Fourier Transforms in Spectroscopy. J. Kauppinen, J. Partanen

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Jyrki Kauppinen, Jari Partanen

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Preface

How much should a good spectroscopist know about Fourier transforms? How well should a professional who uses them as a tool in his/her work understand their behavior? Our belief is, that a profound insight of the characteristics of Fourier transforms is essential for their successful use, as a superficial knowledge may easily lead to mistakes and misinterpretations. But the more the professional knows about Fourier transforms, the better he/she can apply all those versatile possibilities offered by them.

On the other hand, people who apply Fourier transforms are not, generally, mathematicians. Learning unnecessary details and spending years in specializing in the heavy mathematics which could be connected to Fourier transforms would, for most users, be a waste of time. We believe that there is a demand for a book which would cover understandably those topics of the transforms which are important for the professional, but avoids going into unnecessarily heavy mathematical details. This book is our effort to meet this demand.

We recommend this book for advanced students or, alternatively, post-graduate students of physics, chemistry, and technical sciences. We hope that they can use this book also later during their career as a reference volume. But the book is also targeted to experienced professionals: we trust that they might obtain new aspects in the use of Fourier transforms by reading it through.

Of the many applications of Fourier transforms, we have discussed Fourier transform spectroscopy (FTS) in most depth. However, all the methods of signal and spectral processing explained in the book can also be used in other applications, for example, in nuclear magnetic resonance (NMR) spectroscopy, or ion cyclotron resonance (ICR) mass spectrometry.

We are heavily indebted to Dr. Pekka Saarinen for scientific consultation, for planning problems for the book, and, finally, for writing the last chapter for us. We regard him as a leading specialist of linear prediction in spectroscopy. We are also very grateful to Mr. Matti Hollberg for technical consultation, and for the original preparation of most of the drawings in this book.

Jyrki Kauppinen and Jari Partanen

Turku, Finland, 13th October 2000

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1 Basic definitions

1.1 Fourier series

If a function $h(t)$, which varies with t, satisfies *the Dirichlet conditions*

1. $h(t)$ is defined from $t = -\infty$ to $t = +\infty$ and is periodic with some period T,
2. $h(t)$ is well-defined and single-valued (except possibly in a finite number of points) in the interval $\left[-\frac{1}{2}T, \frac{1}{2}T\right]$,
3. $h(t)$ and its derivative $\mathrm{d}h(t)/\mathrm{d}t$ are continuous (except possibly in a finite number of step discontinuities) in the interval $\left(-\frac{1}{2}T, \frac{1}{2}T\right)$, and
4. $h(t)$ is absolutely integrable in the interval $\left[-\frac{1}{2}T, \frac{1}{2}T\right]$, that is, $\int_{-T/2}^{T/2} |h(t)|\,\mathrm{d}t < \infty$,

then the function $h(t)$ can be expressed as a *Fourier series* expansion

$$h(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t)\right] = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t}, \tag{1.1}$$

where

$$\left\{\begin{array}{lcl}
\omega_0 & = & \dfrac{2\pi}{T} = 2\pi f_0, \\
a_n & = & \dfrac{2}{T}\displaystyle\int_{-T/2}^{T/2} h(t)\cos(n\omega_0 t)\,\mathrm{d}t, \\
b_n & = & \dfrac{2}{T}\displaystyle\int_{-T/2}^{T/2} h(t)\sin(n\omega_0 t)\,\mathrm{d}t, \\
c_n & = & \dfrac{1}{T}\displaystyle\int_{-T/2}^{T/2} h(t)e^{-in\omega_0 t}\,\mathrm{d}t = \dfrac{1}{2}(a_n - ib_n), \\
c_{-n} & = & c_n^* = \dfrac{1}{2}(a_n + ib_n).
\end{array}\right. \tag{1.2}$$

f_0 is called the *fundamental frequency* of the system. In the Fourier series, a function $h(t)$ is analyzed into an infinite sum of harmonic components at multiples of the fundamental frequency. The coefficients a_n, b_n and c_n are the *amplitudes* of these harmonic components.

At every point where the function $h(t)$ is continuous the Fourier series converges uniformly to $h(t)$. If the Fourier series is truncated, and $h(t)$ is approximated by a sum of only a finite number of terms of the Fourier series, then this approximation differs somewhat from $h(t)$. Generally, the approximation becomes better and better as more and more terms are included.

