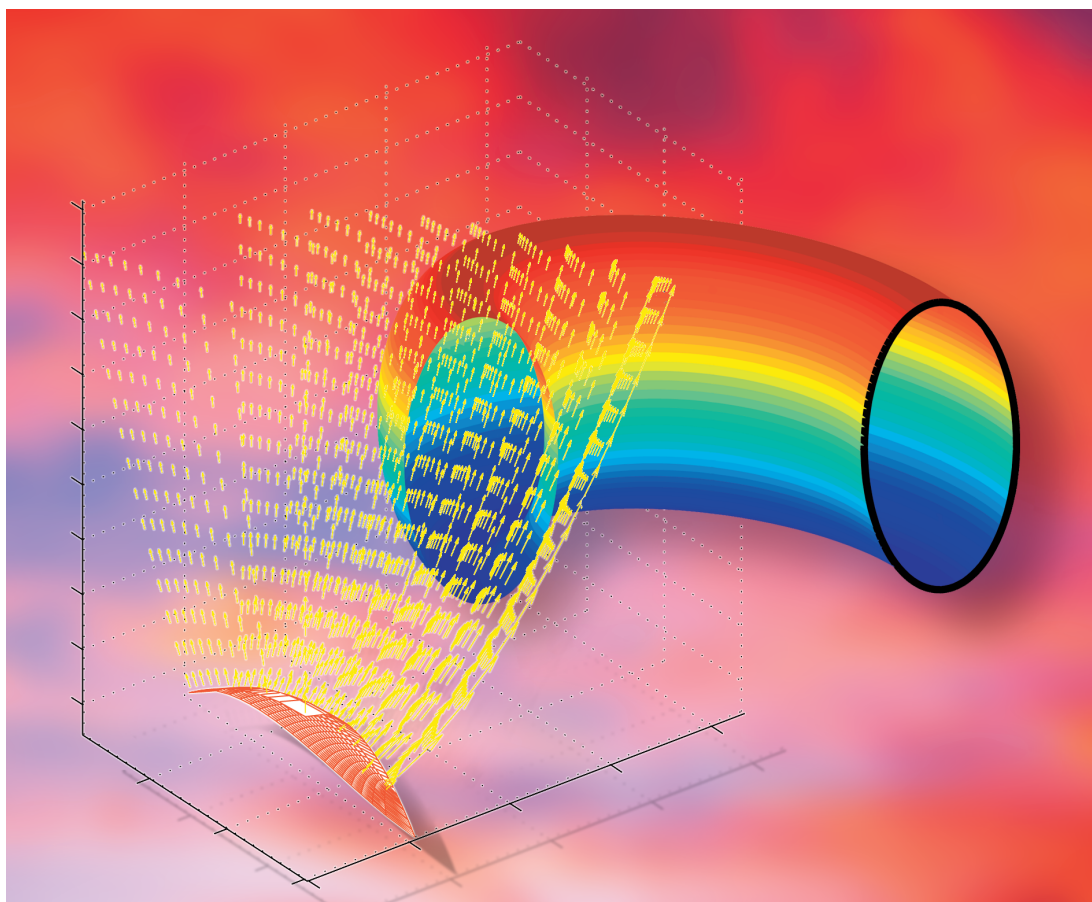


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Lok C. Lew Yan Voon

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Contents

Preface *XXI*

Part One Preliminaries 1

1 Introduction 3

2 General Theory 7

2.1 Introduction 7

2.2 Canonical Partial Differential Equations 7

2.3 Differential Operators in Curvilinear Coordinates 8

2.3.1 Metric 8

2.3.2 Gradient 9

2.3.3 Divergence 9

2.3.4 Circulation 9

2.3.5 Laplacian 9

2.3.5.1 Example 9

2.4 Separation of Variables 10

2.4.1 Two Dimensions 11

2.4.1.1 Rectangular Coordinate System 11

2.4.1.2 Other Coordinate Systems 12

2.4.2 Three Dimensions 15

2.4.2.1 Stäckel Matrix 16

2.4.2.2 Helmholtz Equation 17

2.4.2.3 Schrödinger Equation 19

2.4.2.4 Separable Coordinate Systems 19

2.5 Series Solutions 20

2.5.1 Singularities 20

2.5.2 Bôcher Equation 21

2.5.2.1 Example 21

2.5.3 Frobenius Method 22

2.5.3.1 One Regular Singular Point 24

2.5.3.2 Two Regular Singular Points 24

2.5.3.3 One Irregular Singular Point 24

2.5.3.4 Three Regular Singular Points 25

| | | |
|---|--|-----------|
| 2.6 | Boundary-Value Problems | 26 |
| 2.6.1 | Boundary Conditions | 26 |
| 2.6.2 | Fourier Expansions | 28 |
| 2.7 | Physical Applications | 30 |
| 2.7.1 | Electrostatics | 30 |
| 2.7.2 | Photonics | 30 |
| 2.7.3 | Heat Conduction | 31 |
| 2.7.4 | Newtonian Gravitation | 32 |
| 2.7.5 | Hydrodynamics | 33 |
| 2.7.6 | Acoustics | 33 |
| 2.7.7 | Quantum Mechanics | 35 |
| 2.8 | Problems | 36 |
| Part Two Two-Dimensional Coordinate Systems 39 | | |
| 3 | Rectangular Coordinates | 41 |
| 3.1 | Introduction | 41 |
| 3.2 | Coordinate System | 41 |
| 3.2.1 | Coordinates (x, y) | 41 |
| 3.2.2 | Constant-Coordinate Curves | 41 |
| 3.3 | Differential Operators | 42 |
| 3.3.1 | Metric | 42 |
| 3.3.2 | Operators | 43 |
| 3.3.2.1 | Gradient | 43 |
| 3.3.2.2 | Divergence | 43 |
| 3.3.2.3 | Laplacian | 43 |
| 3.4 | Separable Equations | 43 |
| 3.4.1 | Laplace Equation | 43 |
| 3.4.2 | Helmholtz Equation | 44 |
| 3.4.3 | Schrödinger Equation | 45 |
| 3.5 | Applications | 46 |
