea of the historical development of the subject. We apologize to those whose works we have neglected to cite and for incorrect attributions. We are indebted to many people for finding errors and misprints in the many versions of the material in this book and for suggesting ways to improve the exposition. In particular we would like to thank Ilka Agricola, Laura Barberis, Bachir Bekka, Enriqueta Rodríguez Carrington, Friedrich Knop, Hanspeter Kraft, Peter Landweber, and Tomasz Przebinda. Chapters of the book have been used in many courses, and the interaction with the students was very helpful in arriving at the final version. We thank them all for their patience, comments, and sharp eyes. During the first year that we were writing our previous book (1989–1990), Roger Howe gave a course at Rutgers University on basic invariant theory. We thank him for many interesting conversations on this subject.

The first-named author is grateful for sabbatical leaves from Rutgers University and visiting appointments at the University of California, San Diego; the Université de Metz; Hong Kong University; and the National University of Singapore that were devoted to this book. He thanks the colleagues who arranged programs in which he lectured on this material, including Andrzei Hulanicki and Ewa Damek (University of Wrocław), Bachir Bekka and Jean Ludwig (Université de Metz), Ngai-Ming Mok (Hong Kong University), and Eng-Chye Tan and Chen-Bo Zhu (National University of Singapore). He also is indebted to the mathematics department of the University of California, Berkeley, for its hospitality during several summers of writing the book. The second-named author would like to thank the University of California, San Diego, for sabbatical leaves that were dedicated to this project, and to thank the National Science Foundation for summer support. We thank David Kramer for a superb job of copyediting, and we are especially grateful to our editor Ann Kostant, who has steadfastly supported our efforts on the book from inception to publication.

New Brunswick, New Jersey — *Roe Goodman*
San Diego, California — *Nolan R. Wallach*
February 2009

Organization and Notation



Dependency Chart among Chapters and Sections

"O," said Maggie, pouting, "I dare say I could make it out, if I'd learned what goes before, as you have." "But that's what you just couldn't, Miss Wisdom," said Tom. "For it's all the harder when you know what goes before: for then you've got to say what Definition 3. is and what Axiom V. is." George Eliot, *The Mill on the Floss*

Some Standard Notation

$\#S$ number of elements in set S (also denoted by $\mathrm{Card}(S)$ and $|S|$)

δ_{ij} Kronecker delta (1 if $i = j$, 0 otherwise)

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ nonnegative integers, integers, rational numbers

$\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ real numbers, complex numbers, quaternions

$\mathbb{C}^\times$ multiplicative group of nonzero complex numbers

$[x]$ greatest integer $\leq x$ if x is real

$\mathbb{F}^n$ $n \times 1$ column vectors with entries in field $\mathbb{F}$

$M_{k,n}$ $k \times n$ complex matrices (M_n when $k = n$)

$M_n(\mathbb{F})$ $n \times n$ matrices with entries in field $\mathbb{F}$

$\mathbf{GL}(n,\mathbb{F})$ invertible $n \times n$ matrices with entries from field $\mathbb{F}$

I_n $n \times n$ identity matrix (or I when n understood)

$\dim V$ dimension of a vector space V

V^* dual space to vector space V

$\langle v^*, v \rangle$ natural duality pairing between V^* and V

$\mathrm{Span}(S)$ linear span of subset S in a vector space.

$\mathrm{End}(V)$ linear transformations on vector space V

$\mathbf{GL}(V)$ invertible linear transformations on vector space V

$\mathrm{tr}(A)$ trace of square matrix A

$\det(A)$ determinant of square matrix A

A^t transpose of matrix A

A^* conjugate transpose of matrix A

$\mathrm{diag}[a_1, \ldots, a_n]$ diagonal matrix

$\bigoplus V_i$ direct sum of vector spaces V_i

$\bigotimes^k V$ k-fold tensor product of vector space V (also denoted by $V^{\otimes k}$)

$S^k(V)$ k-fold symmetric tensor product of vector space V

$\bigwedge^k(V)$ k-fold skew-symmetric tensor product of vector space V

$\mathcal{O}[X]$ regular functions on algebraic set X

Other notation is generally defined at its first occurrence and appears in the index of notation at the end of the book.

Chapter 1
Lie Groups and Algebraic Groups

Abstract Hermann Weyl, in his famous book *The Classical Groups, Their Invariants and Representations* [164], coined the name *classical groups* for certain families of matrix groups. In this chapter we introduce these groups and develop the basic ideas of Lie groups, Lie algebras, and linear algebraic groups. We show how to put a Lie group structure on a closed subgroup of the general linear group and determine the Lie algebras of the classical groups. We develop the theory of complex linear algebraic groups far enough to obtain the basic results on their Lie algebras, rational representations, and Jordan–Chevalley decompositions (we defer the deeper results about algebraic groups to Chapter 11). We show that linear algebraic groups are Lie groups, introduce the notion of a *real form* of an algebraic group (considered as a Lie group), and show how the classical groups introduced at the beginning of the chapter appear as real forms of linear algebraic groups.