At every point $t = t_0$ where the function $h(t)$ has a step discontinuity the Fourier series converges to the average of the limiting values of $h(t)$ as the point is approached from above and from below:

$$\left[\lim_{\varepsilon\to 0+} h(t_0+\varepsilon) + \lim_{\varepsilon\to 0+} h(t_0-\varepsilon)\right] \Big/ 2.$$

Around a step discontinuity, a truncated Fourier series overshoots at both sides near the step, and oscillates around the true value of the function $h(t)$. This oscillation behavior in the vicinity of a point of discontinuity is called the *Gibbs phenomenon*.

The coefficients c_n in Equation 1.1 are the *complex amplitudes* of the harmonic components at the frequencies $f_n = nf_0 = n/T$. The complex amplitudes c_n as a function of the corresponding frequencies f_n constitute a *discrete complex amplitude spectrum*.

Example 1.1: Examine the Fourier series of the *square wave* shown in Figure 1.1.

Solution. Applying Equation 1.2, the square wave can be expressed as the Fourier series

$$\begin{aligned} h(t) &= \frac{4}{\pi}\left[\cos(\omega_0 t) - \frac{1}{3}\cos(3\omega_0 t) + \frac{1}{5}\cos(5\omega_0 t) - \cdots\right] \\ &= \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n} \cos(n\omega_0 t). \end{aligned}$$

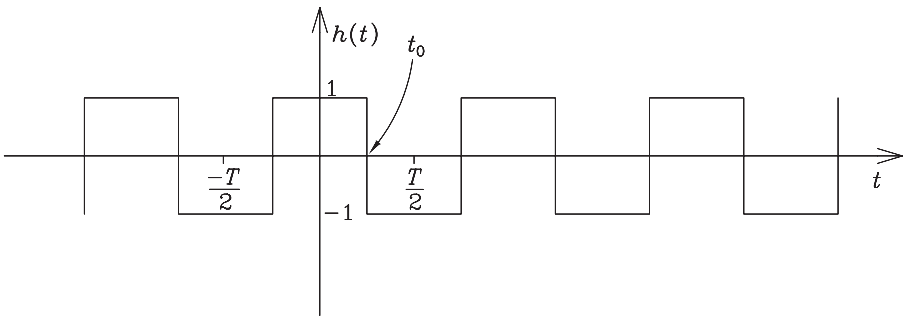

Figure 1.1: Square wave $h(t)$.

If this Fourier series is truncated, and the function is approximated by a finite sum, then this approximation differs from the original square wave, especially around the points of discontinuity. Figure 1.2 illustrates the Gibbs oscillation around the point $t = t_0$ of the square wave of Figure 1.1.

The amplitude spectrum of the square wave of Figure 1.1 is shown in Figure 1.3. The amplitude coefficients of the square wave are $c_n = \frac{1}{2}\,a_n = 0, \frac{2}{\pi}, 0, -\frac{2}{3\pi}, 0, \frac{2}{5\pi}, 0, \ldots$

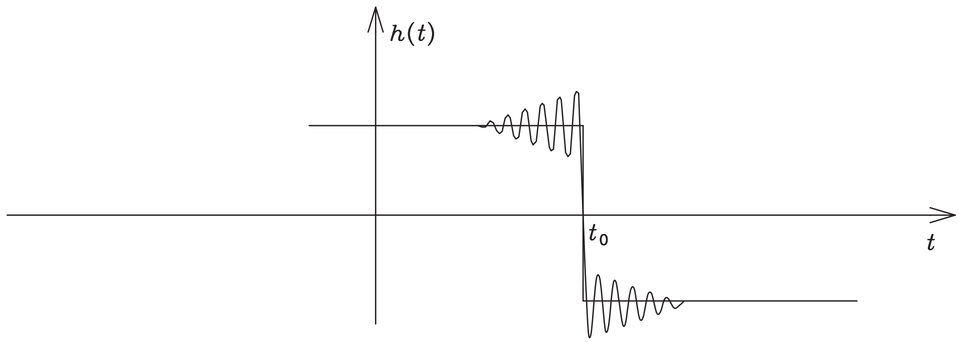

Figure 1.2: The principle how the truncated Fourier series of the square wave $h(t)$ of Fig. 1.1 oscillates around the true value in the vicinity of the point of discontinuity $t = t_0$.

