Cauchy's *Cours d'analyse*

An Annotated Translation

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Sources and Studies in the History of Mathematics and Physical Sciences

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Cauchy's *Cours d'analyse*

An Annotated Translation

Springer

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We dedicate this volume to Ronald Calinger, Victor Katz and Frederick Rickey, who taught us the importance and satisfaction of reading original sources, and to our friends in ARITHMOS, with whom we enjoy putting those lessons into practice.

Translators' Preface

Modern mathematics strives to be rigorous. Ancient Greek geometers had similar goals, to prove absolute truths by using perfect deductive logic starting from incontrovertible premises.

Often in the history of mathematics, we see a pattern where the ideas and applications come first and the rigor comes later. This happened in ancient times, when the practical geometry of the Mesopotamians and Egyptians evolved into the rigorous efforts of the Greeks. It happened again with calculus. Calculus was discovered, some say invented, almost independently by Isaac Newton (1642–1727) about 1666 and by Gottfried Wilhelm von Leibniz (1646–1716) about 10 years later, but its rigorous foundations were not established, despite several attempts, for more than 150 years.

In 1821, Augustin-Louis Cauchy (1789–1857) published a textbook, the *Cours d'analyse*, to accompany his course in analysis at the École Polytechnique. It is one of the most influential mathematics books ever written. Not only did Cauchy provide a workable definition of limits and a means to make them the basis of a rigorous theory of calculus, but also he revitalized the idea that all mathematics could be set on such rigorous foundations. Today, the quality of a work of mathematics is judged in part on the quality of its rigor; this standard is largely due to the transformation brought about by Cauchy and the *Cours d'analyse*.

The 17th century brought the new calculus. Scientists of the age were convinced of the truth of this calculus by its impressive applications in describing and predicting the workings of the natural world, especially in mechanics and the motions of the planets. The foundations of calculus, what Colin Maclaurin (1698–1746) and Jean le Rond d'Alembert (1717–1783) later called its *metaphysics*, were based on the intuitive geometric ideas of Leibniz and Newton. Some of their contemporaries, especially Bishop George Berkeley (1685–1753) in England and Michel Rolle (1652–1719) in France, recognized the problems in the foundations of calculus. Rolle, for example, said that calculus was "a collection of ingenious fallacies," and Berkeley ridiculed infinitely small quantities, one of the basic notions of early calculus, as "the ghosts of departed quantities." Both Berkeley and Rolle freely admitted the practicality of calculus, but they challenged its lack of rigorous foundations. We should note that Rolle's colleagues at the Paris Academy eventually convinced him to change his mind, but Berkeley remained skeptical for his entire life.

Later in the 18th century, only a few mathematicians tried to address the questions of foundations that had been raised by Berkeley and Rolle. Over the years, three main schools of thought developed: infinitesimals, limits, and formal algebra of series. We could consider the British ideas of fluxions and evanescent quantities either to be a fourth school or to be an ancestor of these others. Leonhard Euler (1707–1783) [Euler 1755] was the most prominent exponent of infinitesimals, though he devoted only a tiny part of his immense scientific corpus to issues of foundations. Colin Maclaurin [Maclaurin 1742] and Jean le Rond d'Alembert [D'Alembert 1754] favored limits. Maclaurin's ideas on limits were buried deep in his *Treatise of Fluxions*, and they were overshadowed by the rest of the opus. D'Alembert's works were very widely read, but even though they were published at almost the same time as Euler's contrary views, they did not stimulate much of a dialog.

We suspect that the largest school of thought on the foundations of calculus was in fact a pragmatic school – calculus worked so well that there was no real incentive to worry much about its foundations.

In *An V* of the French Revolutionary calendar, 1797 to the rest of Europe, Joseph-Louis Lagrange (1736–1813) [Lagrange 1797] returned to foundations with his book, the full title of which was *Théorie des fonctions analytiques, contenant les principes du calcul différentiel, dégagés de toute considération d'infiniment petits ou d'évanouissans, de limites ou de fluxions, et réduits à l'analyse algébrique des quantités finies* (Theory of analytic functions containing the principles of differential calculus, without any consideration of infinitesimal or vanishing quantities, of limits or of fluxions, and reduced to the algebraic analysis of finite quantities). The book was based on his analysis lectures at the École Polytechnique. Lagrange used power series expansions to define derivatives, rather than the other way around. Lagrange kept revising the book and publishing new editions. Its fourth edition appeared in 1813, the year Lagrange died. It is interesting to note that, like the *Cours d'analyse*, Lagrange's *Théorie des fonctions analytiques* contains no illustrations whatsoever.

Just two years after Lagrange died, Cauchy joined the faculty of the École Polytechnique as professor of analysis and started to teach the same course that Lagrange had taught. He inherited Lagrange's commitment to establish foundations of calculus, but he followed Maclaurin and d'Alembert rather than Lagrange and sought those foundations in the formality of limits. A few years later, he published his lecture notes as the *Cours d'analyse de l'École Royale Polytechnique; I.re Partie. Analyse algébrique*. The book is usually called the *Cours d'analyse*, but some catalogs and secondary sources call it the *Analyse algébrique*. Evidently, Cauchy had intended to write a second part, but he did not have the opportunity. The year after its publication, the École Polytechnique changed the curriculum to reduce its emphasis on foundations [Lützen 2003, p. 160]. Cauchy wrote new texts, *Résumé des leçons données a l'École Polytechnique sur le calcul infinitesimal, tome premier* in 1823 and *Leçons sur le calcul différentiel* in 1829, in which he reduced the material in the *Cours d'analyse* about foundations to just a few dozen pages.

