# Modeling Students' Mathematical Modeling Competencies

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Editors

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# Chapter 1
# Introduction: ICTMA and the Teaching of Modeling and Applications

**Gabriele Kaiser**

Applications and modeling and their learning and teaching in school and university have become a prominent topic in the last decades in view of the growing worldwide importance of the usage of mathematics in science, technology and everyday life. Given the worldwide impending shortage of youngsters who are interested in mathematics and science it is highly necessary to discuss possibilities to change mathematics education in school and tertiary education towards the inclusion of real world examples and the competencies to use mathematics to solve real world problems.

These concerns were the starting point for the establishment of the ICTMA group – an international community of scholars and researchers supporting the International Conferences on the Teaching of Mathematical Modeling and Applications. ICTMA has been concerned with research, teaching and practice of mathematical modeling since 1983. From the beginning, the aim of the ICTMA group has been to foster both teaching and research about mathematical modeling and the ability to apply mathematics to genuine real world problems – in primary, secondary, and tertiary educational environments – as well as in teacher education, and education in professional and workplace environments.

From the outset ICTMA has maintained the integrity of its focus, which has both a mathematical and an educational component. This makes a distinction from a mathematical focus on the one hand, and a mathematics education context in which the mathematics need not to have a connection with applications and modeling on the other hand. Thus a distinctive aspect of ICTMA is the interface it provides for collaboration between those whose main activity lies in applying mathematics, but who have an informed interest in sharing these abilities and developing relevant skills in others (an educational focus), and those whose principal affiliations are within education, but who have a commitment to supporting the effective application of mathematics to problems outside itself.

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These goals have been promoted by biennial conferences – the International Conferences on the Teaching of Mathematical Modeling and Applications – taking place since 1983 in various parts of the world, namely Great Britain (1983, 1985, 1995, 2005), USA (1993, 2003, 2007), China (2001), Australia (1997), Germany (1987), Denmark (1989), Netherlands (1991), Portugal (1999).

At ICTMA conferences various themes and topics have been discussed, amongst others:

- How far are, at an international level applications and modeling examples, implemented in mathematics education at school and tertiary level?
- How is instruction to structure which is adequate for the teaching and learning of applications and modeling and considering country-specific differences?
- How can the development of modeling competencies be evaluated and promoted?
- How can authentic applications of mathematics in industry and technology be introduced into mathematics instruction at various levels?
- How can modern technologies, necessary for authentic modeling examples, be introduced in mathematics instruction at school and tertiary level?

At the last conferences and especially at the conference ICTMA13, of which important chapters are presented in this book, various answers to these questions and topics have been explored. However, it is obvious that real world and modeling examples still do not have a high level of importance in mathematics education at school and tertiary level, indispensable for a modern society. So, amongst other topics, it seems necessary to investigate affective barriers of teachers and students against the inclusion of these kinds of examples at various levels. Furthermore cognitive barriers which prevent students to develop modeling competencies need to be explored. First case-studies exist, but large-scale studies seem to be necessary in order to broaden and deepen our knowledge about these aspects.

In order to broaden the audience ICTMA became, in 2003, an affiliated study group of the International Commission on Mathematical Instruction, which gives the opportunity to run special sessions during ICME and to carry out ICMI-study 14 on that theme ICMI Study. With these strong relations to the mathematics education debate, and to the intense connection to the engineering and applied mathematics discussion, ICTMA has a unique flavour and will hopefully be able to bring in genuine real world and modeling examples into mathematics instruction around the world. This book expresses this flavour in a special way, it combines a design approach to mathematics instruction at various levels with many empirical studies from all over the world, not only from Western countries, but also in growing number from Asian countries and countries of the southern hemisphere. This growing internationality of the debate shows the joint worldwide concern about the lack of real world and modeling examples in mathematics education at school and tertiary level. But despite this growing internationality and the joint topics discussed all over the world, approaches from various parts of the world are still influenced by their underlying educational philosophies, their socio-cultural conditions and the studies and results presented in the articles of this book displays the variety of this debate. Future trends concerning the teaching and learning of applications and modeling should build on this variety and include it in its future debates.