1.1 The Classical Groups

The *classical groups* are the groups of invertible linear transformations of finite-dimensional vector spaces over the real, complex, and quaternion fields, together with the subgroups that preserve a volume form, a bilinear form, or a sesquilinear form (the forms being nondegenerate and symmetric or skew-symmetric).

1.1.1 General and Special Linear Groups

Let $\mathbb{F}$ denote either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$, and let V be a finite-dimensional vector space over $\mathbb{F}$. The set of invertible linear transformations from V to V will be denoted by $\mathbf{GL}(V)$. This set has a group structure under composition of transformations, with identity element the identity transformation $I(x) = x$ for all $x \in V$. The group $\mathbf{GL}(V)$ is the first of the classical

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groups. To study it in more detail, we recall some standard terminology related to linear transformations and their matrices.

Let V and W be finite-dimensional vector spaces over $\mathbb{F}$. Let $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_m\}$ be bases for V and W, respectively. If $T : V \longrightarrow W$ is a linear map then

$$Tv_j = \sum_{i=1}^{m} a_{ij} w_i \qquad \text{for } j = 1, \ldots, n$$

with $a_{ij} \in \mathbb{F}$. The numbers a_{ij} are called the *matrix coefficients* or *entries* of T with respect to the two bases, and the $m \times n$ array

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is the *matrix* of T with respect to the two bases. When the elements of V and W are identified with column vectors in $\mathbb{F}^n$ and $\mathbb{F}^m$ using the given bases, then action of T becomes multiplication by the matrix A.

Let $S : W \longrightarrow U$ be another linear transformation, with U an l-dimensional vector space with basis $\{u_1, \ldots, u_l\}$, and let B be the matrix of S with respect to the bases $\{w_1, \ldots, w_m\}$ and $\{u_1, \ldots, u_l\}$. Then the matrix of $S \circ T$ with respect to the bases $\{v_1, \ldots, v_n\}$ and $\{u_1, \ldots, u_l\}$ is given by BA, with the product being the usual product of matrices.

We denote the space of all $n \times n$ matrices over $\mathbb{F}$ by $M_n(\mathbb{F})$, and we denote the $n \times n$ identity matrix by I (or I_n if the size of the matrix needs to be indicated); it has entries $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. Let V be an n-dimensional vector space over $\mathbb{F}$ with basis $\{v_1, \ldots, v_n\}$. If $T : V \longrightarrow V$ is a linear map we write $\mu(T)$ for the matrix of T with respect to this basis. If $T, S \in \mathbf{GL}(V)$ then the preceding observations imply that $\mu(S \circ T) = \mu(S)\mu(T)$. Furthermore, if $T \in \mathbf{GL}(V)$ then $\mu(T \circ T^{-1}) = \mu(T^{-1} \circ T) = \mu(\mathrm{Id}) = I$. The matrix $A \in M_n(\mathbb{F})$ is said to be *invertible* if there is a matrix $B \in M_n(\mathbb{F})$ such that $AB = BA = I$. We note that a linear map $T : V \longrightarrow V$ is in $\mathbf{GL}(V)$ if and only if its matrix $\mu(T)$ is invertible. We also recall that a matrix $A \in M_n(\mathbb{F})$ is invertible if and only if its determinant is nonzero.

We will use the notation $\mathbf{GL}(n, \mathbb{F})$ for the set of $n \times n$ invertible matrices with coefficients in $\mathbb{F}$. Under matrix multiplication $\mathbf{GL}(n, \mathbb{F})$ is a group with the identity matrix as identity element. We note that if V is an n-dimensional vector space over $\mathbb{F}$ with basis $\{v_1, \ldots, v_n\}$, then the map $\mu : \mathbf{GL}(V) \longrightarrow \mathbf{GL}(n, \mathbb{F})$ corresponding to this basis is a group isomorphism. The group $\mathbf{GL}(n, \mathbb{F})$ is called the *general linear group of rank* n.