Because it became obsolete as a textbook just a year after it was published, the *Cours d'analyse* saw only one French edition in the 19th century. That first edition, published in 1821, was 568 pages long. The second edition, published as Volume 15 (also identified as Series 2, Volume III) of Cauchy's *Oeuvres complètes*, appeared in 1897. Its content is almost identical to the 1821 edition, but its pagination is quite different, there are some different typesetting conventions, and it is only 468 pages long. The *Errata* noted in the first edition are corrected in the second, and a number of new typographical errors are introduced. At least two facsimiles of the first edition were published during the second half of the 20th century, and digital versions of both editions are available on line, for example, through the Bibliothèque Nationale de France. There were German editions published in 1828 and 1885, and a Russian edition published in Leipzig in 1864. A Spanish translation appeared in 1994, published in Mexico by UNAM. The present edition is apparently the first edition in any other language.

The *Cours d'analyse* begins with a short Introduction, in which Cauchy acknowledges the inspiration of his teachers, particularly Pierre Simon Laplace (1749–1827) and Siméon Denis Poisson (1781–1840), but most especially his colleague and former tutor André Marie Ampère (1775–1836). It is here that he gives his oft-cited intent in writing the volume, "As for the methods, I have sought to give them all the rigor which one demands from geometry, so that one need never rely on arguments drawn from the generality of algebra."

The Introduction is followed by 16 pages of "Preliminaries," what today might be called "Chapter Zero." Here, Cauchy takes pains to define his terms, carefully distinguishing, for example, between *number* and *quantity*. To Cauchy, numbers had to be positive and real, but a quantity could be positive, negative or zero, real or imaginary, finite, infinite or infinitesimal.

Beyond the Preliminaries, the book naturally divides into three major parts and a couple of short topics. The first six chapters deal with real functions of one and several variables, continuity, and the convergence and divergence of series.

In the second part, Chapters 7 to 10, Cauchy turns to complex variables, what he calls *imaginary quantities*. Much of this parallels what he did with real numbers, but it also includes a very detailed study of roots of imaginary equations. We find here the first use of the words *modulus* and *conjugate* in their modern mathematical senses. Chapter 10 gives Cauchy's proof of the fundamental theorem of algebra, that a polynomial of degree n has n real or complex roots.

Chapters 11 and 12 are each short topics, partial fraction decomposition of rational expressions and recurrent series, respectively. In this, Cauchy's structure reminds us of Leonhard Euler's 1748 text, the *Introductio in analysin infinitorum* [Euler 1748], another classic in the history of analysis. In Euler, we find 11 chapters on real functions, followed by Chapters 12 and 13, "On the expansion of real functions into fractions," i.e., partial fractions, and "On recurrent series," respectively.

The third major part of the *Cours d'analyse* consists of nine "Notes," 140 pages in the 1897 edition. Cauchy describes them in his Introduction as "... several notes placed at the end of the volume [where] I have presented the derivations which may be useful both to professors and students of the Royal Colleges, as well as to those who wish to make a special study of analysis."

Though Cauchy was only 32 years old when he published the *Cours d'analyse*, and had been only 27 when he began teaching the analysis course on which it was based, he was already an accomplished mathematician. This should not be surprising, as it was not easy to earn an appointment as a professor at the École Polytechnique. Indeed, by 1821, Cauchy had published 28 memoirs, but the *Cours d'analyse* was his first full-length book.

Cauchy's first original mathematics concerned the geometry of polyhedra and was done in 1811 and 1812. Louis Poinsot (1777–1859) had just established the existence of three new nonconvex regular polyhedra. Cauchy, encouraged to study the problem by Lagrange, Adrien-Marie Legendre (1752–1833) and Étienne Louis Malus (1775–1812), [Belhoste 1991, pp. 25–26] extended Poinsot's results, discovered a generalization of Euler's polyhedral formula, $V-E+F=2$, and proved that a convex polyhedron with rigid faces must be rigid. These results became his earliest papers, the two-part memoir "*Recherches sur les polyèdres*" and "*Sur les polygones et les polyèdres*." [Cauchy 1813] Despite his early success, Cauchy seldom returned to geometry, and these are his only significant results in the field.

After Cauchy's success with the problems of polyhedra, his father encouraged him to work on one of Fermat's (1601–1665) problems, to show that every integer is the sum of at most three triangular numbers, at most four squares, at most five pentagonal numbers, and, in general, at most n n-gonal numbers. He presented his solution to the Institut de France on November 13, 1815 and published it under the title "*Démonstration générale du théorème de Fermat sur les nombres polygones*" [Cauchy 1815]. Belhoste [Belhoste 1991, p. 46] tells us that this was the article "that made him famous," and suggests that "[t]he announcement of his proof may have supported his appointment to the École Polytechnique a few days later."

Just a month later, on December 26, 1815, the Academy's judgment was confirmed when Cauchy won the *Grand Prix de Mathématiques* of the Institut de France, and its prize of 3000 francs, for an essay on the theory of waves.

With his career established, Cauchy married Aloïse de Bure (1795?–1863) in 1818. They had two daughters. It is a measure of Cauchy's later fame and success that one of his daughters married a count, the other a viscount. Indeed, Freudenthal [DSB Cauchy, p. 135] says that Cauchy "was one of the best known people of his time."

The de Bure family were printers and booksellers. The title page of the *Cours d'analyse*, published by de Bure frères, describes them as "*Libraires du Roi et de la bibliothèque du Roi.*"

It seems that Cauchy was an innovative but unpopular teacher at the École Polytechnique. He, along with Ampère and Jacques Binet (1786–1856), proposed substantial revisions in the analysis, calculus and mechanics curricula. Cauchy wrote the *Cours d'analyse* to support the new curriculum.

In 1820, though, before the *Cours d'analyse* was published, but apparently after it had been written and the publisher had committed to printing it, the *Conseil*

Fig. 1 Cauchy, by Susan Petry, 18×28 cm, bas relief in tulip wood, 2008. An interpretation of portraits by Boilly (1821) and Roller ($\sim$ 1840). Photograph by Eliz Alahverdian, 2008. Reprinted with permission of Susan Petry and Aliz Alahverdian.

d'Instruction, more or less a Curriculum Committee, largely influenced by de Prony (1755–1839) and Navier (1785–1836), ordered that Cauchy and Ampère change the curriculum again. As a consequence, the *Cours d'analyse* was never used as a textbook. A more complete account of this episode is found in [Belhoste 1991, pp. 61–66].