# Part I
# The Nature of Models & Modeling

# Chapter 2
# Introduction to Part I Modeling: What Is It? Why Do It?

**Richard Lesh and Thomas Fennewald**

At ICTMA-13, where the chapters in this book were first presented, a variety of views were expressed about an appropriate definition of the term *model* – and about appropriate ways to think about the nature of *modeling activities*. So, it is not surprising that some participants would consider this lack of consensus to be a priority problem that should be solved by a research community that claims to be investigating models and modeling.

We certainly agree that conceptual fuzziness is not a virtue in a research community – especially if it impedes communication among members of the community. Furthermore, we agree that increasing clarity about key constructs is an important goal of research. Nonetheless, we also believe that, especially at early stages of theory development, a certain amount of diversity in thinking is as healthy for research communities as it is for (for example) engineers who are at early "brainstorming" stages in the design of space shuttles, sky scrapers, or transportation systems. Furthermore, we believe that the mathematics education research community in particular has suffered from more than enough pressure for premature ideological orthodoxy.

The theme of ICTMA-13 was *modeling students' modeling competencies*; and, in many of the research methodologies that are described throughout this book, students develop models to describe or design "real life" artifacts or tools; teachers develop models to describe students' modeling competencies – or to design productive learning environments; and, researchers develop models of interactions among students, teachers, and learning environments. So, the goal of many of our most productive studies focus on the development of powerful, sharable, and re-useable models – which then, in turn, influence theory development. But, model

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development research is not the same as theory development research. For example, in theory-driven research, the theory determines what questions are appropriate to ask, what kind of evidence needs to be collected, how the information should be interpeted, analyzed, and assessed – and when the question has been answered or the issue resolved. But, in model-development research (or design research), the problem arises from a "real world" decision-making situation. Thus, this work is more like engineering than a "pure" science. As engineers (and other design scientists) often emphasize (see Zawojewski et al., 2009):

- Engineering is the science of solving "real life" problems where you don't have enough time, or money, or other resources – and when multiple stake holders often hold partly conflicting views about the nature success (low costs versus high quality, simplicity versus completeness, and so on).
- Realistic solutions to realistically complex problems usually need to integrate ideas and procedures drawn from more than a single discipline, or theory, or textbook topic area.
- When high stakes decision making issues arise in "real life" situations, it usually is important to design for success (power, sharability, re-usability) – not simply to test for it.
- In design research, many of the "things" that we most need to understand and explain are "things" that we ourselves are in the process of developing or designing. For example, in math/science education, these "things" range from students' conceptual systems to curriculum materials which include systems of activities for learning or assessment. And, in each of these cases, as soon as we come to better understand the "thing" being developed or designed, we tend to change them – so that another cycle of adaptation is needed. (Note: This is one reason why scientists do not speak of developing a single theory of space shuttles.)

In the book, *Beyond Constructivism: Models & Modeling Perspectives on Mathematics Problem Solving, Learning & Teaching* (Lesh and Doerr, 2003), we describe a variety of reasons why we consider MMP to be a "blue color" perspective which is like engineering more than it is like "pure" sciences such as physics, chemistry, or mathematics. In fact, for many of the same reasons why *Pragmatists* such as James, Pierce, or Dewey considered *Pragmatism* to be more of a framework for developing theories rather than being a theory in itself, we consider MMP to be a framework for developing models of students' modeling.

According to MMP, students, teachers, and researchers – all are in engaged in model development. Consequently, MMP research assumes that: (a) similar principles apply not only to the model development activities of students, but also to the model-development activities of teachers and researchers, and (b) researchers' early-iteration models are expected to be characterized by conceptual inadequacies similar to those that characterize the early-iteration interpretation systems of students or teachers. ... MMP research recognizes that we – the researchers who study modeling – are humbly still in the early phases of *our own* model development. And, just as in the model development activities of the students and teachers that we study, we assume that we will only make progress if we go through multiple cycles of expressing > testing > revising our current ways of thinking. So just as Darwin emphasized in the development of other kinds of complex adaptive systems (1859), evolution of ways of thinking about teaching and students' learning is only likely to occur if provisions are made to encourage diversity, selection, communication, and accumulation (Sawyer, 2006).