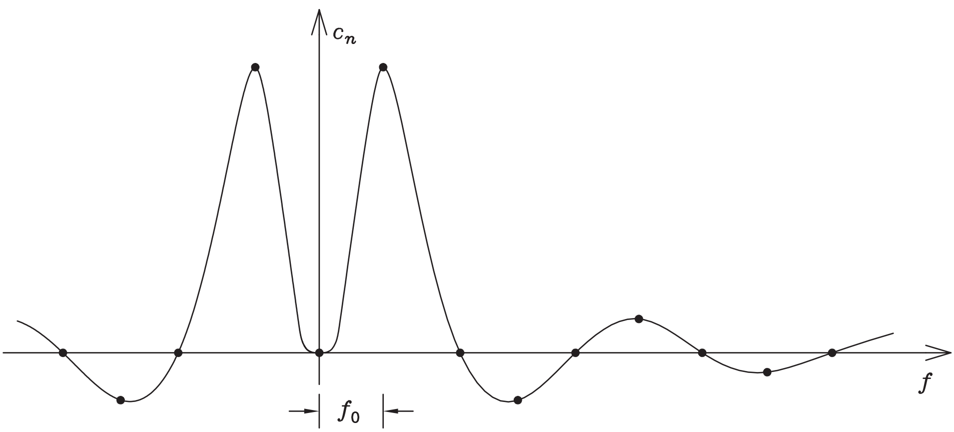

Figure 1.3: Discrete amplitude spectrum of the square wave $h(t)$ of Fig. 1.1, formed by the amplitude coefficients c_n. f_0 is the fundamental frequency.

1.2 Fourier transform

The Fourier series, Equation 1.1, can be used to analyze periodic functions of a period T and a fundamental frequency $f_0 = \frac{1}{T}$. By letting the period tend to infinity and the fundamental frequency to zero, we can obtain a generalization of the Fourier series which also is suitable for analysis of non-periodic functions.

According to Equation 1.1,

$$h(t) = \sum_{n=-\infty}^{\infty} \underbrace{\frac{1}{T} \int_{-T/2}^{T/2} h(t')e^{-i2\pi n f_0 t'}\, \mathrm{d}t'}_{c_n}\, e^{i12\pi n f_0 t}. \tag{1.3}$$

We shall replace nf_0 by f, and let $T \to \infty$ and $1/T = f_0 = \mathrm{d}f \to 0$. In this case,

$$\sum_{n=-\infty}^{\infty} \frac{1}{T} \to \int_{-\infty}^{\infty} \mathrm{d}f, \tag{1.4}$$

and

$$h(t) = \int_{-\infty}^{\infty} \underbrace{\left[\int_{-\infty}^{\infty} h(t')e^{-i2\pi f t'}\, \mathrm{d}t' \right]}_{H(f)} e^{i2\pi f t}\, \mathrm{d}f. \tag{1.5}$$

We can interpret this formula as the sum of the waves $H(f)\,\mathrm{d}f\, e^{i2\pi f t}$.

With the help of the notation $H(f)$, we can write Equation 1.5 in the compact form

$$h(t) = \int_{-\infty}^{\infty} H(f)e^{i2\pi f t}\, \mathrm{d}f = \mathcal{F}\{H(f)\}. \tag{1.6}$$

The operation $\mathcal{F}$ is called the *Fourier transform*. From above,

$$H(f) = \int_{-\infty}^{\infty} h(t)e^{-i2\pi f t}\, \mathrm{d}t = \mathcal{F}^{-1}\{h(t)\}. \tag{1.7}$$

The operation $\mathcal{F}^{-1}$ is called the *inverse Fourier transform*.

Functions $h(t)$ and $H(f)$ which are connected by Equations 1.6 and 1.7 constitute a *Fourier transform pair.* Notice that even though we have used as the variables the symbols t and f, which often refer to time [s] and frequency [Hz], the Fourier transform pair can be formed for *any variables, as long as the product of their dimensions is one* (the dimension of one variable is the inverse of the dimension of the other).

In the literature, it is possible to find several, slightly differing ways to define the Fourier integrals. They may differ in the constant coefficients in front of the integrals and in the exponents. In this book we have chosen the definitions in Equations 1.6 and 1.7, because they are the most convenient for our purposes. *In our definition, the exponential functions inside the integrals carry the coefficient* 2π*, because, in this way, we can avoid the coefficients in front of the integrals.* We have noticed that coefficients in front of Fourier integrals are a constant source of mistakes in calculations, and, by our definition, these mistakes can be avoided. Also the theorems of Fourier transform are essentially simpler, if this definition is chosen: in this way even they, except the derivative theorem, have no front coefficients.