Lectures at the École Polytechnique were scheduled to be 50 lectures per term, each consisting of 30 minutes "revision" then 60 minutes of lecture. On April 12, 1821, Cauchy was delivering the 65th lecture of the term. When the lecture neared the end of its second hour, students began to jeer, and some walked out. A formal investigation followed, and eventually both the students and Cauchy were found responsible, but nobody was punished. Fuller accounts are found in [Belhoste 1991, pp. 71–74] and [Grattan-Guinness 1990, pp. 709–712].

From 1824 to 1830, Cauchy also taught part-time at the Collège de France, where he presented, among other techniques, methods of differential equations, and gave lectures on the theory of light. At the same time he worked also as a substitute professor on the Faculté des Sciences de Paris, where he replaced Poisson, and lectured on the mechanics of solids, fluid mechanics and on his general theory of elasticity.

By 1826, Cauchy had grown impatient with the time it took for the Academy to publish his articles and memoirs. That year they published only 11 of his memoirs, up from six in 1825, so he founded a private journal, the *Exercises de mathématiques*, published by his in-laws, Debure frères. By 1830, he had published five volumes of the *Exercises*, containing 51 of his articles. These comprise volumes 18 to 21 of the *Oeuvres complètes*.

The July Revolution of 1830 deposed the Bourbon monarch, Charles X. Cauchy refused to take a loyalty oath to his Orleans successor, Louis-Philippe, and went into 8 years of voluntary exile. He taught at the University of Turin from 1831 to 1833, where he continued his journal under the new name, *Résumés analytiques* (*Oeuvres complètes*, volume 22), and then spent the rest of his exile tutoring in Prague in the exile court of Charles X. While in Prague, his king awarded Cauchy the title "Baron."

In 1838, Cauchy returned to Paris, but because he had not taken the loyalty oath, he was not allowed to teach, either at the École Polytechnique or at his part-time jobs. He was still an active member of the Académie des Sciences, though, and over the next 10 years he submitted over 400 items to the *Comptes rendus*, the published notes and articles presented at the weekly meetings of the Academy. Because the Academy took breaks and vacations, "weekly" meetings did not actually take place every week. Over these 10 years, Cauchy averaged an article for each week the Academy was in session. These articles occupy most of volumes 4 to 10 of Cauchy's 27-volume *Oeuvres complètes*. At the same time, he continued his private journal under yet another title, the *Exercises d'analyse et de physique mathématique*. These 47 articles fill volumes 23 to 26 of the *Oeuvres complètes*. During his decade away from the classroom, 1838 to 1848, Cauchy produced about half of his published works by item count, about a third of them by page count. It was a remarkable decade.

The February Revolution of 1848 ended the reign of Louis-Philippe and established the Second Republic. Loyalty oaths were not required, so Cauchy returned to the Faculté des Sciences as professor of mathematical astronomy. When loyalty oaths were reestablished in 1852, Napoleon III made an exception for Cauchy.

Cauchy's last 9 years were active. In 1853, he published one last volume of the *Exercises d'analyse et de physique mathématique*. He did a good deal of research on the theory of light and bickered with his colleagues. He made another 159 contributions to the *Comptes rendus*. The last of his 589 contributions to that journal came on May 4, 1857. [*Oeuvres* 12, p. 435] It was a short note on mathematical astronomy, and he closed it with the words *C'est ce que j'expliquerais plus au long dans un prochain Mémoire*, "I will explain this at greater length in a future Memoir." Clearly, he was not expecting to die just 18 days later.

Many studies give more detailed accounts of Cauchy's life, works and times than we give here. For a full biography of Cauchy, we refer our readers to [Belhoste 1991]. The entry in the *Dictionary of Scientific Biography* [DSB Cauchy] is much briefer; it contains many inaccurate citations to Cauchy's work and in general seems to suffer from "hero worship." For example, we find no other source that describes Cauchy as "one of the best-known people of his time, and must have been often mentioned in newspapers, letters and memoirs." Still, its basic facts are correct.

For accounts of Cauchy's work and its importance, we recommend [Grabiner 2005] and [Grattan-Guinness 2005] as good places to begin. See also [Grattan-Guinness 1990] for a comprehensive account of the French mathematical community in the time of Cauchy.

Grattan-Guinness first presents his case that Cauchy "plagiarized" Bolzano in [Grattan-Guinness 1970a]. This assertion precipitated a controversy that raged through [Grattan-Guinness 1970b], [Freudenthal 1971b], and still echoed in [Grabiner 2005].

Other modern contributions to Cauchy scholarship are more numerous than we wish to describe, but we will mention in particular [Jahnke 2003], [Lützen 2003], [Ferraro 2008] and [Bottazzini 1990]. Starting with these references, the interested reader can find a great many more.

As we translated the *Cours d'analyse*, we laid out the text and formulas, used italics, bold face and punctuation, and, as much as possible, adopted the styles of the 1897 edition of the text. We have also added an index (neither the 1821 nor the 1897 editions have indices), and we have used our footnotes to note passages that are quoted, cited or translated in certain important secondary sources. We have not made note of errors cited in the *Errata* of the 1821 edition, all of which were corrected in the 1897 edition, but we have noted errors not mentioned in the *Errata*, as well as new errors introduced in the second edition. We distinguish such footnotes with the signature "(tr.)." Expository footnotes are unsigned.

We believe that the primary purpose of a translation such as this one is to make the work available in English, and not to provide a platform for our opinions on how this work should be interpreted. Towards this end, we have generally limited our commentary to expository remarks rather than interpretative ones. For those passages that are controversial and subject to a variety of interpretations, we try to refer the interested reader to appropriate entry-point sources and do not try to be comprehensive.