In our own "blue collar" research on models and modeling, we take the following to be a useful first-iteration definition of a model. *A model is a system for describing (or explaining, or designing) another system(s) for some clearly specified purpose.*

The preceding definition seems simple and straightforward, but one thing that we like about it is that it has a clear history of pressing us toward important researchable questions. Furthermore, best of all, and unlike terms such as *cognitive structures* or *schemes* or other terms that are favored by cognitive scientists or constructivist philosophers, the preceding definition uses the term *model* in the same way that the term is used in mature science fields like physics or engineering. That is, a model is system that is used to describe (or interpret) another system of interest in a purposeful way. However, whereas physicists and engineers tend to focus on only the written symbolic aspects of the models they develop, when math/science educators investigate what it means to "understand" these models, it inevitably becomes apparent that the written and spoken embodiments that scientists emphasize scientists usually represent only something like the tip of an iceberg. For example, in order for scientific models to be useful, understanding them usually involves a variety of diagrams, concrete models, experience-based metaphors, and other expressive media – in addition to technical spoken language and symbol systems, each of which emphasize some aspects (but deemphasize others) for the "thing" that they are used to describe, or explain, or design. Furthermore, model development often involves dimensions of development such as intuition-to-formalization, concrete-to-abstract, situated-to-decontextualized, specific-to-general, implicit-to-explicit, global-to-analytic,[1] and so on. Furthermore, during early stages of development, models often function in ways that are more like "windows" that we look through rather than as "objects" that we look at; and, they often function in ways that are rather unstable. That is, when the model developer focuses on "forest-level" or "big picture" interpretations of the "thing" being described, they often neglect to keep in mind "tree-level" details about the "thing" being described. In other words, even for experienced scientists, but especially for young students, and especially during early stages of model development, a model developer's early interpretation of the "thing" being described is often far more situated, piecemeal, and non-analytic than most traditional theories of learning consider it to be.

Should early interpretations (such as those that function intuitively, informally, or in piecemeal or unstable ways) be referred to as models? We believe that such

[1] When we speak of these dimensions of development, we recognize that, in specific "real life" situations where model development is needed, the most useful model is not necessarily the one that is most complex, most formal, most abstract, or most decontextualized.

questions should be resolved through research – not through political consensus building. This point brings out a point that we especially like about the term "models and modeling". That is the term allows room for debates among researchers with opposing points of view. And, we believe that debates about the nature of various models have exhibited a clear history of leading to researchable questions, testable hypotheses, and a variety of model validation activities. This is why we say that Models and Modeling is not so much a view of how learning works as it is a methodological approach and framework for investigating learning.

Finally, we should not neglect to mention some other reasons why authors throughout this book view research on models and modeling to be especially important for mathematics education researchers. First, in nearly every field where researchers have investigated similarities and differences between experts and novices (or between successful learners or problem solvers versus those who are less successful), results have shown that expertise not only involves *doing* things differently (or better), it also involves *seeing* or *interpreting* things differently (or better). And, in mathematics and the natural sciences, interpretation development means model development. Furthermore, in students' lives beyond school, the ability to describe or explain things mathematically is one of the main factors that is needed in order for mathematics to be useful.

Throughout this volume, even though most of the authors generally support the ideas outlined above, conflicting views emerged – even during ICTMA-13. One reason why this happened is that the organizers of ICTMA-13 made special efforts to attract not only mathematics educators from more than 25 different countries, but equal efforts also were made to attract leading science educators, engineering educators, and mathematics educators focusing on mathematical and scientific thinking beyond school. Yet, even with such diversity, we know of no cases where differences in perspectives did not appear to be leading toward productive adaptations for the community as a whole.

This volume beings with chapters that aim to answer the fundamental question: what are models? Hestenes outlines important relationships among various forms of modeling and fields that apply modeling theory. In his plenary article, he distinguishes between uses of the term "model" to refer to a conceptual mental model and uses of "model" to refer to a publically shared model. By so doing, he sets the stage for a discussion that relates the act of modeling in professional and non-professional so-called "everyday life" to research in both science and mathematics education, thus giving a full view of modeling theory as describing the process by which a form of knowledge, modeling, is developed through lived acts that are both mental and social in nature. Niss continues this examination of the question of what models are and what modeling is with studies into how to model the learning of modeling itself. His study hints at perhaps the deepest and most philosophical of all questions related to the origin of modeling competencies and knowledge: What is the origin of understanding how to model? From his study of three different modeling problems, he suggests that the capability to model is a learned capacity. The teaching of modeling, he notes, can be approached through activities that encourage cognitive dissonance of a type that drives the emergent modeling of Gravemeijer –

or the model eliciting activities described by Zawojewski or Carmona, or the kind of scaffolded approaches described by Kaiser. Niss considers each as possible modes for teaching modeling. Larson and colleagues also address the question of what models are while also looking ahead to the future of modeling in the curriculum and examining ways modeling can meet the challenge to provide students opportunities to “create and refine ideas for themselves.”