The definition of the Fourier transform pair remains sensible, if a constant c is added in front of one integral and its inverse, constant $1/c$, is added in front of the other integral. The product of the front coefficients should equal one. We strongly encourage *not* to use definitions which do not fulfill this condition. An example of this kind of definition, sometimes encountered in literature, is obtained by setting $f = \omega/2\pi$ in Equations 1.6 and 1.7. We obtain

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) e^{i\omega t}\, \mathrm{d}\omega = \mathcal{F}\{H(\omega)\}, \tag{1.8}$$

and

$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-i\omega t}\, \mathrm{d}t = \mathcal{F}^{-1}\{h(t)\}. \tag{1.9}$$

We do *not* recommend these definitions, since they easily lead to difficulties.

In our definition, the exponent inside the Fourier transform $\mathcal{F}$ *carries a positive sign, and the exponent inside the inverse Fourier transform* $\mathcal{F}^{-1}$ *carries a negative sign.* It is more common to make the definition vice versa: generally the exponent inside the Fourier transform $\mathcal{F}$ has a negative sign, and the exponent inside the inverse Fourier transform $\mathcal{F}^{-1}$ has a positive sign. Our book mainly discusses symmetric functions, and the Fourier transform and the inverse Fourier transform of a symmetric function are the same. Consequently, the choice of the sign has no scientific meaning. In our opinion, our definition is more logical, simpler, and easier to memorize: $+$ *sign in the exponent corresponds to* $\mathcal{F}$ *and* $-$ *sign in the exponent corresponds to* $\mathcal{F}^{-1}$!

We recommend that, while reading this book, the reader forgets all other definitions and uses only this simplest definition. In other contexts, the reader should always check, which definition of Fourier transforms is being used.

Table 1.1 lists a few important Fourier transform pairs, which will be useful in this book, as well as the full width at half maximum, FWHM, of these functions.

Table 1.1: Fourier transform pairs $h(t)$ and $H(f)$, and the FWHM of these functions.

Name of $h(t)$	$h(t)$	FWHM of $h(t)$	$\mathcal{F}^{-1}, \mathcal{F}$ $\Leftrightarrow$	Name of $H(t)$	$H(f)$	FWHM of $H(f)$
boxcar	$\Pi_{2T}(t) = \begin{cases} 1, & \lvert t\rvert \le T, \\ 0, & \lvert t\rvert > T \end{cases}$	$2T$		sinc	$2T\,\mathrm{sinc}(\pi 2Tf)$	$\frac{1.2067}{2T}$
triangular	$\Lambda_T(t) = \begin{cases} 1 - \frac{\lvert t\rvert}{T}, & \lvert t\rvert \le T, \\ 0, & \lvert t\rvert > T \end{cases}$	T		sinc^2	$T\,\mathrm{sinc}^2(\pi T f)$	$\frac{1.7718}{2T}$
Lorentzian	$\frac{\sigma/\pi}{\sigma^2 + t^2}$	2σ		exponential	$\exp(-\pi 2\sigma \lvert f\rvert)$	$\frac{\ln 2}{\pi\sigma}$
Gaussian	$\sqrt{\frac{\alpha}{\pi}}\exp(-\alpha t^2)$	$2\sqrt{\frac{\ln 2}{\alpha}}$		Gaussian	$\exp\left(\frac{-\pi^2 f^2}{\alpha}\right)$	$\frac{2}{\pi}\sqrt{\alpha \ln 2}$
Dirac's delta	$\delta(t - t_0)$	0	$\mathcal{F}$ $\Rightarrow$	exponential wave	$\exp(i 2\pi t_0 f)$	—
Dirac's delta	$\delta(t - t_0)$	0	$\mathcal{F}^{-1}$ $\Rightarrow$	exponential wave	$\exp(-i 2\pi t_0 f)$	—

Example 1.2: Applying Fourier transforms, compute the integral $\int_0^\infty \frac{\sin(px)\cos(qx)}{x}\,\mathrm{d}x$, where $p > 0$ and $p \neq q$.