| 3.5.1 | Electrostatics: Dirichlet Problem for a Conducting Strip | 46 |
| 3.5.2 | Quantum Mechanics: Dirichlet Problem for a Rectangular Box | 47 |
| 3.6 | Problems | 49 |
| 4 | Circular Coordinates | 51 |
| 4.1 | Introduction | 51 |
| 4.2 | Coordinate System | 51 |
| 4.2.1 | Coordinates | 51 |
| 4.2.2 | Constant-Coordinate Curves | 51 |
| 4.3 | Differential Operators | 52 |
| 4.3.1 | Metric | 52 |
| 4.3.2 | Operators | 52 |
| 4.3.2.1 | Gradient | 52 |
| 4.3.2.2 | Divergence | 53 |
| 4.3.2.3 | Laplacian | 53 |

| | | |
|----------|--|-----------|
| 4.4 | Separable Equations | 53 |
| 4.4.1 | Laplace Equation | 53 |
| 4.4.2 | Helmholtz Equation | 54 |
| 4.4.3 | Schrödinger Equation | 55 |
| 4.5 | Applications | 56 |
| 4.5.1 | Quantum Mechanics: Dirichlet and Neumann Problems for a Disk | 56 |
| 4.5.1.1 | Infinite-Barrier Solutions | 56 |
| 4.5.1.2 | Finite-Barrier Solutions | 57 |
| 4.5.1.3 | Infinite-Barrier Pie | 58 |
| 4.6 | Problems | 59 |
| 5 | Elliptic Coordinates | 61 |
| 5.1 | Introduction | 61 |
| 5.2 | Coordinate System | 61 |
| 5.2.1 | Coordinates (u, v) | 61 |
| 5.2.2 | Constant-Coordinate Curves | 62 |
| 5.3 | Differential Operators | 63 |
| 5.3.1 | Metric | 63 |
| 5.3.2 | Operators | 63 |
| 5.3.2.1 | Gradient | 63 |
| 5.3.2.2 | Divergence | 63 |
| 5.3.2.3 | Laplacian | 63 |
| 5.4 | Separable Equations | 64 |
| 5.4.1 | Laplace Equation | 64 |
| 5.4.2 | Helmholtz Equation | 64 |
| 5.4.3 | Schrödinger Equation | 65 |
| 5.5 | Applications | 66 |
| 5.5.1 | Quantum Mechanics: Dirichlet Problem for an Ellipse | 66 |
| 5.5.1.1 | Finite-Barrier Solutions | 66 |
| 5.6 | Problems | 68 |
| 6 | Parabolic Coordinates | 71 |
| 6.1 | Introduction | 71 |
| 6.2 | Coordinate System | 71 |
| 6.2.1 | Coordinates (μ, ν) | 71 |
| 6.2.2 | Constant-Coordinate Curves | 71 |
| 6.3 | Differential Operators | 72 |
| 6.3.1 | Metric | 72 |
| 6.3.2 | Operators | 72 |
| 6.3.2.1 | Gradient | 73 |
| 6.3.2.2 | Divergence | 73 |
| 6.3.2.3 | Laplacian | 73 |
| 6.4 | Separable Equations | 73 |
| 6.4.1 | Laplace Equation | 73 |
| 6.4.2 | Helmholtz Equation | 73 |
| 6.4.3 | Schrödinger Equation | 75 |