With a notion about what models are, discussion then proceeds to examinations of where models and the modelers who make them are found. Noss and Hoyles take research about the pedagogical aspects of modeling beyond the classroom into the workplace, performing an investigation as to what “Techno-mathematical Literacies” are needed to understand manufacturing processes. Cardella examines the modeling of engineering undergraduates and graduates who might not otherwise see the connection of the math they are learning in the classroom to the real world work they will perform. This study is followed by Alpers who examines the modeling of mechanical engineers outside of school contexts, looking at math and modeling performed on the job. Together, these studies show both the application and real-world relevance of modeling in studies beyond the school setting – while at the same time demonsrating as well as to in-school modeling studies have to studies in other disciplines, and in higher level mathematics course.

The modeling process is examined by Larson by employing one of Lesh’s original modeling activities, “Summer Jobs”, in a study of how students develop quantitative reasoning needed in real-world problems, supporting concluding that changes in the perception students have of relationships among quantities leads to the development of better quantitative reasoning ability along a progression of Piagetian-like stages. Using a related activity, “University Cafeteria”, Mousoulides and Sriraman examine the progression in mathematical understandings made by middle school students over the course of their schooling.

The question of what is needed for modeling to occur is taken on by Galbraith, Stillman, and Brown, who provide insight into what has been recognized as one of the most critical aspects of setting up a modeling experience for learners – that is, creating a meaningful context in which it is hoped the modeling learners will engage in significant concepts. They explore this from a uniquely Australian perspective, investigating the response of Australian students to Australian themed modeling activities. Haines and Crouch further investigate this issue while exploring the intricacies of modeling cycles that distinguish modeling activities from other assessments, as they expand their discussion to include the assessment of modeling competencies in general. Amit and Jan then present an “extension of model-eliciting problems into model-eliciting environments which are designed to optimize the chances that significant modeling activities will occur.” Speiser and Walter then investigate another aspect critical to modeling activities and models in general. That is, models are tools created and applied for a purpose. And, purposes are strongly influenced by communities and societies in which designers and modelers operate. Clark and colleagues then discuss lessons learned from a pilot course they designed to elicit systems thinking in which they employed and modified MEAs designed to create a need for modeling. Davis then concludes the section by exploring the need for tools that can help teachers make sense of data collected in generative activities.

The question of how models develop is examined by Riede who studies students as they explore and rediscover Weber's law, progressing through modeling cycles, during their activities. Amit and Neria then examine the express > test > revise cycles of students engaged in generalizing pictorial linear pattern problems, finding that many students applied recursive strategies despite the appropriateness of global strategies. Work in the study of conceptual and model development is also important in the work of Dominguez who uses modeling activities to reveal multiplicity in students' ways of thinking – even when one final answer is agreed upon.

Finally the question of what ways modeling is different from solving traditional textbook word problems is addressed starting with Zawojewski, who asks what research implications and distinctions between the two can be drawn. She is followed by Carmona and Greenstein who find that modeling is suggestive of a spiral curriculum where powerful and underlying themes are revisited constantly – not arrived at permanently. Jensen then shows and investigates distinctions between modeling and problem solving competencies. Greefrath concludes with the examination of students planning processes, noting the differences in strategies used between modeling and problem solving questions.

These chapters reveal many facets of both modeling activities and the modeling research being conducted around student modeling activities. Although this collection of contributions is not intended to provide a comprehensive overview of the work being done to study the nature of modeling, this collection certainly gives the reader a diverse introduction to some of the most important frontiers in research on modeling and applications.

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# Section 1
# What Are Models?

# Chapter 3
# Modeling Theory for Math and Science Education

**David Hestenes**

**Abstract** Mathematics has been described as the science of patterns. Natural science can be characterized as the investigation of patterns in nature. Central to both domains is the notion of model as a unit of coherently structured knowledge. Modeling Theory is concerned with models as basic structures in cognition as well as scientific knowledge. It maintains a sharp distinction between mental models that people think with and conceptual models that are publicly shared. This supports a view that cognition in science, math, and everyday life is basically about making and using mental models. We review and extend elements of Modeling Theory as a foundation for R&D in math and science education.