Solution. Knowing that the imaginary part of e^{iqx} is antisymmetric, we can write

$$\begin{aligned}\int_0^\infty \frac{\sin(px)\cos(qx)}{x}\,\mathrm{d}x &= \frac{p}{2}\int_{-\infty}^{\infty}\frac{\sin(px)}{px}e^{iqx}\,\mathrm{d}x = \frac{p}{2}\int_{-\infty}^{\infty}\mathrm{sinc}(px)e^{i2\pi\frac{q}{2\pi}x}\,\mathrm{d}x \\ &= \frac{\pi}{2}h(\frac{q}{2\pi}),\end{aligned}$$

where the function

$$h(t) = \mathcal{F}\{\frac{p}{\pi}\ \mathrm{sinc}(\pi\frac{p}{\pi}f)\}.$$

From Table 1.1, we know that the Fourier transform of a sinc function is a boxcar function. Consequently,

$$\frac{\pi}{2}h(\frac{q}{2\pi}) = \frac{\pi}{2}\Pi_{p/\pi}(\frac{q}{2\pi}) = \begin{cases}\pi/2, & |q| < p, \\ 0, & |q| > p.\end{cases}$$

1.3 Dirac's delta function

Dirac's delta function, $\delta(t)$, also called the *impulse function*, is a concept which is frequently used to describe quantities which are localized in one point. Even though real physical quantities cannot be truly localized in exactly one point, the concept of Dirac's delta function is very useful.

Dirac's delta function is defined with the following equation:

$$\int_{-\infty}^{\infty} F(t)\delta(t-t_0)\,\mathrm{d}t = F(t_0), \tag{1.10}$$

where $F(t)$ is an arbitrary function of t, continuous at the point $t = t_0$.

By inserting the function $F(t) \equiv 1$ in Equation 1.10, we can see that the area of Dirac's delta function is equal to unity, that is,

$$\int_{-\infty}^{\infty} \delta(t-t_0)\,\mathrm{d}t = 1. \tag{1.11}$$

In the usual sense, $\delta(t)$ is not really a function at all. In practice,

$$\lim_{t\to t_0}\delta(t-t_0) = \infty. \tag{1.12}$$

At points $t \neq t_0$ either $\delta(t-t_0) = 0$ or $\delta(t)$ oscillates with infinite frequency.

It can be shown that Dirac's delta function has the following properties:

$$\begin{cases} \delta(-t) & \triangleq & \delta(t), \\ t\delta(t) & \triangleq & 0, \\ \delta(at) & \triangleq & \frac{1}{a}\,\delta(t), \quad \text{if } a > 0, \\ \frac{\mathrm{d}\delta(t)}{\mathrm{d}t} & \triangleq & -\frac{1}{t}\,\delta(t), \\ F(t)\delta(t-t_0) & \triangleq & F(t_0)\delta(t-t_0), \end{cases} \tag{1.13}$$

where the correspondence relation $\triangleq$ means that the value of the integral in Equation 1.10 remains the same whichever side of the relation is inserted in the integral.

The "shape" of Dirac's delta function is not uniquely defined. There are infinitely many representations of $\delta(t)$ which satisfy Equation 1.10. One of them, very useful with Fourier transforms, is

$$\boxed{\delta(t) = \int_{-\infty}^{\infty} e^{i2\pi ts}\,\mathrm{d}s = \mathcal{F}\{1\} = \int_{-\infty}^{\infty} \cos(2\pi ts)\,\mathrm{d}s.} \tag{1.14}$$

Equivalently, we can write

$$\boxed{\delta(t) = \int_{-\infty}^{\infty} e^{-i2\pi ts}\,\mathrm{d}s = \mathcal{F}^{-1}\{1\} = \int_{-\infty}^{\infty} \cos(2\pi ts)\,\mathrm{d}s.} \tag{1.15}$$

A few other useful representations for Dirac's delta function are, for example,

$$\begin{aligned} \delta(t) \quad &\triangleq \quad \lim_{a\to\infty} \frac{\sin(a\pi t)}{\pi t} \triangleq \lim_{\sigma\to 0+} \frac{\sigma/\pi}{t^2+\sigma^2} \\ &\triangleq \quad \lim_{a\to\infty} \frac{a}{2}\, e^{-a|t|} \triangleq \lim_{a\to 0+} \frac{1}{a\sqrt{2\pi}}\, e^{-t^2/(2a^2)}. \end{aligned} \tag{1.16}$$

Two rather simple representations are

$$\delta(t) \triangleq \lim_{a\to 0+} f(t,a) \triangleq \lim_{a\to 0+} g(t,a), \tag{1.17}$$

where $f(t,a)$ and $g(t,a)$ are the functions illustrated in Figure 1.4.