- 6.5 Applications 75
- 6.5.1 Heat Conduction: Dirichlet Problem for the Laplace Equation 75
- 6.6 Problems 76

Part Three Three-Dimensional Coordinate Systems 79

7 Rectangular Coordinates 81

- 7.1 Introduction 81
- 7.2 Coordinate System 81
 - 7.2.1 Coordinates (x, y, z) 81
 - 7.2.2 Constant-Coordinate Surfaces 81
- 7.3 Differential Operators 82
 - 7.3.1 Metric 82
 - 7.3.2 Operators 82
 - 7.3.2.1 Gradient 82
 - 7.3.2.2 Divergence 82
 - 7.3.2.3 Circulation 83
 - 7.3.2.4 Laplacian 83
 - 7.3.3 Stäckel Matrix 83
- 7.4 Separable Equations 83
 - 7.4.1 Laplace Equation 83
 - 7.4.2 Helmholtz Equation 85
 - 7.4.3 Schrödinger Equation 86
- 7.5 Applications 87
 - 7.5.1 Electrostatics: Dirichlet Problem for a Rectangular Box 87
- 7.6 Problems 89

8 Circular Cylinder Coordinates 91

- 8.1 Introduction 91
- 8.2 Coordinate System 91
 - 8.2.1 Coordinates (r, ϕ, z) 91
 - 8.2.2 Constant-Coordinate Surfaces 92
- 8.3 Differential Operators 92
 - 8.3.1 Metric 92
 - 8.3.2 Operators 93
 - 8.3.2.1 Gradient 93
 - 8.3.2.2 Divergence 93
 - 8.3.2.3 Circulation 93
 - 8.3.2.4 Laplacian 93
 - 8.3.3 Stäckel Theory 93
- 8.4 Separable Equations 94
 - 8.4.1 Laplace Equation 94
 - 8.4.2 Helmholtz Equation 95
 - 8.4.3 Schrödinger Equation 95
- 8.5 Applications 96
 - 8.5.1 Heat Conduction: Dirichlet Problem for a Cylinder 96

| | | |
|-----------|---|------------|
| 8.5.2 | Quantum Mechanics: Dirichlet Problem for a Cylinder | 97 |
| 8.5.2.1 | Infinite Barrier | 97 |
| 8.6 | Problems | 97 |
| 9 | Elliptic Cylinder Coordinates | 99 |
| 9.1 | Introduction | 99 |
| 9.2 | Coordinate System | 99 |
| 9.2.1 | Coordinates (u, v, z) | 99 |
| 9.2.2 | Constant-Coordinate Surfaces | 100 |
| 9.3 | Differential Operators | 101 |
| 9.3.1 | Metric | 101 |
| 9.3.2 | Operators | 101 |
| 9.3.2.1 | Gradient | 101 |
| 9.3.2.2 | Divergence | 101 |
| 9.3.2.3 | Circulation | 102 |
| 9.3.2.4 | Laplacian | 102 |
| 9.3.3 | Stäckel Matrix | 102 |
| 9.4 | Separable Equations | 102 |
| 9.4.1 | Laplace Equation | 102 |
| 9.4.2 | Helmholtz Equation | 104 |
| 9.4.3 | Schrödinger Equation | 105 |
| 9.5 | Applications | 105 |
| 9.5.1 | Hydrodynamics: Dirichlet Problem for an Elliptic Pipe | 106 |
| 9.5.2 | Quantum Mechanics: Dirichlet Problem for an Elliptic Cylinder | 106 |
| 9.6 | Problems | 107 |
| 10 | Parabolic Cylinder Coordinates | 109 |
| 10.1 | Introduction | 109 |
| 10.2 | Coordinate System | 109 |
| 10.2.1 | Coordinates (μ, ν, z) | 109 |
| 10.2.2 | Constant-Coordinate Surfaces | 110 |
| 10.2.3 | Other Geometrical Parameters | 111 |
| 10.3 | Differential Operators | 112 |
| 10.3.1 | Metric | 112 |
| 10.3.2 | Operators | 112 |
| 10.3.2.1 | Gradient | 112 |
| 10.3.2.2 | Divergence | 112 |
| 10.3.2.3 | Circulation | 112 |
| 10.3.2.4 | Laplacian | 112 |
| 10.3.3 | Stäckel Matrix | 113 |
| 10.4 | Separable Equations | 113 |
| 10.4.1 | Laplace Equation | 113 |
| 10.4.2 | Helmholtz Equation | 114 |
| 10.4.3 | Schrödinger Equation | 115 |
| 10.5 | Applications | 115 |
| 10.5.1 | Acoustics: Neumann Problem for a Cavity | 116 |