If $\{w_1, \ldots, w_n\}$ is another basis of V, then there is a matrix $g \in \mathbf{GL}(n, \mathbb{F})$ such that

$$w_j = \sum_{i=1}^{n} g_{ij} v_i \quad \text{and} \quad v_j = \sum_{i=1}^{n} h_{ij} w_i \quad \text{for } j = 1, \ldots, n\ ,$$

with $[h_{ij}]$ the inverse matrix to $[g_{ij}]$. Suppose that T is a linear transformation from V to V, that $A = [a_{ij}]$ is the matrix of T with respect to a basis $\{v_1, \ldots, v_n\}$, and that $B = [b_{ij}]$ is the matrix of T with respect to another basis $\{w_1, \ldots, w_n\}$. Then

$$\begin{aligned} Tw_j &= T\Big(\sum_i g_{ij} v_i\Big) = \sum_i g_{ij} T v_i \\ &= \sum_i g_{ij} \Big(\sum_k a_{ki} v_k\Big) = \sum_l \Big(\sum_k \sum_i h_{lk} a_{ki} g_{ij}\Big) w_l \end{aligned}$$

for $j = 1, \ldots, n$. Thus $B = g^{-1}Ag$ is similar to the matrix A.

Special Linear Group

The special linear group $\mathbf{SL}(n, \mathbb{F})$ is the set of all elements A of $M_n(\mathbb{F})$ such that $\det(A) = 1$. Since $\det(AB) = \det(A)\det(B)$ and $\det(I) = 1$, we see that the special linear group is a subgroup of $\mathbf{GL}(n, \mathbb{F})$.

We note that if V is an n-dimensional vector space over $\mathbb{F}$ with basis $\{v_1, \ldots, v_n\}$ and if $\mu : \mathbf{GL}(V) \longrightarrow \mathbf{GL}(n, \mathbb{F})$ is the map previously defined, then the group

$$\mu^{-1}(\mathbf{SL}(n, \mathbb{F})) = \{T \in \mathbf{GL}(V) : \det(\mu(T)) = 1\}$$

is independent of the choice of basis, by the change of basis formula. We denote this group by $\mathbf{SL}(V)$.

1.1.2 Isometry Groups of Bilinear Forms

Let V be an n-dimensional vector space over $\mathbb{F}$. A bilinear map $B : V \times V \longrightarrow \mathbb{F}$ is called a *bilinear form*. We denote by $\mathbf{O}(V, B)$ (or $\mathbf{O}(B)$ when V is understood) the set of all $g \in \mathbf{GL}(V)$ such that $B(gv, gw) = B(v, w)$ for all $v, w \in V$. We note that $\mathbf{O}(V, B)$ is a subgroup of $\mathbf{GL}(V)$; it is called the *isometry group* of the form B.

Let $\{v_1, \ldots, v_n\}$ be a basis of V and let $\Gamma \in M_n(\mathbb{F})$ be the matrix with $\Gamma_{ij} = B(v_i, v_j)$. If $g \in \mathbf{GL}(V)$ has matrix $A = [a_{ij}]$ relative to this basis, then

$$B(gv_i, gv_j) = \sum_{k,l} a_{ki} a_{lj} B(v_k, v_l) = \sum_{k,l} a_{ki} \Gamma_{kl} a_{lj} .$$

Thus if A^t denotes the transposed matrix $[c_{ij}]$ with $c_{ij} = a_{ji}$, then the condition that $g \in \mathbf{O}(B)$ is that

$$\Gamma = A^t \Gamma A . \tag{1.1}$$

Recall that a bilinear form B is *nondegenerate* if $B(v, w) = 0$ for all w implies that $v = 0$, and likewise $B(v, w) = 0$ for all v implies that $w = 0$. In this case we have $\det \Gamma \neq 0$. Suppose B is nondegenerate. If $T : V \longrightarrow V$ is linear and satisfies

$B(Tv, Tw) = B(v, w)$ for all $v, w \in V$, then $\det(T) \neq 0$ by formula (1.1). Hence $T \in \mathbf{O}(B)$. The next two subsections will discuss the most important special cases of this class of groups.

Orthogonal Groups

We start by introducing the matrix groups; later we will identify these groups with isometry groups of certain classes of bilinear forms. Let $\mathbf{O}(n, \mathbb{F})$ denote the set of all $g \in \mathbf{GL}(n, \mathbb{F})$ such that $gg^t = I$. That is, $g^t = g^{-1}$. We note that $(AB)^t = B^t A^t$ and if $A, B \in \mathbf{GL}(n, \mathbb{F})$ then $(AB)^{-1} = B^{-1}A^{-1}$. It is therefore obvious that $\mathbf{O}(n, \mathbb{F})$ is a subgroup of $\mathbf{GL}(n, \mathbb{F})$. This group is called the *orthogonal group* of $n \times n$ matrices over $\mathbb{F}$. If $\mathbb{F} = \mathbb{R}$ we introduce the *indefinite orthogonal groups*, $\mathrm{O}(p, q)$, with $p + q = n$ and $p, q \in \mathbb{N}$. Let

$$I_{p,q} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$$

and define

$$\mathbf{O}(p, q) = \{g \in M_n(\mathbb{R}) \, : \, g^t I_{p,q} g = I_{p,q}\} \, .$$