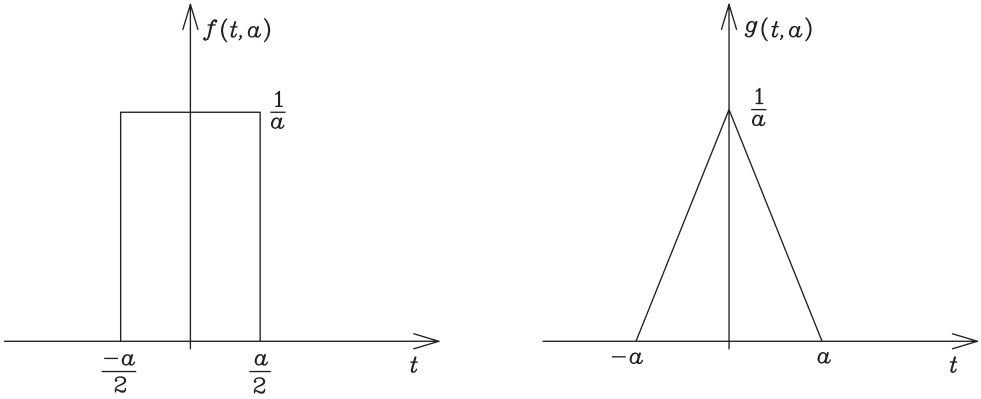

Figure 1.4: Two representations for Dirac's delta function: $\delta(t) \triangleq \lim_{a\to 0+} f(t,a) \triangleq \lim_{a\to 0+} g(t,a)$.

Example 1.3: Using Dirac's delta function in the form of Equation 1.14, prove that Equation 1.6 yields Equation 1.7.

Solution. If $h(t) = \int_{-\infty}^{\infty} H(f)e^{i2\pi ft}\,\mathrm{d}f$, then

$$\begin{aligned}
\mathcal{F}^{-1}\{h(t)\} &= \int_{-\infty}^{\infty} h(t)e^{-i2\pi f't}\,\mathrm{d}t = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(f)e^{i2\pi t(f-f')}\,\mathrm{d}f\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} H(f)\left[\int_{-\infty}^{\infty} e^{i2\pi t(f-f')}\,\mathrm{d}t\right]\mathrm{d}f \overset{(1.14)}{=} \int_{-\infty}^{\infty} H(f)\delta(f-f')\,\mathrm{d}f \\
&\overset{(1.10)}{=} H(f').
\end{aligned}$$

Consequently,

$$H(f) = \int_{-\infty}^{\infty} h(t)e^{-i2\pi ft}\,\mathrm{d}t,$$

that is, Equation 1.6 $\Rightarrow$ Equation1.7.