## 3.1 Introduction

Why should a theoretical physicist be concerned about mathematics education? My answer will be a long one, but let me begin by introducing you to some of my esteemed colleagues in Box 3.1. These fellows are such good physicists that most if not all of them would be worthy candidates for a Nobel Prize if they were alive today. You may know that they are quite good at mathematics as well! Indeed, mathematics textbooks often count them as mathematicians without mentioning that they are physicists. I dare say, however, that they would be mightily offended to hear that they are not counted as physicists. Likewise, I am more than miffed when reviewers of my math-science education proposals discount my qualifications as a mathematician because my doctorate is in physics. Like the fellows in the list, I regard my scientific research as equal parts mathematics and physics. The fact that the education establishment does not recognize that theoretical physicists are uniquely well-qualified to address education at the interface between mathematics and science is traceable to a serious problem within the mathematics profession itself.

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**Box 3.1 Some distinguished theoretical physicists**

| | |
|---|---|
| Newton | Cauchy |
| Euler | Poincaré |
| Gauss | Hilbert |
| Lagrange | Weyl |
| Laplace | Von Neumann |

Of course, the list of *physicist/mathematicians* in Box 3.1 is far from complete. Many of my favorite colleagues are omitted. But there are at least two good reasons why the list ends in the middle of the twentieth century. The distinguished Russian mathematician Arnold (1997) put his finger on both in a widely circulated diatribe *On Teaching Mathematics*, wherein he asserts

> Mathematics is a part of physics. Physics is an experimental science, a part of natural science. Mathematics is the part of physics where experiments are cheap.
>
> In the middle of the 20th century it was attempted to divide physics and mathematics. The consequences turned out to be catastrophic. Whole generations of mathematicians grew up without knowing half of their science and, of course in total ignorance of other sciences.

*Mathematician-cum-historian* Morris Kline (1980) has thoroughly documented "the disastrous divorce" of the mathematics profession from physics, which began in the latter part of the nineteenth century. He estimated that, by 1980, eighty percent of active mathematicians were ignorant of science and perfectly happy to remain that way.

The divorce is thus an incontrovertible fact, but how disastrous can it be if only a minority of mathematicians like Arnold and Kline are alarmed? Isn't it a natural consequence of necessary specialization in an increasingly complex society? And isn't Arnold's claim of intimacy between math and physics merely a personal opinion? Surely the majority of mathematicians believe that mathematics is a completely autonomous discipline.

I claim that the answer to all these questions is a resounding NO! Indeed, *I submit that the single most serious deficiency in US math education is the divorce of mathematics from physics* in the education of mathematicians, in the training of math teachers, and in the structure of the K-12 (-16-20) curriculum. Moreover, this is not a simple deficiency in the breadth of education; it is a fundamental problem in conceptual learning and cognition. I claim that *cognitive processes for understanding math and physics are intimately linked and fundamentally the same*! Indeed, I claim that *Physics is cognitively basic to quantitative science in all domains!!*

Before delving into the deep cognitive issues, let us note some obvious academic consequences of the math/physics divorce. Training in mathematics is essential for all physicists, amounting to the equivalent of a dual major in mathematics for theoretical physicists. But math courses have become increasingly irrelevant to physics, so physics departments offer their own courses in "Methods of mathematical physics" at both graduate and undergraduate levels, with additional courses in more specialized topics like group theory. One consequence is a narrowing of the physicist's appreciation of mathematics. But a far more serious consequence is the reduction in opportunity for math majors to learn about vital connections to physics. This continues through graduate school, so the typical math PhD is ill-prepared for work in applied mathematics. Some math departments have attempted to remedy this deficiency with courses in mathematical modeling, but mathematical modeling without science is like the Cheshire cat: form without content! This is one of the deep cognitive issues that we need to address.

*Far and away the most serious consequence of the math/physics divide is the deficient preparation of K-12 math teachers!* The neglect of geometry and excess of formalism that Arnold (1997) deplores in the university math curriculum has propagated to teacher preparation. There is abundant evidence that most teachers see their job as teaching formal rules and algorithms. Few have even a minimal understanding of Newtonian physics, so most are inept at applying algebra and calculus even to simple problems of motion. Consequently, high school physics courses are forced to revisit the prerequisite math knowledge that students are supposed to bring from years of math instruction. As the math courses lack the intuitive base necessary for conceptual understanding, students are forced to rote learning, which has a short half-life, so their recollection of math has decayed to nearly to zero by the time they get to college.

