Neil F. Cramer

The Physics of Alfvén Waves

WILEY-VCH

Berlin · Weinheim · New York · Chichester · Brisbane · Singapore · Toronto

This page intentionally left blank

Neil F. Cramer

The Physics of Alfvén Waves

The Physics of Alfvén Waves. Neil F. Cramer

ISBN: 3-527-40293-4

Neil F. Cramer

The Physics of Alfvén Waves

WILEY-VCH

Berlin · Weinheim · New York · Chichester · Brisbane · Singapore · Toronto

Dr Neil F. Cramer
School of Physics,
University of Sydney, Australia
e-mail: cramer@physics.usyd.edu.au

1st edition

Library of Congress Card No: applied for

British Library Cataloguing-in-Publication Data: A catalogue record for this book is available from the British Library.

Die Deutsche Bibliothek – CIP Cataloguing-in-Publication-Data
A catalogue record for this publication is available from Die Deutsche Bibliothek

ISBN 3-527-40293-4

Printed on non-acid paper.

Printing: Strauss Offsetdruck GmbH, D-69509 Mörlenbach
Bookbinding: Wilhelm Osswald & Co., D-67433 Neustadt (Weinstraße)

Printed in the Federal Republic of Germany.

WILEY-VCH Verlag Berlin GmbH
Bühringstrasse 10
D-13086 Berlin

Dedicated to
Christine

This page intentionally left blank

Preface

This book was written with the aim of providing a unified treatment, suitable for beginning researchers, of the properties and applications of Alfvén waves and related waves in magnetized plasmas, that is, ionized gases in a magnetic field strong enough to affect the behaviour of the ions and electrons. These waves are among the most fundamental features of magnetized plasmas. The term "Alfvén wave" is used with a number of different meanings in the literature, and it is intended in this book to provide a guide to these different usages.

The book covers the basic properties of the low–frequency wave modes in magnetized plasmas, the Alfvén waves and magnetoacoustic waves. (Henceforth in this book the general term "Alfvén waves" will be understood to encompass magnetoacoustic waves as well). In addition, it covers results of the latest research in applications of the waves in the contexts of laboratory, space and astrophysical plasmas, in particular some of the results achieved since the publication over a decade ago of the two "standard" references on Alfvén waves, *The Alfvén Wave* by Hasegawa & Uberoi (1982), and *An Introduction to Alfvén Waves* by Cross (1988). This book also covers a somewhat broader range of topics than the two earlier books.

There is much potential for cross–fertilization between the different areas of application of plasma physics. This book aims to facilitate this cross–fertilization by showing the common features of the physics of Alfvén waves across the various plasma environments. There is, of course, an enormous volume of work on Alfvén waves in the literature, so I have primarily selected topics in which I have research experience. Hopefully this selection will give starting researchers a flavour of this interesting field of physics, and at least help to point them in the right direction for their specific interests.

The book deals especially with nonideal effects, such as multi–species, collisional and kinetic effects. These physical processes have consequences for the dispersion relations and absorption properties of linear and nonlinear Alfvén waves that are not predicted by ideal or pure magnetohydromagnetic (MHD) theory. It has been common in the literature to use ideal MHD theory to treat the behaviour of the waves, particularly when treating problems of propagation in nonuniform plasmas. This approach has the virtue that the basic equations are relatively simple differential equations, which may often be solved analytically. They lead to many interesting mathematical problems, such as the treatment of the absorption of wave energy at the "Alfvén resonance" by analysing the behaviour of the solutions at the singularities of the differential equations, as well as nonlinear properties of the waves.

However, the emphasis of this book is the analysis of the effects of nonideal corrections to the MHD theory on the properties of the waves. These corrections are intended to reflect the physics of the realistic applications of the theory more closely.

Some of the corrections which have been treated in the literature are: finite frequency effects (i.e. allowance for the wave frequency to approach the ion cyclotron frequency), inclusion of minority ion species and charged dust grains, resistivity, viscosity, friction with neutral particles, and kinetic theory effects. All of these modifications of the plasma change the dispersion relation of the waves, and most cause damping of the waves. The disadvantage of including nonideal effects compared to the MHD approach is that the starting equations are higher-order fluid differential equations, in the case of finite frequency, resistivity and viscosity effects, or are coupled integro-differential equations in the case of kinetic theory effects. The greater complexity of the equations reflects the greater number of wave modes allowable in the plasma when nonideal effects are taken into account, and they can often only be solved numerically.

A unique feature of the "shear" Alfvén wave in ideal MHD is the fact that wave energy propagates along the magnetic field, regardless of the angle of the wave front with the magnetic field. This feature leads to several fascinating phenomena, the discussion of which in the literature has at times been controversial, such as localized propagation, the absence of discrete eigenmodes in nonuniform plasmas, and resonance absorption. This book endeavours to provide a coherent and unified explanation of such phenomena in terms of the above–mentioned nonideal effects.

The plan of the book is as follows. In the first chapter we establish the basic models used to describe the plasma, and give the equations used in the subsequent analysis of the waves. The multi–fluid, ideal MHD and kinetic theory models are discussed. Chapter 2 deals with the waves in a uniform plasma, employing the different plasma models, and then Chapter 3 discusses the waves in nonuniform plasmas, in particular the case of stratified plasmas. Chapter 4 follows on from Chapter 3 with a treatment of surface waves in strongly nonuniform plasmas. The instabilities of nonequilibrium plasmas that produce Alfvén and magnetoacoustic waves, and the theory of nonlinear Alfvén waves, and the instabilities of the waves themselves, are reviewed in Chapter 5. The applications of Alfvén waves in laboratory plasmas are discussed in Chapter 6, including the use of the waves in devices for controlled nuclear fusion. Chapter 7 investigates the natural occurrence of the waves in space and solar plasmas, ranging from the Earth's magnetosphere to the interplanetary plasma and the Sun's atmosphere. Finally, in Chapter 8 we look at two problems in astrophysical plasmas: waves in partially ionized and dusty interstellar clouds, and in the relativistic and very strongly magnetized plasma of pulsar magnetospheres.