We note that $\mathbf{O}(n, 0) = \mathbf{O}(0, n) = \mathbf{O}(n, \mathbb{R})$. Also, if

$$s = \begin{bmatrix} 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{bmatrix}$$

is the matrix with entries 1 on the skew diagonal ($j = n + 1 - i$) and all other entries 0, then $s = s^{-1} = s^t$ and $sI_{p,q}s^{-1} = sI_{p,q}s = sI_{p,q}s = -I_{q,p}$. Thus the map

$$\varphi : \mathbf{O}(p, q) \longrightarrow \mathbf{GL}(n, \mathbb{R})$$

given by $\varphi(g) = sgs$ defines an isomorphism of $\mathbf{O}(p, q)$ onto $\mathbf{O}(q, p)$.

We will now describe these groups in terms of bilinear forms.

Definition 1.1.1. Let V be a vector space over $\mathbb{R}$ and let M be a symmetric bilinear form on V. The form M is *positive definite* if $M(v, v) > 0$ for every $v \in V$ with $v \neq 0$.

Lemma 1.1.2. *Let V be an n-dimensional vector space over $\mathbb{F}$ and let B be a symmetric nondegenerate bilinear form over $\mathbb{F}$.*

1. *If $\mathbb{F} = \mathbb{C}$ then there exists a basis $\{v_1, \ldots, v_n\}$ of V such that $B(v_i, v_j) = \delta_{ij}$.*
2. *If $\mathbb{F} = \mathbb{R}$ then there exist integers $p, q \geq 0$ with $p + q = n$ and a basis $\{v_1, \ldots, v_n\}$ of V such that $B(v_i, v_j) = \varepsilon_i \delta_{ij}$ with $\varepsilon_i = 1$ for $i \leq p$ and $\varepsilon_i = -1$ for $i > p$. Furthermore, if we have another such basis then the corresponding integers (p, q) are the same.*

Remark 1.1.3. The basis for V in part (2) is called a *pseudo-orthonormal basis* relative to B, and the number $p-q$ is called the *signature* of the form (we will also call the pair (p,q) the signature of B). A form is positive definite if and only if its signature is n. In this case a pseudo-orthonormal basis is an orthonormal basis in the usual sense.

Proof. We first observe that if M is a symmetric bilinear form on V such that $M(v,v)=0$ for all $v\in V$, then $M=0$. Indeed, using the symmetry and bilinearity we have

$$4M(v,w)=M(v+w,\,v+w)-M(v-w,\,v-w)=0 \quad \text{for all } v,w\in V\ . \tag{1.2}$$

We now construct a basis $\{w_1,\ldots,w_n\}$ of V such that

$$B(w_i,\,w_j)=0 \quad \text{for } i\neq j \text{ and } \quad B(w_i,\,w_i)\neq 0$$

(such a basis is called an *orthogonal basis* with respect to B). The argument is by induction on n. Since B is nondegenerate, there exists a vector $w_n\in V$ with $B(w_n,w_n)\neq 0$ by (1.2). If $n=1$ we are done. If $n>1$, set

$$V'=\{v\in V\,:\,B(w_n,\,v)=0\}\ .$$

For $v\in V$ set

$$v'=v-\frac{B(v,\,w_n)}{B(w_n,\,w_n)}w_n\ .$$

Clearly, $v'\in V'$; hence $V=V'+\mathbb{F}w_n$. In particular, this shows that $\dim V'=n-1$. We assert that the form $B'=B|_{V'\times V'}$ is nondegenerate on V'. Indeed, if $v\in V'$ satisfies $B(v',\,w)=0$ for all $w\in V'$, then $B(v',\,w)=0$ for all $w\in V$, since $B(v',\,w_n)=0$. Hence $v'=0$, proving nondegeneracy of B'. We may assume by induction that there exists a B'-orthogonal basis $\{w_1,\ldots,w_{n-1}\}$ for V'. Then it is clear that $\{w_1,\ldots,w_n\}$ is a B-orthogonal basis for V.

If $\mathbb{F}=\mathbb{C}$ let $\{w_1,\ldots,w_n\}$ be an orthogonal basis of V with respect to B and let $z_i\in\mathbb{C}$ be a choice of square root of $B(w_i,\,w_i)$. Setting $v_i=(z_i)^{-1}w_i$, we then obtain the desired normalization $B(v_i,\,v_j)=\delta_{ij}$.