| | | |
|-----------|---|------------|
| 10.5.1.1 | Case (a) | 119 |
| 10.5.1.2 | Case (b) | 119 |
| 10.5.1.3 | Case (c) | 119 |
| 10.5.1.4 | Relation between k and α_3 | 120 |
| 10.5.1.5 | Results | 120 |
| 10.6 | Problems | 124 |
| 11 | Spherical Polar Coordinates | 125 |
| 11.1 | Introduction | 125 |
| 11.2 | Coordinate System | 125 |
| 11.2.1 | Coordinates (r, θ, ϕ) | 125 |
| 11.2.2 | Constant-Coordinate Surfaces | 126 |
| 11.3 | Differential Operators | 126 |
| 11.3.1 | Metric | 126 |
| 11.3.2 | Operators | 126 |
| 11.3.2.1 | Gradient | 126 |
| 11.3.2.2 | Divergence | 127 |
| 11.3.2.3 | Circulation | 127 |
| 11.3.2.4 | Laplacian | 127 |
| 11.3.3 | Stäckel Matrix | 127 |
| 11.4 | Separable Equations | 127 |
| 11.4.1 | Laplace Equation | 127 |
| 11.4.2 | Helmholtz Equation | 128 |
| 11.4.3 | Schrödinger Equation | 129 |
| 11.5 | Applications | 130 |
| 11.5.1 | Quantum Mechanics: Dirichlet Problem | 130 |
| 11.5.1.1 | Infinite-Barrier Spherical Dot | 131 |
| 11.5.1.2 | Finite-Barrier Spherical Dot | 131 |
| 11.5.1.3 | Quantum Ice Cream – Infinite Barrier | 132 |
| 11.5.1.4 | $\nu(-\mu) = \nu(\mu)$ | 133 |
| 11.5.1.5 | $E(-\mu) = E(\mu)$ | 133 |
| 11.5.1.6 | $\nu \geq -1/2$ | 133 |
| 11.5.1.7 | $\nu \geq \mu $ | 133 |
| 11.5.1.8 | Additional Constraints | 134 |
| 11.6 | Problems | 137 |
| 12 | Prolate Spheroidal Coordinates | 139 |
| 12.1 | Introduction | 139 |
| 12.2 | Coordinate System | 139 |
| 12.2.1 | Coordinates $(\alpha, \beta, \phi$ and $\xi, \eta, \phi)$ | 139 |
| 12.2.2 | Constant-Coordinate Surfaces | 140 |
| 12.3 | Differential Operators | 141 |
| 12.3.1 | Metric | 141 |
| 12.3.2 | Operators | 141 |
| 12.3.2.1 | Gradient | 141 |
| 12.3.2.2 | Divergence | 141 |

- 12.3.2.3 Circulation 142
- 12.3.2.4 Laplacian 142
- 12.3.3 Stäckel Matrix 142
- 12.4 Separable Equations 142
- 12.4.1 Laplace Equation 142
- 12.4.2 Helmholtz Equation 143
- 12.4.3 Schrödinger Equation 144
- 12.5 Applications 144
- 12.5.1 Dirichlet Problem for the Laplace Equation 145
- 12.5.2 Gravitation: Dirichlet–Neumann Problem 146
- 12.5.3 Quantum Mechanics: Dirichlet Problem 147
- 12.5.3.1 Infinite-Barrier Problem 147
- 12.5.3.2 Finite-Barrier Problem 150
- 12.6 Problems 154
- 13 Oblate Spheroidal Coordinates 155**
- 13.1 Introduction 155
- 13.2 Coordinate System 155
- 13.2.1 Coordinates $(\alpha, \beta, \varphi$ and $\xi, \eta, \varphi)$ 155
- 13.2.2 Constant-Coordinate Surfaces 156
- 13.3 Differential Operators 157
- 13.3.1 Metric 157
- 13.3.2 Operators 157
- 13.3.2.1 Gradient 157
- 13.3.2.2 Divergence 158
- 13.3.2.3 Circulation 158
- 13.3.2.4 Laplacian 158
- 13.3.3 Stäckel Matrix 158
- 13.4 Separable Equations 159
- 13.4.1 Laplace Equation 159
- 13.4.2 Helmholtz Equation 159
- 13.4.3 Schrödinger Equation 160
- 13.5 Applications 161
- 13.5.1 Dirichlet Problem for the Laplace Equation 161
- 13.5.2 Asymptotic Solutions 162
- 13.6 Problems 163
- 14 Parabolic Rotational Coordinates 165**
- 14.1 Introduction 165
- 14.2 Coordinate System 165
- 14.2.1 Coordinates (ξ, η, ϕ) 165
- 14.2.2 Constant-Coordinate Surfaces 166
- 14.2.3 Other Geometrical Parameters 166
- 14.3 Differential Operators 167
- 14.3.1 Metric 167
- 14.3.2 Operators 167