I owe a great deal to my close research collaborators on the physics of Alfvén waves in Sydney, initially Ian Donnelly and, more recently, Sergey Vladimirov, for their stimulation. Amongst my overseas collaborators, I would like to particularly thank Frank Verheest and Jun-Ichi Sakai. I have also drawn much inspiration over the years from the group of Australian Alfvén wave enthusiasts, including the late Frank Paoloni, Rod Cross, Bob Dewar and Robin Storer. Don Melrose has inspired me through his vast knowledge of plasma physics, and has provided a stimulating research environment in Sydney. Dave Galloway has been my valued local expert on solar Alfvén waves. I thank Bob May and Les Woods for introducing me to Alfvén waves at the beginning of my career. Lastly, my students George Rowe, Robert Winglee, Ken Wessen and Lap Yeung, have made invaluable contributions to our research in Sydney on Alfvén waves.

Part of the writing of this book was accomplished during a study leave spent at Toyama University, Japan, the University of Gent, Belgium, and the Max Planck Institute for Extraterrestrial Physics, Garching, Germany, and I thank Professors Sakai, Verheest and Morfill for their hospitality. The work encompassed in this book has also been supported by grants from the Australian Research Council.

Sydney
June 2001

Neil Cramer

This page intentionally left blank

Contents

1 Descriptions of Magnetized Plasmas 1
1.1 Introduction 1
1.2 The Multi–Fluid Equations 2
1.3 The Magnetohydrodynamic Model 3
1.4 The Hall-MHD Model 6
1.5 Fourier Transforms 8
1.6 The Kinetic Theory 8

2 Waves in Uniform Plasmas 11
2.1 Introduction 11
2.2 Waves with the MHD Model 11
2.2.1 The Alfvén Mode 14
2.2.2 The Fast and Slow Magnetoacoustic Modes 16
2.3 The Hall-MHD Model 17
2.3.1 Cold Plasma 17
2.3.2 Warm Plasma 19
2.4 Cold Collisionless Plasmas 21
2.5 Collisional Damping 26
2.5.1 Low Frequency 27
2.5.2 Hall Effects 30
2.6 Multiple Ion Species 33
2.7 Kinetic Theory of Waves 37
2.7.1 Parallel Propagation 37
2.7.2 Low Plasma Beta 40
2.7.3 High Plasma Beta 44
2.8 Kinetic Alfvén Wave and Inertial Alfvén Wave 52
2.8.1 Fluid Theory 53
2.8.2 Parallel Electron Temperature Effects 57
2.8.3 Two–Potential Theory 58
2.8.4 Kinetic Theory 60
2.8.5 Localized Alfvén Waves 62

3 Waves in Nonuniform Plasmas 65
3.1 Introduction 65
3.2 Stratified Plasmas 66

3.2.1 Ideal MHD . . . 66
3.2.2 Hall MHD . . . 69
3.2.3 Multi–Ion Plasmas . . . 76
3.2.4 Effects of Collisions . . . 80
3.3 Waves in Smooth Nonuniformities . . . 83
3.3.1 Ideal MHD . . . 84
3.3.2 Cold Plasma . . . 88
3.4 Alfvén Resonance Absorption . . . 89
3.4.1 Narrow Interfaces . . . 90
3.4.2 Analytic Derivation . . . 95

4 Surface Waves **97**
4.1 Introduction . . . 97
4.2 Surface Waves at Density Jumps . . . 98
4.2.1 Cold Plasma . . . 99
4.2.2 Low–Frequency Surface Waves . . . 101
4.3 Finite Ion Cyclotron Frequency Effects . . . 102
4.3.1 Surface Wave Frequency . . . 102
4.3.2 Resonance Damping . . . 104
4.4 Multiple Ion Species . . . 105
4.4.1 Surface Wave Solutions . . . 106
4.4.2 Resonance Damping . . . 109
4.5 Ideal MHD . . . 111
4.5.1 Dispersion Relation . . . 111
4.5.2 Resonance Damping . . . 114
4.6 Magnetic Field Rotation . . . 115
4.6.1 Ideal MHD . . . 116
4.6.2 Cyclotron Effects . . . 124
4.7 Radiative and Collisional Damping . . . 126
4.7.1 Radiative Damping . . . 127
4.7.2 Collisional Damping . . . 129
4.8 Kinetic Theory . . . 131

5 Instabilities and Nonlinear Waves **135**
5.1 Introduction . . . 135
5.2 Instabilities . . . 135
5.2.1 Macroinstabilities . . . 136
5.2.2 Microinstabilities . . . 137
5.3 Acceleration of Charged Particles . . . 141
5.4 Nonlinear Waves . . . 142
5.4.1 Wave Equations . . . 142
5.4.2 Low Frequency . . . 143
5.4.3 Higher Frequency . . . 144
5.4.4 Oblique Propagation . . . 149
5.5 Parametric and Modulational Instabilities . . . 151

5.5.1 Excitation by a Magnetoacoustic Pump . . . 152
5.5.2 Instability of Alfvén waves . . . 156
5.6 Nonlinear Kinetic and Inertial Alfvén Waves . . . 161
5.7 Nonlinear Surface Waves . . . 162
5.7.1 Nonlinear Surface Waves with Hall Dispersion . . . 162
5.7.2 Surface Alfvén Wave Solitons . . . 164

6 Laboratory Plasmas **167**
6.1 Introduction . . . 167
6.2 Modes of Bounded Plasmas . . . 168
6.2.1 Resistive Plasmas . . . 169
6.3 Cylindrical Geometry . . . 171
6.3.1 Uniform Plasma . . . 172
6.3.2 Bounded Plasma . . . 173
6.3.3 Nonideal Effects . . . 180
6.4 Nonuniform Plasmas . . . 182
6.5 Effects of Current . . . 188
6.6 Discrete Alfvén Waves . . . 193
6.6.1 Slab Plasma . . . 194
6.6.2 Cylindrical Plasma . . . 199
6.7 Toroidal Alfvén Eigenmodes . . . 201
6.8 Current Drive . . . 205
6.9 Localized Alfvén Waves . . . 206