For a variety of reasons, we decided to follow Grabiner [Grabiner 2005], Freudenthal [DSB Cauchy] and others, rather than Kline [Kline 1990], and to make our translation, as well as to cite page numbers, from the second edition. Although electronic copies of both editions are freely available on the World Wide Web, bound copies of the 1821 edition are rather hard to find, while the second edition is found in many university libraries. The on-line library catalog WorldCat reports 57 copies of the 1821 edition in North America, and only seven copies of the facsimiles. Yet they report at least 117 copies of the 1897 edition in North America. We say "at least" because there are several different kinds of catalog entries, and it is difficult to tell how much duplication there is. We would estimate at least 200 copies. The two editions are identical in content, notation and format, but differ in pagination, page layout and some punctuation. In general, we found the typography and page layout of the 1821 edition somewhat cluttered, even quirky, particularly in the ways that formulas were cut into many lines to be arranged on the page. Weighing all these circumstances, it seemed more reasonable to follow the more accessible version.

In general, we resisted the temptation to modernize Cauchy's notation and terminology. When he uses the word *limites* to mean both what we call "limits" and what we call "bounds," we translate it as "limits" in both cases. In the index, citations of the word "limit" direct the reader to instances in which the limit process is being used, and not to instances meaning "bounds." Moreover, when he fails to distinguish between open intervals and closed intervals, or between "less than" and "less than or equal to," we translate it as Cauchy wrote it, and do not attempt to force upon Cauchy distinctions he himself did not make.

There are two conspicuous exceptions. Cauchy wrote lx, or sometimes Lx to denote the logarithm of x to a given base A. We modernize this to $\log x$ or $\mathrm{Log}\, x$ to avoid unnecessary confusion. Likewise, we write $\ln x$ to denote the natural logarithm of x, rather than using Cauchy's lx. Also, Cauchy used periods at the end of the abbreivated names of trignonometric functions (such as cos. x) and denoted the tangent and arctangent functions tang. x and arc tang. x. Following modern usage, we omit the periods and use $\tan x$ and $\arctan x$.

Cauchy did not adopt Euler's innovation of the 1770s, to write i for $\sqrt{-1}$, so we write $\sqrt{-1}$ as well.[1]

Within our translation of the text, numbers in square brackets, like [116], mark where new pages begin in the 1897 edition. Thus, for example, when we find the notation [116] in the midst of the statement of the Cauchy Convergence Criterion, we know that Cauchy's statement of that criterion appeared on pages 115 and 116

[1] Many people attribute Euler's first use of the symbol i to denote $\sqrt{-1}$ to his 1748 text, the *Introductio in analysin infinitorum* [Euler 1748] , but readers who check Volume II, Chapter 21, § 515 will see that the quantity Euler denotes there as i is actually $\ln(-n)$, for some positive value of n, and not the imaginary unit, $\sqrt{-1}$.

of the 1897 edition. We give a page concordance of the two French editions in an appendix.

Cauchy seemed to enjoy choosing his words carefully and precisely, and then once the correct words were chosen, using those very words over and over again. For example, in Chapter VII, § III, he studies the n-th roots of unity, or, as he calls them, 1 to the fractional power $\frac{1}{n}$. He states his theorems and gives his proofs about these objects. Later in that same section, when he studies other fractional powers of 1, $-\frac{1}{n}$, $\frac{m}{n}$, and $-\frac{m}{n}$, the words in his theorems and proofs are almost identical, changing only what must be changed. We have taken care to do the same in our translation.

Our ambition is, as much as the very idea of translation allows, to let Cauchy speak for himself.

We are grateful to Emili Bifet, David Burns, Larry D'Antonio, Ross Gingrich, Andy Perry, Kim Plofker, Fred Rickey, Chuck Rocca and Jeff Suzuki who, as participants in the ARITHMOS reading group, read early drafts of portions of this translation. Likewise, we are grateful to our students Shannon Abernathy, Erik Gundel, Amanda Peterson and Joseph Piraneo, who read parts of this manuscript in a history of mathematics seminar at Western Connecticut State University in the Spring of 2008. Careful proofreading and helpful suggestions by both groups have greatly improved this translation. We also acknowledge the assistance of the editorial staff at Springer, particularly Ann Kostant and Charlene Cruz Cerdas. Most importantly, we thank our wives Susan Petry and Terry Sandifer for supporting and encouraging our efforts, and for being understanding about the many long days that this project occupied.

Garden City, New York, *Robert E. Bradley*
Danbury, Connecticut, *C. Edward Sandifer*
March 2009

Contents

Introduction

[i] Because several people, who were so good as to guide the first steps of my scientific career, and among whom I would cite with recognition Messieurs Laplace[1] and Poisson[2] have expressed the desire to have me publish the *Cours d'analyse* of the École Royale Polytechnique, I have decided to put this Course in writing for the greatest usefulness to students. I offer here the first part of it[3] known by the name *Algebraic analysis*,[4] and in which I successively treat the various kinds of real and imaginary functions, [ii] convergent and divergent series, the resolution of equations and the decomposition of rational fractions.[5] In speaking of the continuity of functions, I could not dispense with a treatment of the principal properties of infinitely small quantities, properties which serve as the foundation of the infinitesimal calculus.[6] Finally, in the preliminaries and in several notes placed at the end of the volume, I have presented the derivations which may be useful both to professors and students of the Royal Colleges, as well as to those who wish to make a special study of analysis.

As for the methods, I have sought to give them all the rigor which one demands from geometry, so that one need never rely on arguments drawn from the generality of algebra.[7] Arguments of this kind, although they are commonly accepted, especially [iii] in the passage from convergent to divergent series, and from real

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[1] Pierre-Simon Laplace (1749–1827).

[2] Siméon Denis Poisson (1781–1840).

[3] Cauchy planned a second part of the *Cours d'analyse*, but no such volume was ever published. When Navier replaced Ampère as the second teacher of analysis, the faculty of the École Polytechnique considered revisions to the analysis curriculum. The changes that were made in 1822 as a result of these reforms took most of the emphasis on foundations out of the course in analysis, making Cauchy's planned second volume obsolete before it was written. This also explains why Cauchy produced no subsequent editions of the *Cours d'analyse*.

[4] *Cours d'analyse* is sometimes referred to as *Analyse algébrique*, for example, in the on-line catalog of the *Bibliotèque Nationale de France*.