Now let $\mathbb{F}=\mathbb{R}$. We rearrange the indices (if necessary) so that $B(w_i,\,w_i)\geq B(w_{i+1},\,w_{i+1})$ for $i=1,\ldots,n-1$. Let $p=0$ if $B(w_1,\,w_1)<0$. Otherwise, let

$$p=\max\{i\,:\,B(w_i,\,w_i)>0\}\ .$$

Then $B(w_i,\,w_i)<0$ for $i>p$. Take z_i to be a square root of $B(w_i,\,w_i)$ for $i\leq p$, and take z_i to be a square root of $-B(w_i,\,w_i)$ for $i>p$. Setting $v_i=(z_i)^{-1}w_i$, we now have $B(v_i,\,v_j)=\varepsilon_i\delta_{ij}$.

We are left with proving that the integer p is intrinsic to B. Take any basis $\{v_1,\ldots,v_n\}$ such that $B(v_i,\,v_j)=\varepsilon_i\delta_{ij}$ with $\varepsilon_i=1$ for $i\leq p$ and $\varepsilon_i=-1$ for $i>p$. Set

$$V_+=\operatorname{Span}\{v_1,\ldots,v_p\}\ ,\qquad V_-=\operatorname{Span}\{v_{p+1},\ldots,v_n\}\ .$$

Then $V = V_+ \oplus V_-$ (direct sum). Let $\pi : V \longrightarrow V_+$ be the projection onto the first factor. We note that $B|_{V_+ \times V_+}$ is positive definite. Let W be any subspace of V such that $B|_{W\times W}$ is positive definite. Suppose that $w \in W$ and $\pi(w) = 0$. Then $w \in V_-$, so it can be written as $w = \sum_{i>p} a_i v_i$. Hence

$$B(w,w) = \sum_{i,j>p} a_i a_j B(v_i, v_j) = -\sum_{i>p} a_i^2 \le 0\,.$$

Since $B|_{W\times W}$ has been assumed to be positive definite, it follows that $w = 0$. This implies that $\pi : W \longrightarrow V_+$ is injective, and hence $\dim W \le \dim V_+ = p$. Thus p is uniquely determined as the maximum dimension of a subspace on which B is positive definite. □

The following result follows immediately from Lemma 1.1.2.

Proposition 1.1.4. *Let B be a nondegenerate symmetric bilinear form on an n-dimensional vector space V over $\mathbb{F}$.*

1. *Let $\mathbb{F} = \mathbb{C}$. If $\{v_1, \ldots, v_n\}$ is an orthonormal basis for V with respect to B, then $\mu : \mathbf{O}(V,B) \longrightarrow \mathbf{O}(n,\mathbb{F})$ defines a group isomorphism.*
2. *Let $\mathbb{F} = \mathbb{R}$. If B has signature $(p, n-p)$ and $\{v_1, \ldots, v_n\}$ is a pseudo-orthonormal basis of V, then $\mu : \mathbf{O}(V,B) \longrightarrow \mathrm{O}(p, n-p)$ is a group isomorphism.*

Here $\mu(g)$, for $g \in \mathbf{GL}(V)$, is the matrix of g with respect to the given basis.

The *special orthogonal group* over $\mathbb{F}$ is the subgroup

$$\mathbf{SO}(n,\mathbb{F}) = \mathbf{O}(n,\mathbb{F}) \cap \mathbf{SL}(n,\mathbb{F})$$

of $\mathbf{O}(n,\mathbb{F})$. The *indefinite special orthogonal groups* are the groups

$$\mathbf{SO}(p,q) = \mathbf{O}(p,q) \cap \mathbf{SL}(p+q,\mathbb{R})\,.$$

Symplectic Group

We set $J = \left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right]$ with I the $n \times n$ identity matrix. The symplectic group of rank n over $\mathbb{F}$ is defined to be

$$\mathbf{Sp}(n,\mathbb{F}) = \{g \in M_{2n}(\mathbb{F}) \,:\, g^t J g = J\}.$$

As in the case of the orthogonal groups one sees without difficulty that $\mathbf{Sp}(n,\mathbb{F})$ is a subgroup of $\mathbf{GL}(2n,\mathbb{F})$.

We will now look at the coordinate-free version of these groups. A bilinear form B is called *skew-symmetric* if $B(v,w) = -B(w,v)$. If B is skew-symmetric and nondegenerate, then $m = \dim V$ must be even, since the matrix of B relative to any basis for V is skew-symmetric and has nonzero determinant.