7 Space and Solar Plasmas **209**
7.1 Introduction . . . 209
7.2 The Magnetosphere . . . 209
7.2.1 Micropulsations . . . 209
7.2.2 Kinetic and Inertial Alfvén Waves . . . 211
7.3 Solar and Stellar Winds . . . 213
7.3.1 Turbulent Waves in the Solar Wind . . . 213
7.3.2 Wind Acceleration . . . 214
7.4 Dusty Space Plasmas . . . 216
7.5 Cometary Plasmas . . . 220
7.5.1 Ion Ring–Beam Instability . . . 220
7.5.2 The Dispersion Equation . . . 221
7.5.3 Unstable Waves . . . 224
7.5.4 The Effects of Dust . . . 226
7.6 The Solar Corona . . . 230
7.6.1 Heating of the Corona . . . 230
7.6.2 Resonant Absorption . . . 230
7.6.3 Phase Mixing . . . 232
7.7 Solar Flux Tubes . . . 232
7.7.1 Modes in Flux Tubes . . . 232
7.7.2 Twisted Flux Tubes . . . 235

7.7.3 The Interaction of Acoustic Waves with Flux Tubes 238

8 Astrophysical Plasmas **239**

8.1 Introduction 239

8.2 Interstellar Clouds 239

8.2.1 Alfvén and Magnetoacoustic Waves in Interstellar Clouds 240

8.2.2 Wave Equations with Collisions 241

8.2.3 The Dispersion Relation 243

8.2.4 Conductivity Tensor 244

8.2.5 Damping and Dispersion of Alfvén Waves 245

8.2.6 Nonlinear Waves 251

8.3 Pulsar Magnetospheres 260

8.3.1 Cold Pair Plasmas 261

8.3.2 Relativistic Plasmas 265

8.3.3 Instabilities 266

8.4 Concluding Remarks 268

Appendix: The Dielectric Tensor **269**

References **273**

Index **291**

1 Descriptions of Magnetized Plasmas

1.1 Introduction

It was discovered, some five decades ago, that low–frequency electromagnetic waves are able to propagate in conducting fluids, such as plasmas, even though they cannot propagate in rigid conductors. Hannes Alfvén, in 1942, investigated the properties of plasmas, assuming the plasma medium to be a highly conducting, magnetized and incompressible fluid. He found that a distinctive wave mode arises in the fluid, propagating along the magnetic field direction (Alfvén 1942). This wave is now called the shear or torsional Alfvén wave. The existence of the wave, in the conducting fluid mercury, was experimentally verified by Lundquist (1949). The importance of the waves discovered by Alfvén for space and astrophysical plasmas was soon realized, and the compressible plasma case, which leads to the fast and slow magnetoacoustic waves in addition to the shear Alfvén wave, was treated by Herlofsen (1950).

The Alfvén and magnetoacoustic waves, which are the basic low–frequency wave modes of magnetized plasmas, have been the subject of intense study in the succeeding decades. The main reason for the great interest in these waves is that they play important roles in the heating of, and the transport of energy in, laboratory, space and astrophysical plasmas. The "Alfvén wave heating" scheme has been investigated theoretically and experimentally as a supplementary heating scheme for fusion plasma devices, and it has been invoked as a model of the heating of the solar and stellar coronae. The waves are believed to underlie the transport of magnetic energy in the solar and stellar winds, transfer angular momentum in interstellar molecular clouds during star formation, play roles in magnetic pulsations in the Earth's magnetosphere, and provide scattering mechanisms for the acceleration of cosmic rays in astrophysical shock waves. These and other applications of Alfvén and magnetoacoustic waves in the fusion, space physics and astrophysics fields are the subject of this book.

In realistic physical problems in all plasma environments, Alfvén and magnetoacoustic waves propagate in nonuniform plasmas. As a result, the waves may be reflected, transmitted or absorbed. The practical question of the heating to high temperatures of laboratory fusion plasmas that are contained in a vessel, and are therefore necessarily nonuniform, involves such processes. The space and astrophysical environments where the waves are found are also inevitably nonuniform.

Wave energy can be concentrated in plasma regions of nonuniform density and/or magnetic field, and in the limiting case of density or magnetic field discontinuities, when a well-defined surface is present, wave eigenmodes exist whose amplitudes decay approximately exponentially in each direction away from the surface. These are the Alfvén surface wave eigenmodes, which have been shown by theory and experiment to play an important role

The Physics of Alfvén Waves. Neil F. Cramer

ISBN: 3-527-40293-4

in Alfvén wave heating, because they can be easily excited by an antenna in a laboratory plasma. Alfvén surface waves are also expected to exist in astrophysical plasmas where jumps in density or magnetic field occur, such as the surfaces of magnetic flux tubes in the solar and stellar atmospheres, or the boundaries between plasmas of different properties in the Earth's magnetosphere. The properties of the waves when they propagate in nonuniform plasmas, including the phenomenon of Alfvén surface waves, are treated here in some depth.

A number of different models of the plasma, namely the multi–fluid, ideal MHD, Hall-MHD and kinetic theory models, are presented in this chapter, as well as a brief summary of Fourier transform theory. The results are used in later chapters to describe the self–consistent response of the plasma medium to the presence of the waves. All the models use Maxwell's equations for the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$:

$$\nabla \cdot \boldsymbol{E} = \frac{\rho_e}{\epsilon_0} \tag{1.1}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{1.2}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{1.3}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}. \tag{1.4}$$

Here ρ_e is the electric charge density, $\boldsymbol{J}$ is the current density, and c is the speed of light *in vacuo*. SI units are used throughout.

1.2 The Multi–Fluid Equations

We consider first the multi–fluid model of a plasma, in which each distinct species of particle is specified by the index α, with mass m_α and charge $Z_\alpha e$, where e is the fundamental unit of electric charge. Each collection of particles of a specific type is supposed to act as a fluid, with its own velocity $\boldsymbol{v}_\alpha$, mass density ρ_α, number density n_α and pressure p_α. Each fluid is "collision dominated". This means that the time for relaxation of each type of species to a Maxwellian velocity distribution with a unique temperature T_α, through collisions of like particles, is short compared with the other time–scales of interest. Each fluid may be acted on by the electric and magnetic fields, and may act on the other fluids via collisions, which may have characteristic times of the order of the time–scales of interest.