[5] As we will see in Chapter I, these are "rational functions" in the modern sense.

[6] It is interesting that Cauchy does not also mention limits here.

[7] This sentence is quoted, in translation, in [DSB Cauchy, p. 135].

quantities to imaginary expressions,[8] may be considered, it seems to me, only as examples serving to introduce the truth some of the time, but which are not in harmony with the exactness so vaunted in the mathematical sciences. We must also observe that they tend to grant a limitless scope to algebraic formulas, whereas, in reality, most of these formulas are valid only under certain conditions or for certain values of the quantities involved. In determining these conditions and these values and in establishing precisely the meaning of the notation that I will be using, I will make all uncertainty disappear, so that the different formulas present nothing but relations among real[9] quantities, relations which will always be easy to verify [iv] by substituting numbers for the quantities themselves. It is true that, in order to remain consistently faithful to these principles, I will have to accept several propositions which may appear to be a bit rigid at first. For example, I state in Chapter VI that *a divergent series does not have a sum*;[10] in Chapter VII that *an imaginary equation is nothing but the symbolic representation of two equations involving real quantities*;[11] in Chapter IX that *if the constants or the variables involved in a function, having first been taken to be real, become imaginary, the notation used to express the function can be kept in the calculation only by virtue of a new convention keeping the sense of the notation in the latter hypothesis*;[12] &c. But those who read my book will recognize, [v] I hope, that propositions of this nature, entailing the happy necessity of putting more precision into the theories and of applying useful restrictions to assertions that are too broad, work in favor of analysis and furnish several research topics which are not without importance. Therefore, before summing any series, I must examine the cases in which the series can be summed, or, in other words, the conditions for its convergence; and I have, on this subject, established general rules which appear to me to merit some attention.

Moreover, if I have sought, on the one hand, to perfect mathematical analysis, yet on the other hand I am far from pretending that this analysis ought to be applied to all the rational sciences. Without a doubt, in those sciences we call "natural," the only [vi] method which we may successfully employ consists in observing the facts and then subjecting those observations to calculation. But it would be a grave error to think that we can find certainty only in geometric proofs, or in the evidence of the senses; and even though nobody has yet tried to prove by analysis the existence of Augustus[13] or that of Louis XIV,[14] all sensible people would admit that their existence is as certain to them as the square of the hypotenuse or the theorem of

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[8] Cauchy never speaks of imaginary *numbers*, but only of imaginary *expressions*. His imaginary expressions correspond to the modern notion of complex numbers. Here and throughout the book, we will use Cauchy's terminology and write "imaginary expressions."

[9] Here, Cauchy is careful to exclude imaginary expressions. As we will see later, imaginary expressions are equal, for example, when their corresponding real quantities are equal.

[10] See p. 85 or [Cauchy 1821, p. 123] or [Cauchy 1897, p. 114].

[11] See p. 119 or [Cauchy 1821, p. 176] or [Cauchy 1897, p. 155].

[12] See p. 159 or [Cauchy 1821, p. 240] or [Cauchy 1897, p. 204].

[13] Probably Caesar Augustus (63 BCE – 14 CE).

[14] Louis XIV (1638–1715). We note that Cauchy, whose given name is Augustin-Louis, may be engaging in a rare dsiplay of humor by choosing these two particular examples.

Maclaurin.[15] Furthermore, the proof of this last theorem is within reach of only a few people, and scientists themselves do not all agree on its scope one ought to attribute to it; whereas everyone knows quite well who ruled France in the 17th century, and no reasonable argument can be raised against this. What I say here [vii] about historical facts applies equally well to a whole range of questions in religion, ethics and politics. We should thus believe that there are truths other than algebraic truths, and realities other than tangible objects. Let us cultivate with ardor the mathematical sciences, without wishing to extend them beyond their domain; and let us not imagine that we are able to attack history with formulas, nor to make moral judgments with theorems of algebra or integral calculus.

In closing this Introduction, I cannot but acknowledge the insights and advice of several people who have been very helpful, particularly Messieurs Poisson, Ampère[16] and Coriolis.[17] I am indebted to this last person for the rule on the convergence of infinite products,[18] among other things, and I have profited many times from [viii] the observations of Monsieur Ampère, as well as from the methods which he develops in his lessons on analysis.[19]

[15] Colin Maclaurin (1698–1746); the reference is probably to Maclaurin series.

[16] André-Marie Ampère (1775–1836).

[17] Gaspard Gustave de Coriolis (1792–1843).

[18] See Note IX, Theorem I, p. 386 or [Cauchy 1821, p. 564] or [Cauchy 1897, p. 460].

[19] These appear to have been collected in *Cours d'analyse et de mécanique l'école polytechnique*, a manuscript of notes taken by G. Vincens of Ampère's course, which is available in the Dibner Collection of the Smtihsonian Institute.

Preliminaries

Cours d'analyse

of

the École Royale Polytechnique

Preliminaries

Review of the various kinds of real quantities which one might consider, be they algebraic or trigonometric, and of the notation we use to represent them. On the averages of several quantities.

[17] To avoid any kind of confusion in algebraic language and notation, we shall establish here in these Preliminaries the meanings of various terms and notation that we will use in ordinary algebra and trigonometry. The explanations that we will give for these terms are necessary so that we will be certain of being perfectly understood by those who read this work. First of all, we will indicate what idea will be appropriate to attach to the two words *number* and *quantity*.