I doubt that these crippling deficiencies in math education can be fully resolved without a "sea change" in the culture of mathematics. To drive such revolutionary change we need a coherent theory of mathematical learning and cognition supported by a substantial body of empirical evidence. My purpose here is to report on progress in that direction.

## 3.2 Origins of Modeling Theory

I have been investigating the epistemology of science and mathematics across the full range of academic disciplines for half a century. As that may sound implausible, let me describe the unusual initial conditions that got me started.

My father was an accomplished mathematician who helped organize American mathematicians to support the war effort in WWII. Consequently, he got to know mathematicians across the country on a first name basis. That served him well when, shortly after the war, he was wooed from the University of Chicago to build a first rank math department at UCLA. He was also appointed director of the Institute for Numerical Analysis (INA), where the National Bureau of Standards installed the first electronic computer in western United States. With a solid background in "pure mathematics" (Calculus of Variations), he blossomed then into a pioneer in the fledgling fields of Control Theory and Numerical Analysis, for which he was posthumously inducted into the Hall of Fame for Engineering, Science and Technology (HOFEST). The well-funded, vigorous research activity at the INA and the rapid emergence of the UCLA math department attracted a steady stream of distinguished mathematicians from around the world, for which my father was usually the host. He was at the acme of his career when I entered graduate school in 1956.

My undergraduate major in philosophy introduced me to the great conundrums of epistemology, and I was inspired by Bertrand Russell to switch to physics in search of answers. When I started graduate school, my father found me an office in the INA where I was surrounded by a whirlwind of excited activity about the beginnings of Computer Science and Artificial Intelligence. That prepared me to follow the evolution of both those fields throughout my career, though my main efforts were concentrated on physics and mathematics. By my third year I hit upon "Geometric Algebra" as the central theme of my scientific research. That induced me to reject Russell's logicist view of mathematics and sharpened my insight and interest in epistemological and cognitive foundations. Throughout my graduate years in physics I spent most of my time in the math department where I imbibed the culture of mathematics. This strong association with mathematics continued throughout my career in physics, anchored by relations with my father.

My diverse interests in cognitive science and theoretical physics converged in the 1980s when I got embroiled in problems of improving introductory physics instruction. Like any confirmed theoretician, I framed the problems in the context of developing a theory with testable empirical consequences. Largely from my own experience as a scientist I identified scientific models and modeling as the core of scientific knowledge and practice, and I proceeded to incorporate it into the design of physics instruction with the help of brilliant graduate students Ibrahim Halloun and Malcolm Wells. Thus began a program of educational R&D guided by an evolving research perspective that I called *Modeling Theory* in a 1987 chapter. That program has continued to evolve beyond all my expectations. An up-to-date review of Modeling Theory is available in a recent chapter (Hestenes, 2007). The present chapter is a continuation, introducing new material with only enough duplication to make it reasonably self-contained. Therefore, it contains many gaps, some of which can be filled by consulting the earlier chapter, and others that I hope will stimulate original research. For Modeling Theory is an enormous enterprise that amounts to a thematic approach to the whole of cognitive science. The best we can do here is sample the major themes.

As schematized in Fig. 3.1, research on Modeling Theory has developed along two complementary strands. The strand on the right investigates *scientific models and modeling practices* that are explicit and observable. It provides *a window to structure and process in scientific and mathematical thinking* that we aim to peek through. That involves us with the strand on the left, which will be our main concern.

You may ask, "Why should one adopt a model-centered epistemology of science?" There are three good reasons:

1. *Theoretical*: Models are *basic units of coherently structured knowledge*, from which one can make *logical inferences, predictions, explanations, plans and*