The equation of motion for the fluid corresponding to the species α is

$$\frac{\mathrm{d}\boldsymbol{v}_\alpha}{\mathrm{d}t} = -\frac{1}{\rho_\alpha}\nabla p_\alpha + \frac{Z_\alpha e}{m_\alpha}(\boldsymbol{E} + \boldsymbol{v}_\alpha \times \boldsymbol{B}) - \sum_{\alpha'} \nu_{\alpha\alpha'}(\boldsymbol{v}_\alpha - \boldsymbol{v}_{\alpha'}) \tag{1.5}$$

where $\nu_{\alpha\alpha'}$ is the collision frequency of a particle of species α with particles of species α'. The continuity equation for each fluid is, if sources and sinks for the particles are neglected,

$$\frac{\partial \rho_\alpha}{\partial t} + \nabla \cdot (\rho_\alpha \boldsymbol{v}_\alpha) = 0. \tag{1.6}$$

A partial pressure $p_\alpha = n_\alpha k_B T_\alpha$, where k_B is Boltzmann's constant, may be associated with each species. The thermal speed for species α is then defined as

$$V_\alpha = \sqrt{k_B T_\alpha / m_\alpha}. \tag{1.7}$$

A strong magnetic field $\boldsymbol{B}$ may enable the pressure parallel to $\boldsymbol{B}$ to be different to the pressure perpendicular to $\boldsymbol{B}$, leading to distinct parallel and perpendicular temperatures for each species. Such concepts are useful if the relaxation times for the pressures in the two directions are longer than the wave periods being considered.

The multi–fluid equations (1.5) and (1.6) may be combined and simplified under various assumptions, as is covered in elementary plasma physics texts such as the books by Schmidt (1979) and Tanenbaum (1967). A careful discussion of the basis of multi–fluid models in terms of relative time–scales is to be found in the book by Woods (1987).

The simplest approximation of the multi–fluid model is that of magnetohydrodynamics (MHD), where the inertia of the electron fluid is neglected, and the motions of the different ion and neutral species are combined such that the plasma is assumed to act like a single fluid. If the collisions between electrons and ions are allowed for, the electron momentum equation reduces to Ohm's law, which relates the electric field $\boldsymbol{E}'$ in the rest frame of the fluid to the current density $\boldsymbol{J}$.

The form of Ohm's law used in "collisional" or "resistive" MHD is

$$\boldsymbol{E}' = \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = \eta \boldsymbol{J} \tag{1.8}$$

where $\boldsymbol{v}$ is the single fluid velocity, and η is the electrical resistivity, related to the electron-ion collision frequency ν_{ei} by

$$\eta = \nu_{ei} / \epsilon_0 \omega_{pe}^2 \tag{1.9}$$

where

$$\omega_{pe} = \left(\frac{n_e e^2}{\epsilon_0 m_e} \right)^{1/2} \tag{1.10}$$

is the electron plasma frequency. Here n_e is the electron density. The "ideal" MHD model, in which the resistivity is neglected in Eq. (1.8), is discussed further in the next section.

1.3 The Magnetohydrodynamic Model

Magnetohydrodynamics, or hydromagnetics, is a fluid model which describes a magnetized plasma in which both the ions and the electrons are said to be strongly magnetized or tied to the magnetic field lines, which is to say that the magnetic field is strong enough that the cyclotron periods of all the charged species are well below all other time–scales of interest. The entire plasma acts like a single normal fluid with a single well–defined temperature and pressure. The equations derived from the multi–fluid equations (1.5) and (1.6) correspond to the plasma being treated as a single fluid of density ρ and velocity $\boldsymbol{v}$, and include the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \tag{1.11}$$

and the equation of fluid motion:

$$\rho \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\nabla p + \boldsymbol{J} \times \boldsymbol{B} \tag{1.12}$$

where p is the thermal pressure of the plasma particles, that is, the sum of the partial pressures due to each species of plasma particle.

The third of the MHD equations, in the presence of resistivity, is Ohm's law (Eq. 1.8). In the derivation of Eqs. (1.8) and (1.11) (for example by Woods, 1987), the plasma is assumed to be charge neutral, and the electron mass is assumed negligible compared to the ion mass. In the absence of resistivity or other collisional processes, the magnetic lines of force are said to be frozen–in to the plasma. This concept is valid provided there is no electric field along the magnetic field direction.

We also need an equation of state linking the pressure to the other state variables, such as the adiabatic equation of state

$$\frac{\mathrm{d}}{\mathrm{d}t}(p\rho^{-\gamma}) = 0 \tag{1.13}$$

with γ the adiabatic index. On the other hand, for an incompressible plasma, the equation of state is

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = 0 \tag{1.14}$$

or equivalently, from Eq. (1.11),

$$\nabla \cdot \boldsymbol{v} = 0. \tag{1.15}$$

In the case of an adiabatic equation of state for a nonionized fluid, the speed of sound is given by

$$c_{\mathrm{s}} = (\gamma p/\rho)^{1/2}. \tag{1.16}$$

We thus see from Eqs. (1.13) and (1.14) that the incompressible equation of state (1.14) corresponds to an infinite adiabatic index and infinite speed of sound.

In Ampère's law Eq. (1.4) in the MHD model we usually neglect the displacement current term, the reason being that the characteristic speeds are normally much less than the speed of light *in vacuo* (an exception is considered in Chapter 8). In this case we may write, for the magnetic force in Eq. (1.12),

$$\boldsymbol{J} \times \boldsymbol{B} = -\frac{1}{2\mu_0}\nabla B^2 + \frac{1}{\mu_0}(\boldsymbol{B} \cdot \nabla)\boldsymbol{B}. \tag{1.17}$$

Table 1.1: Representative values of the electron density n_e, temperature T, magnetic field B, Alfvén speed v_A, sound speed c_s and plasma β, in different physical regimes.

	$n_e(\mathrm{m}^{-3})$	$T(\mathrm{K})$	$B(\mathrm{T})$	$v_A(\mathrm{ms}^{-1})$	$c_s(\mathrm{ms}^{-1})$	β
Laboratory plasma	$10^{18}-10^{24}$	10^6	1	10^7-10^4	10^5	$10^{-4}-10^2$
Ionosphere	10^8-10^{12}	10^3	$10^{-4.5}$	10^7-10^5	10^3	$< 10^{-4}$
Solar corona	10^{13}	10^6	10^{-4}	10^6	10^5	10^{-2}
Solar atmosphere	10^{18}	10^4	10^{-1}	10^6	10^4	10^{-4}
Gaseous nebula	10^9	10^2	10^{-9}	10^3	10^3	1
Interstellar gas	10^6	10^2	10^{-9}	10^4	10^3	10^{-2}