We always take the meaning of *numbers* in the sense that is used in arithmetic, where numbers arise from the absolute measure of magnitudes, and we will only apply the term *quantities* to real positive or negative quantities, that is to say to numbers preceded by the signs $+$ or $-$. Furthermore, we regard these quantities as intended to express increase and decrease, so that a given magnitude will simply be represented by a number if we only mean to compare it to another magnitude of the same type taken as a unit, and by the same number preceded by the sign $+$ or the sign $-$, if we consider it [18] as being capable of increasing or decreasing a given magnitude of the same kind. Given this, the signs $+$ or $-$ placed in front of a number modify its meaning, more or less as an adjective modifies the meaning of a noun. We call the *numerical value* of a quantity that number which forms its basis.[1] We say that two quantities are *equal* if they have the same sign and the same numerical value, and two quantities are *opposites*[2] if their numerical values are the same but with opposite signs. From these principles, it is easy to give an account of the various operations that one may perform on these quantities. For example, given two quantities, one may always find a third quantity which, taken as increasing a fixed number, if it is positive, and as decreasing, if it is negative, brings us to the same result as the two given quantities, applied one after the other in the same way. This third quantity, which by itself produces the same effect as the other two is what we call their *sum*. For example, the two quantities -10 and $+7$ have as their sum -3, given that a decrease of 10 units followed by an increase of 7 units is equivalent to a decrease of 3 units.[3] To *add* two quantities is to form their sum. The difference between a first quantity and a second is a third quantity which, added to the second, gives the first. Finally, we say that one quantity is *larger* or *smaller* than another depending on whether the difference between the first and the second is positive or negative. It follows from this definition that positive numbers are always larger than negative numbers, and the latter ought to be considered as being as small as their numerical values are large.[4]

In algebra, we use letters to represent quantities as well as numbers. Since it is customary to classify the a numbers as positive quantities, we may denote the positive quantity that has as its numerical value the number A by $+A$ or just by A, whereas the opposite negative quantity is denoted by $-A$. Likewise, when the letter a represents a quantity, it is customary to regard [19] the two expressions a and $+a$ as synonyms, and to denote by $-a$ the quantity that is opposite to $+a$. These remarks suffice to establish what we call the *rule of signs* (see Note I).

We call a quantity *variable* if it can be considered as able to take on successively many different values. We normally denote such a quantity by a letter taken from the end of the alphabet. On the other hand, a quantity is called *constant*, ordinarily denoted by a letter from the beginning of the alphabet, if it takes on a fixed and determined value. When the values successively[5] attributed to a particular variable indefinitely approach a fixed value in such a way as to end up by differing from it by as little as we wish, this fixed value is called the *limit* of all the other values.[6] Thus, for example, an irrational number is the limit of the various fractions that give better and better approximations to it.[7] In Geometry, the area of a circle is the limit

[1] That is, the absolute value of the quantity.

[2] Later in Note III, Cauchy uses "contrary" rather than "opposite" to represent this idea.

[3] Cauchy discusses arithmetic operations and signs in some detail in Note I.

[4] In the 18th century, opposite numbers were considered to be the same size. Here, when Cauchy proposes that negative numbers are smaller than positive ones, it is a relatively new idea.

[5] This adverb "successively" (*successivement*) seems appropriate to a discussion of convergence of sequences, although perhaps less so in the case of continuous variables.

[6] This passage is translated in [Kline, p. 951].

[7] [DSB Cauchy, p. 136] cites this passage, but incorrectly states that it defines "convergence and absolute convergence of series, and limits of sequences and functions." It clearly does less than that.

towards which the areas of the inscribed polygons converge when the number of their sides grows more and more, etc.

When the successive numerical values of such a variable decrease indefinitely, in such a way as to fall below any given number, this variable becomes what we call *infinitesimal*, or an *infinitely small quantity*. A variable of this kind has zero as its limit.[8]

When the successive numerical values of a given variable increase more and more in such a way as to rise above any given number, we say that this variable has *positive infinity* as its limit,[9] denoted by the symbol ∞, if it is a positive variable, and *negative infinity*, denoted by $-\infty$, if it is a negative variable. The infinities, positive and negative, are designated together by the name of *infinite quantities*.

The quantities that arise in calculation as the result of operations made on one or more constant or variable quantities can be divided into various kinds, depending on the [20] nature of the operations that produce them. In algebra, for example, we distinguish sums and differences, products and quotients, powers and roots, and exponentials and logarithms. In trigonometry, we distinguish sines and cosines, secants and cosecants, tangents and cotangents, and the arcs of a circle for which a trigonometric line is given.[10] To better understand what is meant by these last kinds of quantities, it is necessary to review the following principles.

A length measured along a straight or curved line may, like any kind of magnitude, be represented either by a number or by a quantity. It would be represented as a number when we consider it only as a measure of this length, and as a quantity, that is to say as a number preceded by a $+$ or a $-$ sign, when we consider the length in question as drawn from a fixed point along the given line in one direction or the other, serving as the increase or the decrease of another constant length that ends at this fixed point. The fixed point in question from which we must measure the variable lengths denoted by these quantities is what we call the *origin* of these same lengths. Two lengths measured from a common origin but in opposite directions must be represented by quantities of different sign. We may choose at will the direction in which lengths are denoted by positive quantities, but once that choice is made, we must necessarily consider lengths denoted by negative numbers as going in the opposite direction.

In a circle, whose plane is assumed to be vertical, we ordinarily take for the origin of the arcs the endpoint of the radius drawn horizontally from left to right, and we measure positive arcs as rising above this point, that is to say, those arcs which we measure by positive quantities. On the same circle, when the radius is assumed to be 1, the sine of an arc, that is to say the projection of the radius which passes through the endpoint of the arc onto the vertical diameter is measured [21] positively from bottom to top, and negatively in the opposite direction, taking the center of the circle as the origin of the sines. The tangent is measured positively in the same direction as the sine but measured from the origin of the arcs along the vertical line drawn

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[8] This passage is also translated in [Kline, p. 951].

[9] This passage is also translated in [Kline, p. 951].

[10] Cauchy means inverse trigonometric functions. Note that they were still called "lines" and that this is implicitly using Euler's unit circle definition of trigonometric functions.

from this origin. Finally, the secant is measured from the center of the circle along the radius drawn through the endpoint of the arc in question[11] and positively in the direction of this radius.