| | | |
|-----------|--|------------|
| 14.3.2.1 | Gradient | 167 |
| 14.3.2.2 | Divergence | 168 |
| 14.3.2.3 | Circulation | 168 |
| 14.3.2.4 | Laplacian | 168 |
| 14.3.3 | Stäckel Matrix | 168 |
| 14.4 | Separable Equations | 168 |
| 14.4.1 | Laplace Equation | 168 |
| 14.4.2 | Helmholtz Equation | 169 |
| 14.4.3 | Schrödinger Equation | 170 |
| 14.5 | Applications | 171 |
| 14.5.1 | Heat Conduction: Boundary-Value Problem for the Laplace Equation | 172 |
| 14.5.1.1 | Dirichlet | 172 |
| 14.5.2 | Quantum Mechanics: Interior Dirichlet Problem | 173 |
| 14.5.2.1 | Numerical Results | 177 |
| 14.6 | Problems | 179 |
| 15 | Conical Coordinates | 181 |
| 15.1 | Introduction | 181 |
| 15.2 | Coordinate System | 181 |
| 15.2.1 | Coordinates (r, θ, λ) | 181 |
| 15.2.2 | Constant-Coordinate Surfaces | 182 |
| 15.3 | Differential Operators | 183 |
| 15.3.1 | Metric | 183 |
| 15.3.2 | Operators | 183 |
| 15.3.2.1 | Gradient | 183 |
| 15.3.2.2 | Divergence | 183 |
| 15.3.2.3 | Circulation | 184 |
| 15.3.2.4 | Laplacian | 184 |
| 15.3.3 | Stäckel Theory | 184 |
| 15.4 | Separable Equations | 184 |
| 15.4.1 | Laplace Equation | 184 |
| 15.4.2 | Helmholtz Equation | 185 |
| 15.4.3 | Schrödinger Equation | 186 |
| 15.5 | Applications | 187 |
| 15.5.1 | Electrostatics: Dirichlet and Neumann Problems on a Plane Angular Sector | 187 |
| 15.6 | Problems | 189 |
| 16 | Ellipsoidal Coordinates | 191 |
| 16.1 | Introduction | 191 |
| 16.2 | Coordinate System | 192 |
| 16.2.1 | Coordinates (ξ_1, ξ_2, ξ_3) | 193 |
| 16.2.2 | Ellipsoid | 194 |
| 16.3 | Differential Operators | 195 |
| 16.3.1 | Metric | 195 |

| | | |
|-----------|--|------------|
| 16.3.2 | Operators | 195 |
| 16.3.2.1 | Gradient | 195 |
| 16.3.2.2 | Divergence | 196 |
| 16.3.2.3 | Circulation | 196 |
| 16.3.2.4 | Laplacian | 196 |
| 16.4 | Separable Equations | 197 |
| 16.4.1 | Laplace Equation | 197 |
| 16.4.2 | Helmholtz Equation | 199 |
| 16.5 | Applications | 200 |
| 16.5.1 | Interior Problem for the Laplace Equation | 200 |
| 16.5.1.1 | Ellipsoidal Harmonic of the First Species | 201 |
| 16.5.1.2 | Ellipsoidal Harmonic of the Second Species | 202 |
| 16.5.1.3 | Ellipsoidal Harmonic of the Third Species | 202 |
| 16.5.1.4 | Ellipsoidal Harmonic of the Fourth Species | 203 |
| 16.5.2 | Elliptic Functions | 203 |
| 16.5.3 | Dirichlet Problem for the Helmholtz Equation: ATZ Algorithm | 204 |
| 16.5.3.1 | First Solution to the Ellipsoidal Wave Equation | 205 |
| 16.5.3.2 | Second Solution to the Ellipsoidal Wave Equation | 208 |
| 16.5.3.3 | Ellipsoidal Domain | 209 |
| 16.5.4 | Quantum Mechanics: Interior Dirichlet Problem for an Ellipsoid | 210 |
| 16.5.4.1 | Characteristic Curves | 211 |
| 16.5.4.2 | Determination of γ Eigenvalues | 212 |
| 16.5.4.3 | Lamé Wave Functions | 213 |
| 16.6 | Problems | 215 |
| 17 | Paraboloidal Coordinates | 217 |
| 17.1 | Introduction | 217 |
| 17.2 | Coordinate System | 217 |
| 17.2.1 | Coordinates (μ, ν, λ) | 217 |
| 17.2.2 | Constant-Coordinate Surfaces | 218 |
| 17.3 | Differential Operators | 219 |
| 17.3.1 | Metric | 219 |
| 17.3.2 | Operators | 219 |
| 17.3.3 | Stäckel Matrix | 220 |
| 17.4 | Separable Equations | 221 |
| 17.4.1 | Laplace Equation | 221 |
| 17.4.1.1 | Separation of Variables | 221 |
| 17.4.1.2 | Series Solutions | 221 |
| 17.4.1.3 | Polynomial Solutions | 223 |
| 17.4.2 | Helmholtz Equation | 227 |
| 17.4.2.1 | Separation of Variables | 227 |
| 17.5 | Applications | 227 |
| 17.5.1 | Electrostatics: Dirichlet Problem for a Paraboloid | 227 |
| 17.6 | Problems | 229 |