The first term on the right–hand side of Eq. (1.17) corresponds to minus the gradient of an effective *magnetic pressure*,

$$p_B = B^2/2\mu_0 \tag{1.18}$$

which may be combined with the particle pressure p in Eq. (1.12) to give an effective total pressure

$$p_T = p + p_B. \tag{1.19}$$

The strength of the particle pressure compared with the magnetic pressure is measured by the plasma beta,

$$\beta = \frac{p}{(B^2/2\mu_0)}. \tag{1.20}$$

β is proportional (with the constant of proportionality depending on the equation of state) to the ratio of the square of the sound speed c_s to the square of the Alfvén speed v_A, which as we shall see is the characteristic speed of low–frequency shear Alfvén waves , and is given by

$$v_A = B/(\mu_0\rho)^{1/2}. \tag{1.21}$$

Thus for an adiabatic equation of state, we have

$$\beta = \frac{2}{\gamma}\frac{c_s^2}{v_A^2}. \tag{1.22}$$

Typical values of v_A, c_s and β, for the particle densities, temperatures and magnetic fields applicable to the various physical regimes to be covered in this book, are shown in Table 1.1.

It is also useful to distinguish the ion and electron betas separately:

$$\beta_i = \frac{p_i}{(B^2/2\mu_0)}, \qquad \beta_e = \frac{p_e}{(B^2/2\mu_0)} \tag{1.23}$$

where p_i and p_e are the partial pressures of the ions and the electrons respectively. We note that in an incompressible plasma, β can be finite, even though c_s becomes infinite.

The second term on the right–hand side of Eq. (1.17), $(\boldsymbol{B} \cdot \nabla)\boldsymbol{B}/\mu_0$, can be decomposed into two components. Defining $\boldsymbol{b}$ as the unit vector in the direction of the magnetic field, the component aligned with the magnetic field may be written as

$$\boldsymbol{b}\boldsymbol{b} \cdot \nabla B^2/2\mu_0 .$$

This component cancels the field–aligned component of the magnetic pressure gradient. Thus only the components of the magnetic pressure gradient perpendicular to the magnetic field exert force on the plasma. If this magnetic pressure gradient force exists, there is said to be *magnetic compression*.

The component of $(\boldsymbol{B} \cdot \nabla)\boldsymbol{B}/\mu_0$ that is perpendicular to the field may be written as (e.g. Kivelson 1995a)

$$-\boldsymbol{n}B^2/\mu_0 R_c \tag{1.24}$$

where $\boldsymbol{n}$ is the outward normal vector and R_c is the local radius of curvature of the magnetic field. This component is antiparallel to the radius of curvature of the field lines, and is called the *magnetic tension* or the *curvature force*. It is present only for curved field lines, and is analogous to the perpendicular force exerted by tension in a curved string. It acts to reduce the curvature of the field line. These concepts of magnetic compression and tension are useful for gaining an intuitive idea of the behaviour of Alfvén waves, as we will find in the next chapter.

A generalization of the MHD model, sometimes used to describe waves in collisionless plasmas with anisotropic pressures and a strong magnetic field, is the CGL or double–adiabatic approximation (Chew, Goldberger & Low 1956, Schmidt 1979). Replacing Eq. (1.13) are two adiabatic equations of state for the pressures parallel and perpendicular to the magnetic field:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p_\perp^2 p_\parallel}{\rho^5}\right) = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p_\perp}{\rho B}\right) = 0. \tag{1.25}$$

The first relation follows from parallel motion of the particles being independent of the perpendicular motion, while the second expresses constancy of magnetic moments (Schmidt 1979). Instead of Eq. (1.12) with Eq. (1.17), we have two equations of motion for the parallel and perpendicular components of $\boldsymbol{v}$:

$$\rho\left(\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\right)_\parallel = -\nabla_\parallel p_\parallel - (p_\perp - p_\parallel)\left(\frac{\nabla B}{B}\right)_\parallel \tag{1.26}$$

$$\rho\left(\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\right)_\perp = -\nabla_\perp\left(p_\perp + \frac{B^2}{2\mu_0}\right) + \frac{1}{\mu_0}((\boldsymbol{B} \cdot \nabla)\boldsymbol{B})_\perp\left(\frac{p_\perp - p_\parallel}{B^2/\mu_0} + 1\right). \tag{1.27}$$

1.4 The Hall-MHD Model

The "two–fluid" model is the next fluid approximation, and assumes the plasma to consist of an electron fluid and a single ion species fluid. The electrons are considered to be magnetized (electron cyclotron period much shorter than time–scales of interest), while the ions are not completely magnetized (ion cyclotron period comparable with the other time–scales). In its simplest form this model is often referred to as Hall MHD, since the Hall term is present on the right–hand side of Ohm's law (assuming resistivity to be negligible):

$$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = \frac{1}{n_i e} \boldsymbol{J} \times \boldsymbol{B}, \tag{1.28}$$

where n_i is the ion number density. The absence of the Hall term in Ohm's law Eq. (1.8) used in MHD means that ion cyclotron effects are absent in the MHD model, in other words the ion cyclotron frequency

$$\Omega_i = \frac{Be}{m_i} \tag{1.29}$$

is assumed much higher than the wave frequency, whereas in the Hall-MHD model the wave frequency can be comparable with the ion cyclotron frequency.

In the static Hall effect, for a conductor of finite extension across the current flow and a static magnetic field applied perpendicular to the current, an electric field across the conductor is observed. In dynamic situations such as waves, alternating Hall currents are produced. The Hall-MHD model allows wave frequencies up to and beyond the ion cyclotron frequency, but below the electron cyclotron frequency, to be considered.

We can employ the single–particle point of view of a plasma to provide a physical explanation of the effect of the Hall term. If the motion of a particle is followed over several gyration orbits, that is if the time–scale of interest, such as the period of a wave, is larger than the gyration period about the magnetic field, the particle drifts in a direction perpendicular to both the electric and magnetic field with the velocity

$$\boldsymbol{u}_D = \boldsymbol{E} \times \boldsymbol{B}/B^2. \tag{1.30}$$

This drift does not introduce currents into the plasma, since $\boldsymbol{u}_D$ is independent of both the charge q and the mass m of the particle. However, the electrons and ions will move in different directions if the wave frequency becomes comparable with the ion cyclotron frequency. Such differential motions of charges constitute a current, the Hall current. Another way of expressing this is to say that the individual electron and ion Hall currents no longer cancel each other.