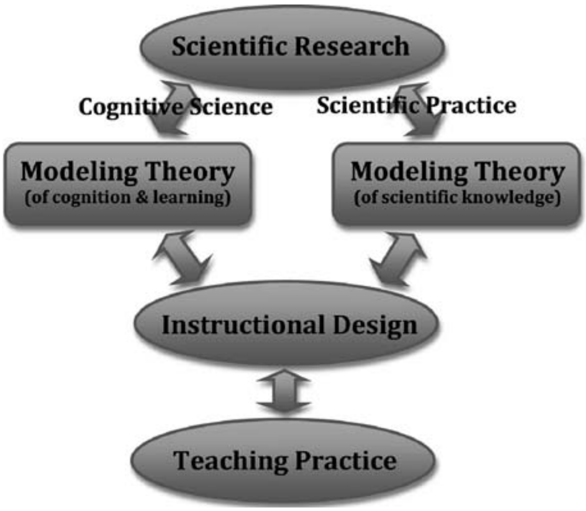


**Fig. 3.1** Modeling theory – a research program

*designs*. One cannot make inferences from isolated facts or theoretical principles. A model can serve as inferential tool for the kind of structure it embodies.

2. *Empirical*: Models can be *directly compared with physical things and processes*. A theoretical hypothesis or general principle cannot be tested empirically except through incorporation in a model. Empirical data is meaningless without interpretation supplied by a model.
3. *Cognitive*: Model structure is concretely *embodied in physical intuition*, where it serves as an element of *physical understanding*.

The third reason is based on the Modeling Theory of cognition set forth in this chapter.

## 3.3 Models and Concepts

The term *model* is usually used informally (hence ambiguously), but to make crucial theoretical distinctions we need precise definitions. Although I have discussed this issue at length before, it is so important that I revisit it with a slightly different slant. I favor the following general definition:

A *model* is a *representation of structure* in a given *system*.

A *system* is a set of related *objects*, which may be real or imaginary, physical or mental, simple or composite. The *structure* of a system is a set of relations among its objects. The system itself is called the *referent* of the model.

We often identify the model with its *representation* in a concrete inscription of words, symbols or figures (such as graphs, diagrams or sketches). But it must not be forgotten that the inscription is supplemented by a system of (mostly tacit) rules and conventions for encoding model structure. As depicted in Fig. 3.2, I use the term *symbolic form* for the triad of elements defining a model. I chose the term deliberately to suggest association with the great work on symbolic forms by philosopher Ernst Cassirer (Cassirer, 1953).

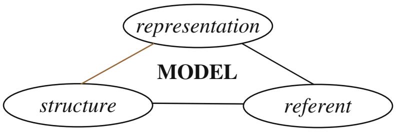


**Fig. 3.2** Symbolic form of a model

We are especially interested in scientific models, for which I have often used the definition: A (scientific) *model* is a *representation of structure* in a physical system or process.

This differs from the general definition only in emphasis and scope. Its scope is limited by assuming that the objects in a physical system are physical *things*. Nevertheless, the definition applies to all the sciences (including biology and social sciences). Models in the various sciences differ in the *kinds of structure* that they attribute to systems. The term *process* is included in the definition only for emphasis; it refers to a change in the structure of a system. Thus a *process model* is an abstraction of structural change from a more complete model including objects in the system.

In most discussions of scientific models the crucial role of structure is overlooked or addressed only incidentally. In Modeling Theory *structure is central* to the concept of model. The *structure* of a system (hence structure in its model) is defined as a *set of relations* among the objects in the system (hence among parts in the model).

*Universal structure types*: From studying a wide variety of examples, I have concluded that *five types of structure suffice* to characterize any scientific model. As this seems to be an important empirical fact, a brief description of each type is in order here.

- *Systemic structure:* Its representation specifies (a) *composition* of the system (b) *links* among the parts (individual objects), (c) links to *external agents* (objects in the environment). A diagrammatic representation is usually best (with objects represented by nodes and links represented by connecting lines) because it provides a wholistic image of the entire structure. Examples: electric circuit diagrams, organization charts, family trees.
- *Geometric structure:* specifies (a) *configuration* (geometric relations among the parts), (b) *location* (position with respect to a reference frame)
- *Object structure: intrinsic properties* of the parts. For example, mass and charge if the objects are material things, or *roles* if the objects are *agents* with complex behaviors. The objects may themselves be systems (such as atoms composed of

electrons and nuclei), but their internal structure is not represented in the model, though it may be reflected in the attributed properties.