| | | |
|------------------|---|-----|
| Part Four | Advanced Formulations | 231 |
| 18 | Differential-Geometric Formulation | 233 |
| 18.1 | Introduction | 233 |
| 18.2 | Review of Differential Geometry | 233 |
| 18.2.1 | Curvilinear Coordinates | 233 |
| 18.2.2 | Gradient, Divergence, and Laplacian | 236 |
| 18.2.3 | Curl and Cross Products | 238 |
| 18.2.4 | Vector Calculus Expressions in General Coordinates | 239 |
| 18.3 | Problems | 239 |
| 19 | Quantum-Mechanical Particle Confined to the Neighborhood of Curves | 241 |
| 19.1 | Introduction | 241 |
| 19.2 | Laplacian in a Tubular Neighborhood of a Curve – Arc-Length Parameterization | 241 |
| 19.2.1 | Arc-Length Parameterization | 241 |
| 19.2.1.1 | Minimal Rotating Frame | 243 |
| 19.2.2 | Laplacian | 245 |
| 19.2.3 | Circular Cross Section | 247 |
| 19.3 | Application to the Schrödinger Equation | 248 |
| 19.3.1 | Solutions to the χ_2 and χ_3 Equations | 249 |
| 19.4 | Schrödinger Equation in a Tubular Neighborhood of a Curve – General Parameterization | 250 |
| 19.5 | Applications | 251 |
| 19.6 | Perturbation Theory Applied to the Curved-Structure Problem | 259 |
| 19.6.1 | Dirichlet Unperturbed Eigenstates | 259 |
| 19.6.2 | Evaluation of $\Delta\lambda_n$ in the Case with Dirichlet Boundary Conditions | 260 |
| 19.6.3 | Eigenstate Perturbations | 263 |
| 19.6.4 | Neumann Unperturbed Eigenstates | 263 |
| 19.6.5 | Evaluation of $\Delta\lambda_n$ in the Case with Neumann Boundary Conditions | 264 |
| 19.6.6 | Perturbation Theory in the General Parameterization Case | 267 |
| 19.6.7 | Comparison between Analytical Results and Perturbation Theory for Circular-Bent Rectangular Domains in Two Dimensions – Dirichlet Boundary Conditions | 268 |
| 19.6.7.1 | Rectangular Domain – No Bending | 268 |
| 19.6.7.2 | Rectangular Domain – With Bending | 268 |
| 19.6.8 | Comparison between Analytical Results and Perturbation Theory for Circular-Bent Rectangular Domains in Two Dimensions – Neumann Boundary Conditions | 268 |
| 19.6.8.1 | Rectangular Domain – No Bending | 269 |
| 19.6.9 | Rectangular Domain – With Bending | 269 |
| 19.7 | Problems | 269 |