Lemma 1.1.5. *Let V be a $2n$-dimensional vector space over $\mathbb{F}$ and let B be a nondegenerate, skew-symmetric bilinear form on V. Then there exists a basis $\{v_1,\ldots,v_{2n}\}$ for V such that the matrix $[B(v_i,v_j)]$ equals J (call such a basis a B-*symplectic basis*).*

Proof. Let v be a nonzero element of V. Since B is nondegenerate, there exists $w\in V$ with $B(v,w)\neq 0$. Replacing w with $B(v,w)^{-1}w$, we may assume that $B(v,w)=1$. Let

$$W=\{x\in V\,:\,B(v,x)=0 \text{ and } B(w,x)=0\}\ .$$

For $x\in V$ we set $x'=x-B(v,x)w-B(x,w)v$. Then

$$B(v,x')=B(v,x)-B(v,x)B(v,w)-B(w,x)B(v,v)=0\ ,$$

since $B(v,w)=1$ and $B(v,v)=0$ (by skew symmetry of B). Similarly,

$$B(w,x')=B(w,x)-B(v,x)B(w,w)+B(w,x)B(w,v)=0\ ,$$

since $B(w,v)=-1$ and $B(w,w)=0$. Thus $V=U\oplus W$, where U is the span of v and w. It is easily verified that $B|_{U\times U}$ is nondegenerate, and so $U\cap W=\{0\}$. This implies that $\dim W=m-2$. We leave to the reader to check that $B|_{W\times W}$ also is nondegenerate.

Set $v_n=v$ and $v_{2n}=w$ with v,w as above. Since $B|_{W\times W}$ is nondegenerate, by induction there exists a B-symplectic basis $\{w_1,\ldots,w_{2n-2}\}$ of W. Set $v_i=w_i$ and $v_{n+1-i}=w_{n-i}$ for $i\leq n-1$. Then $\{v_1,\ldots,v_{2n}\}$ is a B-symplectic basis for V. □

The following result follows immediately from Lemma 1.1.5.

Proposition 1.1.6. *Let V be a $2n$-dimensional vector space over $\mathbb{F}$ and let B be a nondegenerate skew-symmetric bilinear form on V. Fix a B-symplectic basis of V and let $\mu(g)$, for $g\in\mathbf{GL}(V)$, be the matrix of g with respect to this basis. Then $\mu:\mathbf{O}(V,B)\longrightarrow\mathbf{Sp}(n,\mathbb{F})$ is a group isomorphism.*

1.1.3 Unitary Groups

Another family of classical subgroups of $\mathbf{GL}(n,\mathbb{C})$ consists of the unitary groups and special unitary groups for definite and indefinite Hermitian forms. If $A\in M_n(\mathbb{C})$ we will use the standard notation $A^*=\overline{A}^t$ for its adjoint matrix, where $\overline{A}$ is the matrix obtained from A by complex conjugating all of the entries. The *unitary group* of rank n is the group

$$\mathbf{U}(n)=\{g\in M_n(\mathbb{C})\,:\,g^*g=I\}\ .$$

The *special unitary group* is $\mathbf{SU}(n)=\mathbf{U}(n)\cap\mathbf{SL}(n,\mathbb{C})$. Let the matrix $I_{p,q}$ be as in Section 1.1.2. We define the *indefinite unitary group* of signature (p,q) to be

$$\mathbf{U}(p,q)=\{g\in M_n(\mathbb{C})\,:\,g^*I_{p,q}g=I_{p,q}\}\ .$$

The special indefinite unitary group of signature (p,q) is $\mathbf{SU}(p,q) = \mathbf{U}(p,q) \cap \mathbf{SL}(n,\mathbb{C})$.

We will now obtain a coordinate-free description of these groups. Let V be an n-dimensional vector space over $\mathbb{C}$. An $\mathbb{R}$ bilinear map $B : V \times V \longrightarrow \mathbb{C}$ (where we view V as a vector space over $\mathbb{R}$) is said to be a *Hermitian* form if it satisfies

1. $B(av,w) = aB(v,w)$ for all $a \in \mathbb{C}$ and all $v,w \in V$.
2. $B(w,v) = \overline{B(v,w)}$ for all $v,w \in V$.

By the second condition, we see that a Hermitian form is nondegenerate provided $B(v,w) = 0$ for all $w \in V$ implies that $v = 0$. The form is said to be *positive definite* if $B(v,v) > 0$ for all $v \in V$ with $v \neq 0$. (Note that if M is a Hermitian form, then $M(v,v) \in \mathbb{R}$ for all $v \in V$.) We define $\mathbf{U}(V,B)$ (also denoted by $\mathbf{U}(B)$ when V is understood) to be the group of all elements $g \in \mathbf{GL}(V)$ such that $B(gv,gw) = B(v,w)$ for all $v,w \in V$. We call $\mathbf{U}(B)$ the *unitary group* of B.