- *Interaction structure:* properties of the links (typically *causal* interactions). Usually represented as binary relations on object pairs. Examples of interactions: forces (momentum exchange), transport of materials in any form, information exchange.
- *Temporal (event) structure: temporal change in the state* of the system. Change in position (motion) is the most fundamental kind of change, as it provides the basic measure of time. Measurement theory specifies how to quantify the properties of a system into property variables. The state of a system is a set of values for its property variables (at a given time). Temporal change can be represented *descriptively* (as in graphs), or *dynamically* (by equations of motion or conservation laws).

Optimal precision in definition and analysis of structure is supplied by *mathematics, the science of structure*. This agrees with the usual notion of a *mathematical model* as a representation in terms of mathematical symbols.

Now, here is a *perplexing question* that bothered me for decades: If the meaning of a scientific model derives from its physical interpretation, *from whence comes the meaning of a mathematical model*? Mathematical models are *abstract*, which means they have *no physical referent!*

It dawned on me during the last decade that the emerging field of *cognitive linguistics* provides a *revolutionary answer*. Cognitive linguistics has revolutionized the field of semantics by maintaining that the actual *referents of language are mental models* in the mind rather than concrete objects in an external world. It follows that, if mathematics is "the language of science," then the *referents of mathematical models must be mental models.* Likewise, *the proper referents of scientific models must be mental models of physical situations*, which are only indirectly related to real physical systems through data, observation and experiment.

This implies a common cognitive foundation for math, science and language: Just as science is about making and using *objective models* of real things and events, so cognition (in mathematics and science as well as everyday life) is about making and manipulating *mental models* of imaginary objects and events!

Let me sum up this revolution in semantics with a modified definition:

A *conceptual model* is a *representation of structure* in a *mental model*.

As before, the representation in a conceptual model is a concrete inscription that encodes structure in the referent. However, we make no commitment as to what the structure of a mental model may represent. Henceforth, scientific and mathematical models are to be regarded as conceptual models. But the referent of a conceptual model is always a mental model, so its structure in the mind is inaccessible to direct observation. How, then, can this be an advance in Modeling Theory?

The answer is: It enables transfer from a Modeling Theory of scientific knowledge to a Modeling Theory of cognition in science and mathematics. Much is known about the structure of scientific models. We seek to solve the inverse problem of inferring the structure of mental models from the objective structure of scientific representations. If that seems like an impossible task, note that it is commonplace to infer thoughts in other minds from social interaction. Can we not make stronger inferences with the full resources of science? Here we have a modeling approach to the theory of cognition, so we can draw on the whole corpus of results in cognitive science for support and critique. I will not duplicate my previous reference to that enormous literature (Hestenes, 2007). However, I should emphasize the special relevance of cognitive linguistics and point out that two recent introductions to the field (Evans and Green, 2006; Croft and Cruse, 2004) provide a comprehensive overview that was difficult to put together only a few years ago.

Let me now propose the *First Principle for a Modeling Theory of Cognition:*

I. *Cognition is basically about making and manipulating mental models.*

I call this the *Primacy (of Models) Principle*, noting I have already tacitly invoked a variant of it for the Modeling Theory of scientific knowledge. Commitment to this principle might seem extreme, for I must admit that it is not to be found in the cognitive science literature from which I draw most of the supporting evidence. However, I contend that for a guiding research principle the standard is not that it is true but that it is *productive*, by which I mean that it leads to significant predictions that are empirically testable. Even if proved wrong, that would be quite an interesting result! In the meantime, we shall see that the primacy principle can carry us a long way.

For a start, the Primacy Principle helps sharpen the definition of a concept, as it implies that concepts must refer to mental models, at least indirectly. As done before, I define a *concept* as a *{form, meaning} pair* represented by a *symbol* (or assembly of symbols). In analogy to Fig. 3.2, I define the *symbolic form* of a concept as the triad in Fig. 3.3. Much like a model, the *form* of a concept is its conceptual structure, including relations among its parts and its place within a conceptual system. The *meaning* of a concept is its relation to mental models. All this is close enough to the usual loose definition of "concept" to conform to common parlance. It provides then a foundation for a more rigorous analysis of important concepts.

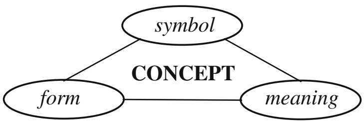


**Fig. 3.3** Symbolic form of a concept

We are now prepared to propose the *Second Principle for a Modeling Theory of Cognition:*

II. *Mental models possess five basic types of structure: systemic, geometric, descriptive, interactive, temporal.*