| | | |
|-----------|---|------------|
| 20 | Quantum-Mechanical Particle Confined to Surfaces of Revolution | 271 |
| 20.1 | Introduction | 271 |
| 20.2 | Laplacian in Curved Coordinates | 271 |
| 20.3 | The Schrödinger Equation in Curved Coordinates | 274 |
| 20.4 | Applications | 274 |
| 20.4.1 | Truncated Cone | 274 |
| 20.4.2 | Elliptic Torus | 277 |
| 20.5 | Problems | 281 |
| 21 | Boundary Perturbation Theory | 283 |
| 21.1 | Nondegenerate States | 283 |
| 21.2 | Degenerate States | 285 |
| 21.3 | Applications | 286 |
| 21.4 | Problems | 293 |
| | Appendix A Hypergeometric Functions | 295 |
| A.1 | Introduction | 295 |
| A.2 | Hypergeometric Equation | 295 |
| A.3 | Hypergeometric Functions | 296 |
| A.3.1 | First Solution | 296 |
| A.3.1.1 | Examples | 297 |
| A.3.1.2 | Properties | 297 |
| A.3.2 | Second Solution | 297 |
| A.4 | Confluent Hypergeometric Equation | 298 |
| A.5 | Confluent Hypergeometric Functions | 298 |
| A.5.1 | First Solution | 298 |
| A.5.1.1 | Examples | 298 |
| A.5.2 | Second Solution | 299 |
| A.5.3 | Properties | 299 |
| A.6 | Whittaker Functions | 299 |
| A.6.1 | Whittaker Equation | 299 |
| A.6.2 | Whittaker Functions | 299 |
| A.7 | Associated Laguerre Functions | 300 |
| A.7.1 | Associated Laguerre Equation | 300 |
| A.7.2 | Associated Laguerre Function | 300 |
| A.7.3 | Laguerre Equation | 300 |
| A.7.3.1 | Alternative Representation | 300 |
| A.7.4 | Generalized Laguerre Polynomials | 300 |
| A.8 | Hermite Polynomial | 301 |
| A.8.1 | Hermite Equation | 301 |
| A.8.2 | Hermite Polynomials | 301 |
| A.8.3 | Properties | 302 |
| A.9 | Airy Functions | 302 |
| A.9.1 | Airy Equation | 303 |
| A.9.2 | Properties | 303 |

Appendix B Baer Functions 305

- B.1 Introduction 305
- B.2 Baer Equation 305
- B.3 Baer Functions 305
- B.4 Baer Wave Equation 306
- B.5 Baer Wave Functions 306
 - B.5.1 Orthogonality 306

Appendix C Bessel Functions 309

- C.1 Introduction 309
- C.2 Bessel Equations 309
- C.3 Bessel Functions 310
 - C.3.1 ν Nonintegral 310
 - C.3.2 ν Integral 310
 - C.3.3 Properties 311
 - C.3.4 Hankel Functions 313
 - C.3.4.1 Properties 313
- C.4 Modified Bessel Functions 314
 - C.4.1 Properties 314
- C.5 Spherical Bessel Functions 315
 - C.5.1 Properties 316
- C.6 Modified Spherical Bessel Functions 316
- C.7 Bessel Wave Functions 316
 - C.7.1 Series Solution 317
 - C.7.2 Orthogonality 318

Appendix D Lamé Functions 321

- D.1 Lamé Equations 321
- D.2 Lamé Functions 322
 - D.2.1 First Kind 322
 - D.2.1.1 $F(z)$ 322
 - D.2.1.2 $F(z) = \sqrt{z^2 - a^2} B(z)$ 324
 - D.2.1.3 $F(z) = \sqrt{z^2 - a^2} \sqrt{z^2 - b^2} B(z)$ 325
 - D.2.2 Second Kind 326
- D.3 Lamé Wave Equation 326
 - D.3.1 Moon–Spencer Form 327
 - D.3.2 Arscott’s Algebraic Form 327

Appendix E Legendre Functions 329

- E.1 Introduction 329
- E.2 Legendre Equation 329
- E.3 Series Solutions 329
 - E.3.1 Recurrence Relation 329
 - E.3.2 Convergence 330
- E.4 Legendre Polynomials 330
 - E.4.1 Normalization 330

| | | |
|---|----------------------------------|-----|
| E.4.2 | Representations | 331 |
| E.4.2.1 | Hypergeometric Function | 331 |
| E.4.2.2 | Rodrigue's Formula | 331 |
| E.4.2.3 | Generating Function | 331 |
| E.4.2.4 | Schaefli Integral Representation | 331 |
| E.4.3 | Special Values | 332 |
| E.4.4 | Orthogonality | 332 |
| E.4.5 | Expansions | 332 |
| E.5 | Legendre Function | 333 |
| E.5.1 | Hypergeometric Representation | 333 |
| E.5.2 | Properties | 333 |
| E.6 | Associated Legendre Functions | 333 |
| E.6.1 | Associated Legendre Equation | 333 |
| E.6.2 | Associated Legendre Functions | 334 |
| E.6.2.1 | Properties | 335 |
| E.6.3 | Associated Legendre Polynomials | 335 |
| E.6.4 | Generating Function | 335 |
| E.6.5 | Recurrence Relations | 335 |
| E.6.6 | Parity | 336 |
| E.6.7 | Orthogonality | 336 |
| E.7 | Spherical Harmonics | 336 |
| E.7.1 | Definition | 336 |
| E.7.2 | Orthogonality | 337 |
| Appendix F Mathieu Functions 339 | | |
| F.1 | Introduction | 339 |
| F.2 | Mathieu Equation | 339 |
| F.3 | Mathieu Function | 340 |
| F.3.1 | Properties | 341 |
| F.3.2 | Orthogonality | 341 |
| F.3.3 | Periodic Solution for Small q | 343 |
| F.4 | Characteristic Equation | 343 |
| F.4.1 | Recurrence Relations | 344 |
| F.4.1.1 | (Even, π) Solutions | 344 |
| F.4.1.2 | (Even, 2π) Solutions | 345 |
| F.4.1.3 | (Odd, π) Solutions | 345 |
| F.4.1.4 | (Odd, 2π) Solutions | 346 |
| F.4.1.5 | (Even, π) Solutions | 346 |
| F.4.1.6 | (Even, 2π) Solutions | 347 |
| F.4.1.7 | (Odd, π) Solutions | 347 |
| F.4.1.8 | (Odd, 2π) Solutions | 347 |
| F.4.2 | Continued Fraction Solution | 347 |
| F.4.2.1 | (Even, π) Solutions | 348 |
| F.4.2.2 | (Even, 2π) Solutions | 348 |
| F.4.2.3 | (Odd, π) Solutions | 348 |