Frequently, the result of an operation performed on a quantity may have several values, different from one another. When we do not wish to distinguish among the various values, we use a notation in which the quantity is enclosed in doubled symbols or double parentheses and we reserve the ordinary notation for the most simple value, or the one that seems to deserve to be distinguished. Therefore, for example, if a is a positive quantity, the square root of a has two values, numerically equal, but with opposite signs, an arbitrary one of which is expressed by the notation

$$((a))^{\frac{1}{2}} \quad \text{or} \quad \sqrt{\sqrt{a}}$$

while the positive value alone is written as

$$a^{\frac{1}{2}} \quad \text{or} \quad \sqrt{a},$$

so that we have

(1) $$\sqrt{\sqrt{a}} = \pm\sqrt{a}$$

or, what amounts to the same thing,

(2) $$((a))^{\frac{1}{2}} = \pm a^{\frac{1}{2}}.$$

Similarly, if we represent a positive or negative quantity by a, the notation[12]

$$\arcsin((a)) \quad \text{or} \quad \arctan((a))$$

denotes an arbitrary arc having the quantity a for its sine or for its tangent, respectively, while the notation

$$\arcsin(a) \quad \text{or} \quad \arctan(a)$$

[22] indicates only that particular arc with the smallest numerical value. With the aid of these conventions, we avoid the confusion that could result from the use of symbols, the values of which have not been determined precisely. In order to remove all difficulty in this matter, I give here the table of notations which we will use for expressing the results of algebraic and trigonometric operations.

The sum of two quantities is usually denoted by the juxtaposition of these two quantities, each of which is expressed by a letter preceded by the sign $+$ or $-$, which we may suppress (if the sign is $+$) in front of the first letter only. And so,

[11] That is, along the radius from the center through the circumference and to the vertical line along which tangents are measured.

[12] Cauchy writes "arc sin$((a))$" and "arc tang$((a))$." Here and subsequently we will use the more modern notation and write "arcsin" and "arctan." (tr.)

$$+a+b, \quad \text{or simply} \quad a+b,$$

denotes the sum of the two quantities $+a$ and $+b$, and

$$+a-b, \quad \text{or simply} \quad a-b,$$

denotes the sum of the two quantities $+a$ and $-b$, which is equivalent to the difference of the two quantities $+a$ and $+b$.

We indicate the equality of two quantities a and b by the sign $=$, written between them, as follows,

$$a = b,$$

and we indicate that the first is greater than the second, that is to say that the difference $a-b$ is positive, by writing

$$a > b \quad \text{or} \quad b < a.$$

As usual, we represent the product of two quantities $+a$ and $+b$ by[13]

$$+a \times +b, \quad \text{or simply} \quad a.b \quad \text{or} \quad ab$$

and their quotient by

$$\frac{a}{b} \quad \text{or} \quad a : b.$$

[23] Now let m and n be two whole numbers, A an arbitrary number and a and b two arbitrary quantities, positive or negative. Then

$$A^m,\ A^{\frac{1}{n}} = \sqrt[n]{A},\ A^{\pm\frac{m}{n}} \quad \text{and} \quad A^b$$

represent the positive quantities which we obtain by raising the number A to the powers denoted respectively by the exponents

$$m,\ \frac{1}{n},\ \pm\frac{m}{n} \quad \text{and} \quad b,$$

and

$$a^{\pm m}$$

denotes the quantity, positive or negative, that arises from taking the quantity a to the power $\pm m$. We use the notation

$$((a))^{\frac{1}{n}} = \sqrt[n]{\sqrt{a}} \qquad \text{and} \quad ((a))^{\pm\frac{m}{n}}$$

to denote not only the positive and negative values, when they exist, of the powers of the quantity a raised to the exponents

[13] In [Cauchy 1821, p. 9, Cauchy 1897, p. 22] Cauchy used a period in *a.b* rather than a centered dot, as we would today.

$$\frac{1}{n} \quad \text{and} \quad \pm\frac{m}{n},$$

but also the imaginary values[14] of these same powers (see Chap. VII for the meaning of *imaginary expressions*). It is helpful to observe that if we let A be the numerical value of a, and if we assume that the fraction $\frac{m}{n}$ is in lowest terms, then the power

$$((a))^{\frac{m}{n}}$$

has a single positive or negative real value, namely

$$+A^{\frac{m}{n}} \quad \text{or} \quad -A^{\frac{m}{n}},$$

as long as $\frac{m}{n}$ is a fraction with an odd denominator, but if the denominator is even, then it has [24] either the two real values just mentioned, or no real values. We could make a similar remark about the expression

$$((a))^{-\frac{m}{n}}.$$

In the particular case where the quantity a is positive and we let $\frac{m}{n} = \frac{1}{2}$, the expression $((a))^{\frac{m}{n}}$ has two real values, given by formula (2) or, what amounts to the same thing, by formula (1).

The notations[15]

$$l(B),\ L(B),\ L'(B),\ \ldots$$

denote the real logarithms of the number B to different bases, whereas each the following,

$$l((b)),\ L((b)),\ L'((b)),\ \ldots$$

denote, in addition to the real logarithm of the quantity b, when it exists, any of the imaginary logarithms of this same quantity (see Chap. IX for the meaning of *imaginary logarithms*).

In trigonometry,

$$\sin a, \quad \cos a, \quad \tan a, \quad \cot a, \quad \sec a, \quad \csc a, \quad \text{siv}\, a \quad \text{and} \quad \text{cosiv}\, a$$

denote, respectively, the sine, cosine, tangent, cotangent, secant, cosecant, versine and vercosine of the arc a.[16] The notations

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[14] Cauchy does not actually define an "imaginary value," but it is clear that it is what we get when we assign particular real values to the real quantities in an imaginary expression.

[15] Here we have reproduced Cauchy's notation for logarithm. Subsequently, we will always use more modern notation, like $\ln(B)$, $\log(B)$, $\text{Log}(B)$.