Lemma 1.1.7. *Let V be an n-dimensional vector space over $\mathbb{C}$ and let B be a nondegenerate Hermitian form on V. Then there exist an integer p, with $n \geq p \geq 0$, and a basis $\{v_1,\dots,v_n\}$ of V such that $B(v_i,v_j) = \varepsilon_i\delta_{ij}$, with $\varepsilon_i = 1$ for $i \leq p$ and $\varepsilon_i = -1$ for $i > p$. The number p depends only on B and not on the choice of basis.*

The proof of Lemma 1.1.7 is almost identical to that of Lemma 1.1.2 and will be left as an exercise.

If V is an n-dimensional vector space over $\mathbb{C}$ and B is a nondegenerate Hermitian form on V, then a basis as in Lemma 1.1.7 will be called a *pseudo-orthonormal basis* (if $p = n$ then it is an *orthonormal basis* in the usual sense). The pair $(p, n-p)$ will be called the *signature* of B. The following result is proved in exactly the same way as the corresponding result for orthogonal groups.

Proposition 1.1.8. *Let V be a finite-dimensional vector space over $\mathbb{C}$ and let B be a nondegenerate Hermitian form on V of signature (p,q). Fix a pseudo-orthonormal basis of V relative to B and let $\mu(g)$, for $g \in \mathbf{GL}(V)$, be the matrix of g with respect to this basis. Then $\mu : U(V,B) \longrightarrow U(p,q)$ is a group isomorphism.*

1.1.4 Quaternionic Groups

We recall some basic properties of the quaternions. Consider the four-dimensional real vector space $\mathbb{H}$ consisting of the 2×2 complex matrices

$$w = \begin{bmatrix} x & -\overline{y} \\ y & \overline{x} \end{bmatrix} \quad \text{with } x,y \in \mathbb{C}\ . \tag{1.3}$$

One checks directly that $\mathbb{H}$ is closed under multiplication in $M_2(\mathbb{C})$. If $w \in \mathbb{H}$ then $w^* \in \mathbb{H}$ and

$$w^*w = ww^* = (|x|^2 + |y|^2)I$$

(where w^* denotes the conjugate-transpose matrix). Hence every nonzero element of $\mathbb{H}$ is invertible. Thus $\mathbb{H}$ is a *division algebra* (or skew field) over $\mathbb{R}$. This division algebra is a realization of the quaternions.

The more usual way of introducing the quaternions is to consider the vector space $\mathbb{H}$ over $\mathbb{R}$ with basis $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$. Define a multiplication so that $\mathbf{1}$ is the identity and

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1} ,$$

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k} , \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j} , \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i} ;$$

then extend the multiplication to $\mathbb{H}$ by linearity relative to real scalars. To obtain an isomorphism between this version of $\mathbb{H}$ and the 2×2 complex matrix version, take

$$\mathbf{1} = I , \quad \mathbf{i} = \begin{bmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{bmatrix} , \quad \mathbf{j} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} , \quad \mathbf{k} = \begin{bmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix} ,$$

where i is a fixed choice of $\sqrt{-1}$. The *conjugation* $w \mapsto w^*$ satisfies $(uv)^* = v^* u^*$. In terms of real components, $(a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k})^* = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$ for $a, b, c, d \in \mathbb{R}$. It is useful to write quaternions in complex form as $x + \mathbf{j}y$ with $x, y \in \mathbb{C}$; however, note that the conjugation is then given as

$$(x + \mathbf{j}y)^* = \bar{x} + \bar{y}\mathbf{j} = \bar{x} - \mathbf{j}y .$$

On the $4n$-dimensional real vector space $\mathbb{H}^n$ we define multiplication by $a \in \mathbb{H}$ on the right:

$$(u_1, \ldots, u_n) \cdot a = (u_1 a, \ldots, u_n a) .$$

We note that $u \cdot \mathbf{1} = u$ and $u \cdot (ab) = (u \cdot a) \cdot b$. We can therefore think of $\mathbb{H}^n$ as a vector space over $\mathbb{H}$. Viewing elements of $\mathbb{H}^n$ as $n \times 1$ column vectors, we define Au for $u \in \mathbb{H}^n$ and $A \in M_n(\mathbb{H})$ by matrix multiplication. Then $A(u \cdot a) = (Au) \cdot a$ for $a \in \mathbb{H}$; hence A defines a quaternionic linear map. Here matrix multiplication is defined as usual, but one must be careful about the order of multiplication of the entries.