The Hall-MHD model is used to study the wave heating of laboratory plasmas at frequencies approaching the ion cyclotron frequency (see Chapter 6). It has also been found to capture important macroscopic effects in the simulation of the interaction of the solar wind with the Earth's magnetosphere (Winglee 1994, Huba 1996) and of magnetic field line reconnection (Lottermoser & Scholer 1997). Ion cyclotron effects may also play a role in solar coronal heating (Cranmer, Field & Kohl 1999) (see Chapter 7).

As an extension of the Hall-MHD model of the plasma, the nonideal effects of resistivity, electron inertia and electron pressure that arise from the species equations of motion can be included in the following generalized Ohm's law:

$$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = \eta \boldsymbol{J} + \frac{m_e}{n_i e^2} \frac{\partial \boldsymbol{J}}{\partial t} + \frac{1}{n_i e} \boldsymbol{J} \times \boldsymbol{B} - \frac{1}{n_i e} \nabla p_e. \tag{1.31}$$

The first term on the right–hand side of Eq. (1.31) is the usual resistive term and the second term is the electron-inertia term, which allows the effect of a finite plasma frequency to be included. The third term is the Hall term and the fourth is an electron pressure gradient term, with p_e the electron gas pressure. This generalized Ohm's law allows the description of wave modes with very short wavelength in a direction perpendicular to the magnetic field, an important aspect of the process of Alfvén resonance absorption as we will see in Chapters 2 and 3.

1.5 Fourier Transforms

Both the fluid and kinetic descriptions of a plasma employ the theory of Fourier transforms , which we summarize here. If the equilibrium state of the plasma is assumed uniform, and the fluid equations are linearized, the coefficients in the resulting wave equations are constants. A Fourier transform in space and time will then yield an algebraic equation in the Fourier amplitude of, for example, the perturbation velocity $\boldsymbol{v}_1$, which is a function of time t and space $\boldsymbol{x}$. The Fourier transform of $\boldsymbol{v}_1$ is defined as

$$\boldsymbol{v}(\omega, \boldsymbol{k}) = \int \mathrm{d}t\mathrm{d}^3\boldsymbol{x} \exp(\mathrm{i}(\omega t - \boldsymbol{k} \cdot \boldsymbol{x}))\boldsymbol{v}_1(t, \boldsymbol{x}) \tag{1.32}$$

with the inverse transform

$$\boldsymbol{v}_1(t, \boldsymbol{x}) = \int \frac{\mathrm{d}\omega\mathrm{d}^3\boldsymbol{k}}{(2\pi)^4} \exp(-\mathrm{i}(\omega t - \boldsymbol{k} \cdot \boldsymbol{x}))\boldsymbol{v}(\omega, \boldsymbol{k}). \tag{1.33}$$

Taking the Fourier transform in the uniform plasma case is simply equivalent to seeking plane wave solutions, that is, to assuming the form

$$\exp[\mathrm{i}(k_x x + k_y y + k_z z - \omega t)] \tag{1.34}$$

for the wave fields, with k_x, k_y and k_z the constant wavenumbers in a Cartesian coordinate system. However, this is not the case for a nonuniform plasma or for nonlinear wave fields, where Fourier methods are not so useful.

1.6 The Kinetic Theory

The disadvantage of a fluid description of a plasma is that some effects, such as Landau and cyclotron damping, caused by a resonance of the wave with particles, cannot be modelled. The description of such effects requires a kinetic theory of the plasma. The theory of collisionless plasmas is well developed, so we can simply quote the relevant results from the theory, for example from the book of Melrose (1986).

The kinetic theory proceeds from the Vlasov theory of the collisionless plasma. After Fourier transforming Maxwell's equations and the Vlasov equations for the ions and electrons, we use expansions in terms of Bessel functions to derive the frequency and wavenumber dependent dielectric tensor K_{ij}, which is defined in terms of the conductivity tensor σ_{ij}:

$$K_{ij}(\omega, \boldsymbol{k}) = \delta_{ij} + \frac{\mathrm{i}}{\varepsilon_0\omega}\sigma_{ij}(\omega, \boldsymbol{k}). \tag{1.35}$$

The results of the calculation of the dielectric tensor from Vlasov theory are given in the Appendix. The plasma is said to be spatially dispersive if K_{ij} depends on $\boldsymbol{k}$.

The Fourier transform of the current density induced by the electric field imposed on the plasma is given in component form by

$$J_i(\omega, \boldsymbol{k}) = \sigma_{ij}(\omega, \boldsymbol{k}) E_j(\omega, \boldsymbol{k}). \tag{1.36}$$

The wave equation for the electric field derived from Maxwell's equations is

$$\nabla \times \nabla \times \boldsymbol{E} + \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = -\mu_0 \frac{\partial \boldsymbol{J}}{\partial t}. \tag{1.37}$$

Fourier transforming Eq. (1.37) and using Eq. (1.35) and Eq. (1.36) yields

$$\Lambda_{ij}(\omega, \boldsymbol{k}) E_j(\omega, \boldsymbol{k}) = 0 \tag{1.38}$$

with the wave tensor

$$\Lambda_{ij}(\omega, \boldsymbol{k}) = \frac{c^2}{\omega^2}(k_i k_j - k^2 \delta_{ij}) + K_{ij}(\omega, \boldsymbol{k}) \tag{1.39}$$

where $k^2 = k_x^2 + k_y^2 + k_z^2$. The dispersion equation is then obtained by setting the determinant, $\Lambda(\omega, \boldsymbol{k})$, of the 3×3 matrix $\Lambda_{ij}(\omega, \boldsymbol{k})$ to zero.