Problems

1. Show that the Fourier transform $\mathcal{F}$ and the inverse Fourier transform $\mathcal{F}^{-1}$ of a symmetric function are the same. Also show that these Fourier transforms are symmetric.
2. Compute $\mathcal{F}\{\mathcal{F}\{h(t)\}\}$ and $\mathcal{F}^{-1}\{\mathcal{F}^{-1}\{h(t)\}\}$.
3. Derive the Fourier transforms of the Gaussian curve, *i.e.*,
$$\mathcal{F}\left\{\sqrt{\frac{\alpha}{\pi}}\exp\left(-\alpha f^2\right)\right\} \text{ and } \mathcal{F}^{-1}\left\{\sqrt{\frac{\alpha}{\pi}}\exp\left(-\alpha f^2\right)\right\},$$
where $\alpha > 0$.
4. Derive the Fourier transform of the Lorentzian function $L(f) = \dfrac{\sigma/\pi}{f^2+\sigma^2}$, $\sigma > 0$. (Function $L(f)$ is symmetric, and hence $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the same.)
Hint: Use the following integral from mathematical tables: $\displaystyle\int_0^\infty \frac{\cos(Cx)}{x^2+a^2}\,\mathrm{d}x = \frac{\pi}{2a}e^{-Ca}$, where $C, a > 0$.
5. Applying Fourier transforms, compute the integral $I = \displaystyle\int_{-T}^{T}(T-|t|)\cos[2\pi f_0(T-t)]\,\mathrm{d}t$.
6. Let us denote $\Pi_{2T'}(t)$ the boxcar function of the width $2T'$, stretching from $-T'$ to T', and of one unit height. The sum of N boxcar functions
$$\frac{1}{N}\Pi_{\frac{1}{N}2T}(t),\ \frac{1}{N}\Pi_{\frac{2}{N}2T}(t),\ \frac{1}{N}\Pi_{\frac{3}{N}2T}(t),\ldots,\ \frac{1}{N}\Pi_{2T}(t)$$
is a one-unit high step-pyramidal function. In the limit $N\to\infty$ this pyramidal function approaches a one-unit high triangular function $\Lambda_T(t)$. Determine the inverse Fourier transform of the pyramidal function, and find the inverse Fourier transform of $\Lambda_T(t)$ by letting $N\to\infty$.
Hint: You may need the trigonometric identity
$$\sin\alpha+\sin(2\alpha)+\cdots+\sin(N\alpha) = \frac{\sin\left[\frac{1}{2}(N+1)\alpha\right]\sin\left(\frac{1}{2}N\alpha\right)}{\sin(\alpha/2)}.$$
7. Show that the function
$$\delta(t) = \int_{-\infty}^{\infty} e^{i2\pi ft}\,\mathrm{d}f$$
satisfies the definition of Dirac's delta function by using the Fourier's integral theorem
$$\mathcal{F}^{-1}\mathcal{F}\{F\} = F.$$

8. Show that the function $f(t) = \lim_{\sigma \to 0} \frac{\sigma/\pi}{t^2 + \sigma^2}$ satisfies the definition of Dirac's delta function.

9. Let us define the step function $u(t)$ as

$$u(t) = \begin{cases} 0, & t < 0, \\ 1, & t \geq 0. \end{cases}$$

Show that the choice $\delta(t - t_0) = \frac{\mathrm{d}}{\mathrm{d}t_0} u(t_0 - t)$ satisfies the condition of Dirac's delta function.

Hint: You can change the order of differentiation and integration.

10. Compute the integral $\int_{-\varepsilon}^{\varepsilon} \delta(t)\,\mathrm{d}t$, where $\varepsilon > 0$, by inserting $\delta(t) = \int_{-\infty}^{\infty} e^{\pm i 2\pi f t}\,\mathrm{d}f$ in it.

11. What are the Fourier transforms (both $\mathcal{F}$ and $\mathcal{F}^{-1}$) of the following functions? Let us assume that $f_0 > 0$.

(a) $\delta(f)$ (Dirac's delta peak in the origin);

(b) $\delta(f - f_0)$ (Dirac's delta peak at f_0);

(c) $\delta(f - f_0) + \delta(f + f_0)$ (Dirac's delta peaks at f_0 and $-f_0$);

(d) $\delta(f - f_0) - \delta(f + f_0)$ (Dirac's delta peak at f_0 and negative Dirac's delta peak at $-f_0$).

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2 General properties of Fourier transforms

The general definitions of the Fourier transform $\mathcal{F}$ and the inverse Fourier transform $\mathcal{F}^{-1}$ are

$$h(t) = \int_{-\infty}^{\infty} H(f)e^{i2\pi ft}\,\mathrm{d}f = \mathcal{F}\{H(f)\}, \tag{2.1}$$

$$H(f) = \int_{-\infty}^{\infty} h(t)e^{-i2\pi ft}\,\mathrm{d}t = \mathcal{F}^{-1}\{h(t)\}. \tag{2.2}$$

$H(f)$ is called the *spectrum* and $h(t)$ is the *signal*. The inverse Fourier transform operator $\mathcal{F}^{-1}$ generates the spectrum from a signal, and the Fourier transform operator $\mathcal{F}$ restores the signal from a spectrum. A single point in the spectrum corresponds to a single exponential wave in the signal, and vice versa. The signal is often defined in the time domain (t-domain), and the spectrum in the frequency domain (f-domain), as above.