[16] We note that Cauchy uses "tang. a" for the trigonometric function as well as the inverse trigonometric function. His notations for secant and cosecant are "séc. *a*" and "coséc. *a*." Note also his use of the obsolete trigonometric functions versed sine and versed cosine. He will later also use the obsolete function chord on p. 45; [Cauchy 1821, p. 63, Cauchy 1897, p. 66].

$$\arcsin((a)),\ \arccos((a)),\ \arctan((a)),$$
$$\operatorname{arccot}((a)),\ \operatorname{arcsec}((a)),\ \operatorname{arccsc}((a))$$

indicate some one of the arcs which have the quantity a as their sine, cosine, tangent, cotangent, secant or cosecant. We use the simple notations

$$\arcsin(a),\ \arccos(a),\ \arctan(a),\ \operatorname{arccot}(a),\ \operatorname{arcsec}(a),\ \operatorname{arccsc}(a),$$

[25] or we may suppress the parentheses and write

$$\arcsin a,\ \arccos a,\ \arctan a,\ \operatorname{arccot} a,\ \operatorname{arcsec} a,\ \operatorname{arccsc} a$$

when, from among the arcs for which a trigonometric function is equal to a,[17] we wish to designate the one with smallest numerical value, or, if there are two such arcs with opposite signs, the one with the positive value. Consequently,

$$\arcsin a,\ \arctan a,\ \operatorname{arccot} a,\ \operatorname{arccsc} a,$$

denote positive or negative arcs between the limits

$$-\frac{\pi}{2} \quad \text{and} \quad +\frac{\pi}{2},$$

where π denotes the semiperimeter of the unit circle, whereas

$$\arccos a \quad \text{and} \quad \operatorname{arcsec} a$$

denote positive arcs between 0 and π.

By virtue of the conventions that we have just established, if we denote by k an arbitrary positive integer, we obviously have, for arbitrary positive or negative values of the quantity a,

$$(3)\qquad \begin{cases} \arcsin((a)) = \frac{\pi}{2} \pm \left(\frac{\pi}{2} - \arcsin a\right) \pm 2k\pi, \\ \arccos((a)) = \pm \arccos a \pm 2k\pi, \\ \arctan((a)) = \arctan a \pm k\pi, \\ \arccos a + \arcsin a = \frac{\pi}{2} \quad \text{and} \\ \operatorname{arccsc} a + \operatorname{arcsec} a = \frac{\pi}{2}. \end{cases}$$

Furthermore, we find that, for positive values of a,

$$(4)\qquad \operatorname{arccot} a + \arctan a = \frac{\pi}{2},$$

[26] and for negative values of a,

[17] Here, Cauchy writes "...parmi les arcs dont un ligne trigonométrique est égale à a." This translates literally "...among the arcs for which a trigonometric line is equal to a." Cauchy is treating trigonometric functions as giving lines, which have signed lengths, rather than in the modern view of giving real numbers.

(5) $$\operatorname{arccot} a + \arctan a = -\frac{\pi}{2}.$$

When a variable quantity converges towards a fixed limit, it is often useful to indicate this limit with particular notation. We do this by placing the abbreviation[18]

$$\lim$$

in front of the variable quantity in question. Sometimes, when one or several variables converge towards fixed limits, an expression containing these variables converges towards several different limits at the same time. We therefore denote an arbitrary one of these limits using the doubled parentheses following the abbreviation lim, so as to enclose the expression under consideration. Specifically, suppose that a positive or negative variable denoted by x converges towards the limit 0, and denote by A a constant number. It is easy to see that each of the expressions

$$\lim A^x \quad \text{and} \quad \lim \sin x$$

has a unique value determined by the equation

$$\lim A^x = 1$$

or

$$\lim \sin x = 0,$$

whereas the expression

$$\lim \left(\left(\frac{1}{x} \right) \right)$$

takes two values, $+\infty$ and $-\infty$, and

$$\lim \left(\left(\sin \frac{1}{x} \right) \right)$$

admits an infinity of values between the limits -1 and $+1$.

We will finish these preliminaries by presenting several theorems on average quantities, the knowledge of which will [27] be extremely useful in the remainder of this work. We call an *average* among several given quantities a new quantity between the smallest and the largest of those under consideration. From this definition it is clear that there are an infinity of averages among several unequal quantities, and that the average among several equal quantities is equal to their common value. Given this, we will easily establish, as one can see in Note II, the following propositions:

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[18] The notation "Lim." for limit was first used by Simon Antoine Jean L'Huilier (1750–1840) in [L'Huilier 1787, p. 31]. Cauchy wrote this as "lim." in [Cauchy 1821, p. 13]. The period had disappeared by [Cauchy 1897, p. 26].

Theorem I.[19] — *Let b, b′, b″, … denote n quantities of the same sign, and a, a′, a″, … be the same number of arbitrary quantities. The fraction*

$$\frac{a+a'+a''+\dots}{b+b'+b''+\dots}$$

is an average of the following quantities,

$$\frac{a}{b},\ \frac{a'}{b'},\ \frac{a''}{b''},\ \dots.$$

Corollary. — If we let

$$b=b'=b''=\dots=1,$$

it follows from the preceding theorem that the quantity

$$\frac{a+a'+a''+\dots}{n}$$

is an average of the quantities

$$a,\ a',\ a'',\ \dots.$$

This particular kind of average is called the *arithmetic mean*.

Theorem II. — *Let A, A′, A″, …; B, B′, B″, …, be two sequences of numbers taken at will, which we suppose contain n terms each. Form from these two sequences the roots*

$$\sqrt[B]{A},\ \sqrt[B']{A'},\ \sqrt[B'']{A''},\ \dots.$$

[28] *Then* $\sqrt[B+B'+B''+\dots]{AA'A''\dots}$ *is a new root which is an average of the other roots.*

Corollary. — If we let

$$B=B'=B''=\dots=1,$$

we find that the positive quantity

$$\sqrt[n]{AA'A''\dots}$$

is an average of

$$A,\ A',\ A'',\ \dots.$$

This particular average is called the *geometric mean*.

Theorem III. — *With the same hypotheses as in theorem I, and if* α, α', α'', … *again denote quantities of the same sign, the fraction*

[19] Cauchy gives a proof of this theorem in Note II, Theorem XII [Cauchy 1821, p. 447, Cauchy 1897, p. 368].