For example, in the next chapter we will assume the wavevector to lie in the x-z plane, so $k_y = 0$, in which case the determinant is

$$\Lambda(\omega, \boldsymbol{k}) = \begin{vmatrix} -c^2 k_z^2/\omega^2 + K_{11} & K_{12} & c^2 k_x k_z/\omega^2 + K_{13} \\ K_{21} & -c^2 k^2/\omega^2 + K_{22} & K_{23} \\ c^2 k_x k_z/\omega^2 + K_{31} & K_{32} & -c^2 k_x^2/\omega^2 + K_{33} \end{vmatrix}. \tag{1.40}$$

The dispersion relation for some mode M is given by a solution, $\omega = \omega_{\mathrm{M}}(\boldsymbol{k})$, of the dispersion equation $\Lambda = 0$. The direction of the electric field $\boldsymbol{E}(\omega_{\mathrm{M}}(\boldsymbol{k}), \boldsymbol{k})$ for waves in the mode M is described by the unimodular polarization vector $e_{\mathrm{M}}(\boldsymbol{k})$, which satisfies the relation

$$e_{\mathrm{M}}(\boldsymbol{k}) \cdot e_{\mathrm{M}}^*(\boldsymbol{k}) = 1. \tag{1.41}$$

An important quantity that characterizes the magnitude and direction of the flow of energy in waves is the group velocity vector, given by

$$\boldsymbol{v}_{\mathrm{g}} = \frac{\partial \omega}{\partial \boldsymbol{k}}. \tag{1.42}$$

The energy flux in the waves is equal to the group velocity times the energy density in the waves. Provided the plasma is not spatially dispersive (i.e., there is no thermal energy in the wave (Stix 1992)), the energy flux is purely due to the electromagnetic energy flux, given by the Poynting vector

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{B}/\mu_0. \tag{1.43}$$

In that case, the group velocity vector has the same direction as the Poynting vector. Further discussion of energy flux and the group velocity is to be found in the books by Cross (1988) and Melrose (1986).

This page intentionally left blank

2 Waves in Uniform Plasmas

2.1 Introduction

In this chapter we consider the properties of small amplitude linear waves in a spatially uniform plasma, using the models of the plasma introduced in Chapter 1. The properties of small amplitude waves in the uniform plasma form a basis for the discussion in following chapters of the waves in nonuniform plasmas, and of nonlinear waves. The waves will be assumed to have frequencies below or of the order of the ion cyclotron frequency, and we shall concentrate on waves that are predominantly electromagnetic. The classification of these waves is not as difficult a task as in the general case for the full range of frequencies covering the electron plasma frequency and the electron cyclotron frequency. In that case recourse may be made to techniques such as the Clemmow-Mullaly-Allis diagram to classify the waves (Stix 1992).

We discuss the waves first with the ideal MHD or hydromagnetic model, suitable for low–frequency waves in thermal plasmas with no interspecies collisions. Then the frequency range is extended to encompass the ion cyclotron frequency, using the Hall-MHD model for both cold and warm plasmas. A description of the cold plasma modes in terms of the dielectric tensor approach follows. The nonideal effects of interspecies collisions and multiple ions are then considered. Next, kinetic effects using the Vlasov theory are included, for both low and high plasma beta. Finally, the "kinetic" and "inertial" Alfvén waves are discussed, employing both fluid and kinetic theory.

2.2 Waves with the MHD Model

Let us assume an equilibrium with the plasma at rest and with no zero–order electric field. The plasma will be modelled in this section by the MHD equations (1.8), (1.11) and (1.12), with the adiabatic equation of state (1.13). We assume initially that the plasma has zero resistivity (the ideal MHD model). If subscripts 0 denote the equilibrium state, and subscripts 1 denote the first–order perturbations associated with the wave motion, the equilibrium satisfies the force balance equation obtained from Eq. (1.12),

$$\nabla p_0 = \boldsymbol{J}_0 \times \boldsymbol{B}_0. \tag{2.1}$$

From Eq. (1.11) and Eq. (1.13), the perturbed density and pressure satisfy

$$\frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_0 \boldsymbol{v}_1) = 0 \tag{2.2}$$

The Physics of Alfvén Waves. Neil F. Cramer

ISBN: 3-527-40293-4

and

$$p_1 = \frac{\gamma p_0}{\rho_0}\rho_1. \tag{2.3}$$

From the equation of fluid motion (1.12), the perturbed fluid velocity satisfies

$$\rho_0 \frac{\partial \boldsymbol{v}_1}{\partial t} = -\nabla p_1 + \boldsymbol{J}_0 \times \boldsymbol{B}_1 + \boldsymbol{J}_1 \times \boldsymbol{B}_0. \tag{2.4}$$

The perturbed electric and magnetic fields satisfy the equations

$$\frac{\partial \boldsymbol{B}_1}{\partial t} = -\nabla \times \boldsymbol{E}_1 \tag{2.5}$$

and, from Eq. (1.8) with $\eta = 0$,

$$\boldsymbol{E}_1 = -\boldsymbol{v}_1 \times \boldsymbol{B}_0. \tag{2.6}$$

If the wavelengths are much shorter than the scale–lengths over which the equilibrium quantities ρ_0, p_0 and B_0 change, these quantities can be assumed to be constants, and the plasma is effectively uniform. The equilibrium current density $\boldsymbol{J}_0$ can therefore be neglected in Eq. (2.4). If we also neglect the displacement current in Eq. (1.3), assuming the characteristic speeds are much less than the speed of light *in vacuo* (we remove this assumption in Chapter 8), we have from Eq. (1.4)

$$\mu_0 \boldsymbol{J}_1 = \nabla \times \boldsymbol{B}_1. \tag{2.7}$$

The uniform equilibrium magnetic field is chosen to lie along the z-axis. It is then convenient to use the following perturbation variables to describe the wave fields:

$$\nabla \cdot \boldsymbol{v}_1, \quad v_{1z}, \quad B_{1z}, \quad J_{1z}, \quad \rho_1, \quad \zeta_{1z} \tag{2.8}$$

where

$$\zeta_{1z} = (\nabla \times \boldsymbol{v}_1)_z \tag{2.9}$$

is the fluid vorticity in the magnetic field direction.