Usually, the signal $h(t)$ is real. The spectrum $H(f)$ can still be complex, because

$$\begin{aligned} H(f) &= \int_{-\infty}^{\infty} h(t)e^{-i2\pi ft}\,\mathrm{d}t = \int_{-\infty}^{\infty} h(t)\cos(2\pi ft)\,\mathrm{d}t - i\int_{-\infty}^{\infty} h(t)\sin(2\pi ft)\,\mathrm{d}t \\ &= \mathcal{F}_{\cos}\{h(t)\} - i\mathcal{F}_{\sin}\{h(t)\} = |H(f)|e^{i\theta(f)}. \end{aligned} \tag{2.3}$$

$\mathcal{F}_{\cos}\{h(t)\}$ and $\mathcal{F}_{\sin}\{h(t)\}$ are called the *cosine transform* and the *sine transform*, respectively, of $h(t)$.

$$|H(f)| = \sqrt{[\mathcal{F}_{\cos}\{h(t)\}]^2 + [\mathcal{F}_{\sin}\{h(t)\}]^2} \tag{2.4}$$

is the *amplitude spectrum* and

$$\theta(f) = -\arctan\frac{\mathcal{F}_{\sin}\{h(t)\}}{\mathcal{F}_{\cos}\{h(t)\}} \tag{2.5}$$

is the *phase spectrum.* (Equation 2.5 is valid only when $-\pi/2 \le \theta \le \pi/2$.) The amplitude spectrum and the phase spectrum are illustrated in Figure 2.1. The inverse Fourier transform and the Fourier transform can be expressed with the help of the cosine transform and the sine transform as

$$\mathcal{F}^{-1}\{h(t)\} = \mathcal{F}_{\cos}\{h(t)\} - i\mathcal{F}_{\sin}\{h(t)\} \tag{2.6}$$

and

$$\mathcal{F}\{H(f)\} = \mathcal{F}_{\cos}\{H(f)\} + i\mathcal{F}_{\sin}\{H(f)\}. \tag{2.7}$$

Often, $i\mathcal{F}_{\sin}\{H(f)\}$ equals zero.

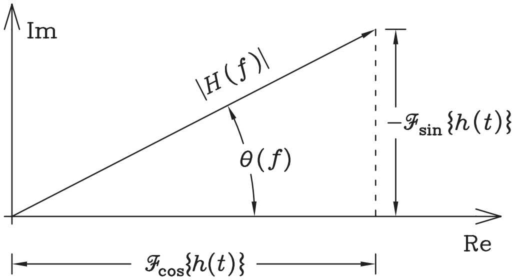

Figure 2.1: The amplitude spectrum $|H(f)|$ and the phase spectrum $\theta(f)$.

In the following, a collection of theorems of Fourier analysis is presented. They contain the most important characteristic features of Fourier transforms.

2.1 Shift theorem

Let us consider how a shift $\pm f_0$ of the frequency of a spectrum $H(f)$ affects the corresponding signal $h(t)$, which is the Fourier transform of the spectrum. We can find this by making a change of variables in the Fourier integral:

$$\begin{aligned}
\mathcal{F}\{H(f \pm f_0)\} &= \int_{-\infty}^{\infty} H(f \pm f_0) e^{i2\pi f t}\,\mathrm{d}f \\
&\overset{\substack{g=f\pm f_0 \\ \mathrm{d}g=\mathrm{d}f}}{=} \int_{-\infty}^{\infty} H(g) e^{i2\pi(g\mp f_0)t}\,\mathrm{d}g = e^{\mp i2\pi f_0 t} \int_{-\infty}^{\infty} H(g) e^{i2\pi g t}\,\mathrm{d}g \\
&= h(t) e^{\pm i2\pi f_0 t}.
\end{aligned}$$

Likewise, we can obtain the effect of a shift $\pm t_0$ of the position of the signal $h(t)$ on the spectrum $H(f)$, which is the inverse Fourier transform of the signal.

These results are called the *shift theorem*. The theorem states how the shift in the position of a function affects its transform. If $H(f) = \mathcal{F}^{-1}\{h(t)\}$, and both t_0 and f_0 are constants, then

$$\boxed{\begin{aligned}
\mathcal{F}\{H(f \pm f_0)\} &= \mathcal{F}\{H(f)\} e^{\mp i2\pi f_0 t} &= h(t) e^{\mp i2\pi f_0 t}, \\
\mathcal{F}^{-1}\{h(t \pm t_0)\} &= \mathcal{F}^{-1}\{h(t)\} e^{\pm i2\pi f t_0} &= H(f) e^{\pm i2\pi f t_0}.
\end{aligned}} \tag{2.8}$$

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