We can make $\mathbb{H}^n$ into a $2n$-dimensional vector space over $\mathbb{C}$ in many ways; for example, we can embed $\mathbb{C}$ into $\mathbb{H}$ as any of the subfields

$$\mathbb{R}\mathbf{1} + \mathbb{R}\mathbf{i} , \quad \mathbb{R}\mathbf{1} + \mathbb{R}\mathbf{j} , \quad \mathbb{R}\mathbf{1} + \mathbb{R}\mathbf{k} . \tag{1.4}$$

Using the first of these embeddings, we write $z = x + \mathbf{j}y \in \mathbb{H}^n$ with $x, y \in \mathbb{C}^n$, and likewise $C = A + \mathbf{j}B \in M_n(\mathbb{H})$ with $A, B \in M_n(\mathbb{C})$. The maps

$$z \mapsto \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and} \quad C \mapsto \begin{bmatrix} A & -\overline{B} \\ B & \overline{A} \end{bmatrix}$$

identify $\mathbb{H}^n$ with $\mathbb{C}^{2n}$ and $M_n(\mathbb{H})$ with the real subalgebra of $M_{2n}(\mathbb{C})$ consisting of matrices T such that

$$JT = \overline{T}J\,, \quad \text{where} \quad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}\,. \tag{1.5}$$

We define $\mathbf{GL}(n,\mathbb{H})$ to be the group of all invertible $n \times n$ matrices over $\mathbb{H}$. Then $\mathbf{GL}(n,\mathbb{H})$ acts on $\mathbb{H}^n$ by complex linear transformations relative to each of the complex structures (1.4). If we use the embedding of $M_n(\mathbb{H})$ into $M_{2n}(\mathbb{C})$ just described, then from (1.5) we see that $\mathbf{GL}(n,\mathbb{H}) = \{g \in \mathbf{GL}(2n,\mathbb{C}) \,:\, Jg = \overline{g}J\}$.

Quaternionic Special Linear Group

We leave it to the reader to prove that the determinant of $A \in \mathbf{GL}(n,\mathbb{H})$ as a complex linear transformation with respect to any of the complex structures (1.4) is the same. We can thus define $\mathbf{SL}(n,\mathbb{H})$ to be the elements of determinant one in $\mathbf{GL}(n,\mathbb{H})$ with respect to any of these complex structures. This group is usually denoted by $\mathbf{SU}^*(2n)$.

The Quaternionic Unitary Groups

For $X = [x_{ij}] \in M_n(\mathbb{H})$ we define $X^* = [x^*_{ji}]$ (here we take the quaternionic matrix entries $x_{ij} \in M_2(\mathbb{C})$ given by (1.3)). Let the diagonal matrix $I_{p,q}$ (with $p+q=n$) be as in Section 1.1.2. The *indefinite quaternionic unitary groups* are the groups

$$\mathbf{Sp}(p,q) = \{g \in \mathbf{GL}(p+q,\mathbb{H}) \,:\, g^* I_{p,q} g = I_{p,q}\}\,.$$

We leave it to the reader to prove that this set is a subgroup of $\mathbf{GL}(p+q,\mathbb{H})$.

The group $\mathbf{Sp}(p,q)$ is the isometry group of the nondegenerate *quaternionic Hermitian form*

$$B(w,z) = w^* I_{p,q} z, \quad \text{for } w,z \in \mathbb{H}^n\,. \tag{1.6}$$

(Note that this form satisfies $B(w,z) = B(z,w)^*$ and $B(w\alpha, z\beta) = \alpha^* B(w,z)\beta$ for $\alpha, \beta \in \mathbb{H}$.) If we write $w = u + \mathbf{j}v$ and $z = x + \mathbf{j}y$ with $u,v,x,y \in \mathbb{C}^n$, and set $K_{p,q} = \operatorname{diag}[I_{p,q}\;\; I_{p,q}] \in M_{2n}(\mathbb{R})$, then

$$B(w,z) = \begin{bmatrix} u^* & v^* \end{bmatrix} K_{p,q} \begin{bmatrix} x \\ y \end{bmatrix} + \mathbf{j} \begin{bmatrix} u^t & v^t \end{bmatrix} K_{p,q} \begin{bmatrix} -y \\ x \end{bmatrix}\,.$$

Thus the elements of $\mathbf{Sp}(p,q)$, viewed as linear transformations of $\mathbb{C}^{2n}$, preserve both a Hermitian form of signature $(2p,2q)$ and a nondegenerate skew-symmetric form.