Equations (2.2)-(2.7) can then be manipulated to yield the following set of six differential equations:

$$\rho_0 \frac{\partial \zeta_{1z}}{\partial t} - B_0 \frac{\partial J_{1z}}{\partial z} = 0 \tag{2.10}$$

$$\mu_0 \frac{\partial J_{1z}}{\partial t} - B_0 \frac{\partial \zeta_{1z}}{\partial z} = 0 \tag{2.11}$$

$$\rho_0 \frac{\partial}{\partial t} \nabla \cdot \boldsymbol{v}_1 + \frac{B_0}{\mu_0} \nabla^2 B_{1z} + c_s^2 \nabla^2 \rho_1 = 0 \tag{2.12}$$

$$\frac{\partial B_{1z}}{\partial t} + B_0 \left(\nabla \cdot \boldsymbol{v}_1 - \frac{\partial v_{1z}}{\partial z} \right) = 0 \tag{2.13}$$

$$\rho_0 \frac{\partial v_{1z}}{\partial t} + c_s^2 \frac{\partial \rho_1}{\partial z} = 0 \tag{2.14}$$

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \boldsymbol{v}_1 = 0. \tag{2.15}$$

It is seen that the two differential equations (2.10) and (2.11) for ζ_{1z} and J_{1z} are uncoupled from the four differential equations (2.12)-(2.15) for $\nabla \cdot \boldsymbol{v}_1$, B_{1z}, v_{1z} and ρ_1. We should also note that in the equations for ζ_{1z} and J_{1z}, the spatial derivatives are only in the equilibrium magnetic field direction. Taking the Fourier transforms, defined in Eq. (1.32), of Eq. (2.10) and Eq. (2.11) (or simply substituting the plane wave solution Eq. (1.34) into the differential equations), we obtain a consistency equation for a nontrivial solution which relates the frequency to the wavenumber. This is the dispersion equation for waves described by the variables ζ_{1z} and J_{1z}:

$$\omega^2 - v_A^2 k_z^2 = 0 \tag{2.16}$$

where the Alfvén speed v_A in the equilibrium plasma is given by

$$v_A = B_0/(\mu_0 \rho_0)^{1/2}. \tag{2.17}$$

The dispersion equation (2.16) is independent of the components of the wavevector perpendicular to the equilibrium magnetic field, and is also independent of the sound speed c_s.

Taking the Fourier transforms of Eqs. (2.12)-(2.15) yields a separate dispersion equation for waves described by the variables $\nabla \cdot \boldsymbol{v}_1$, B_{1z}, v_{1z} and ρ_1:

$$\omega^4 - \omega^2 (v_A^2 + c_s^2) k^2 + v_A^2 c_s^2 k^2 k_z^2 = 0 \tag{2.18}$$

where $k = |\boldsymbol{k}|$. This dispersion equation does involve the perpendicular components of the wavevector, and the sound speed, in contrast to Eq. (2.16).

It is evident that the two dispersion equations (2.16) and (2.18), together with their corresponding sets of characteristic wave field variables, correspond to two distinct types of wave mode. The waves described by Eq. (2.16) are called *Alfvén* waves, and the waves described by Eq. (2.18) are called *magnetoacoustic* (or *magnetosonic*) waves (see Table 2.1). The magnetoacoustic mode may be further split up into two distinct modes, the fast and slow magnetoacoustic waves. An arbitrary low–frequency disturbance can be represented as a superposition of the Alfvén wave and the fast and slow magnetoacoustic waves.

Let us define the angle θ between the wavevector and the magnetic field $\boldsymbol{B}_0$, so that $k_z = k \cos\theta$. The first dispersion equation (2.16) then gives the positive frequency solution

$$\omega_A = v_A |k_z| = v_A k |\cos\theta| \tag{2.19}$$

of the Alfvén mode. The second dispersion equation (2.18) gives two positive frequency solutions: the fast magnetoacoustic mode, with

$$\omega_F^2 = \frac{k^2}{2} \left(v_A^2 + c_s^2 + \left((v_A^2 + c_s^2)^2 - 4 v_A^2 c_s^2 \cos^2\theta \right)^{1/2} \right) \tag{2.20}$$

Table 2.1: The dispersion equations and characteristic variables for the Alfvén and magnetoacoustic modes in the ideal MHD model.

	Dispersion equation	Characteristic variables
Alfvén wave	$\omega^2 - v_A^2 k_z^2 = 0$	$J_{1z},\ \zeta_{1z}$
Magnetoacoustic waves	$\omega^4 - \omega^2(v_A^2 + c_s^2)k^2$ $+ v_A^2 c_s^2 k^2 k_z^2 = 0$	$\nabla \cdot \boldsymbol{v}_1,\ v_{1z},\ B_{1z},\ \rho_1$

and the slow magnetoacoustic mode, with

$$\omega_S^2 = \frac{k^2}{2}\left(v_A^2 + c_s^2 - \left((v_A^2 + c_s^2)^2 - 4v_A^2 c_s^2 \cos^2\theta\right)^{1/2}\right). \tag{2.21}$$

We note that the phase velocity $v_{ph} = \omega/k$ is independent of k for all three modes, so all the modes are nondispersive, although they are anisotropic because v_{ph} depends on the angle of propagation θ. The characteristic phase velocity surfaces, that is, polar plots of the phase velocities of the three modes against the angle θ, have often been presented in texts on MHD (see for example Shercliff (1965)).

Defining $\boldsymbol{b}$ as the unit vector in the direction of $\boldsymbol{B}_0$, and $\boldsymbol{\kappa}$ as the unit vector along the wavevector $\boldsymbol{k}$, we have $\boldsymbol{\kappa} \cdot \boldsymbol{b} = \cos\theta$. To discuss the polarization properties of the three modes, it is convenient to define two mutually orthogonal unit vectors, each orthogonal to the $\boldsymbol{\kappa}$ vector:

$$\boldsymbol{a} = -\frac{\boldsymbol{k} \times \boldsymbol{B}_0}{|\boldsymbol{k} \times \boldsymbol{B}_0|} \tag{2.22}$$

and

$$\boldsymbol{t} = \boldsymbol{a} \times \boldsymbol{\kappa}. \tag{2.23}$$

Without loss of generality, for a uniform plasma we can choose the $\boldsymbol{k}$ vector to lie in the x-z plane. We then have

$$\boldsymbol{b} = (0, 0, 1), \qquad \boldsymbol{\kappa} = (\sin\theta, 0, \cos\theta) \tag{2.24}$$

and the $\boldsymbol{a}$ and $\boldsymbol{t}$ vectors become

$$\boldsymbol{a} = (0, 1, 0), \qquad \boldsymbol{t} = (\cos\theta, 0, -\sin\theta). \tag{2.25}$$

These vectors are shown in Figure 2.1. We now proceed to discuss the three modes in some detail.

2.2.1 The Alfvén Mode

If the wave is purely in the Alfvén mode, we can assume the characteristic variables listed in Table 2.1 for the magnetoacoustic mode to be zero. Thus we have (with $k_y = 0$),

$$v_{1z} = 0 \quad \text{and} \quad \nabla \cdot \boldsymbol{v}_1 = \mathrm{i}k_x v_{1x} = 0. \tag{2.26}$$