| | | |
|---------|---------------------------------------|-----|
| F.4.2.4 | (Odd, 2π) Solutions | 348 |
| F.4.2.5 | (Even, π) Solutions | 348 |
| F.4.2.6 | (Even, 2π) Solutions | 349 |
| F.4.2.7 | (Odd, π) Solutions | 349 |
| F.4.2.8 | (Odd, 2π) Solutions | 349 |
| F.5 | Mathieu Functions of Fractional Order | 349 |
| F.6 | Nonperiodic Second Solutions | 350 |

Appendix G Spheroidal Wave Functions 351

| | | |
|---------|---------------------------|-----|
| G.1 | Introduction | 351 |
| G.2 | Spheroidal Wave Equation | 351 |
| G.3 | Spheroidal Wave Functions | 352 |
| G.3.1 | Prolate Angular Functions | 352 |
| G.3.1.1 | Recurrence Relation | 352 |
| G.3.1.2 | Eigenvalue Problem | 354 |
| G.3.1.3 | Continued Fractions | 355 |

Appendix H Weber Functions 357

| | | |
|-------|-----------------|-----|
| H.1 | Weber Equation | 357 |
| H.2 | Weber Functions | 358 |
| H.2.1 | Properties | 359 |

Appendix I Elliptic Integrals and Functions 361

| | | |
|---------|--------------------------------------|-----|
| I.1 | Elliptic Integrals | 361 |
| I.1.1 | Elliptic Integral of the First Kind | 361 |
| I.1.2 | Elliptic Integral of the Second Kind | 362 |
| I.1.3 | Elliptic Integral of the Third Kind | 362 |
| I.1.4 | Complete Elliptic Integrals | 363 |
| I.1.4.1 | Limiting Values | 363 |
| I.2 | Jacobian Elliptic Functions | 363 |
| I.2.1 | Notation | 364 |
| I.2.2 | Degeneracy | 365 |
| I.2.2.1 | $k \rightarrow 0$ | 365 |
| I.2.2.2 | $k \rightarrow 1$ | 365 |
| I.2.3 | Relations | 365 |
| I.2.4 | Derivatives | 365 |
| I.2.5 | Parity | 365 |
| I.2.6 | Addition Theorems | 365 |
| I.2.6.1 | $\operatorname{sn} z$ | 365 |
| I.2.6.2 | $\operatorname{cn} z$ | 365 |
| I.2.6.3 | $\operatorname{dn} z$ | 366 |
| I.2.7 | K' | 366 |
| I.2.8 | Special Values | 366 |
| I.2.9 | Period | 366 |

I.2.10 Behavior near the Origin and iK' 366

I.2.10.1 Near the Origin 367

I.2.10.2 Near iK' 367

References 369

Index 375

Preface

We became interested in the research that has led to this book in 2000, when the two of us met at the ferry terminal in Tsim Tsai Tsui, Hong Kong, and discussed the problem of separability of partial differential equations. This was followed by a research visit by L.C.L.Y.V. to the Mads Clausen Institute at Syddansk Universitet in 2003, a visit funded by the Balslev Foundation. It is only fitting that L.C.L.Y.V. was invited back to the Mathematical Modeling Group of the Mads Clausen Institute on the beautiful new campus of Syddansk Universitet at Alsion to finish work on the book.

Our interaction during that time has led to numerous publications, including a few on the topic of this book and another book on the electronic properties of semiconductors. Whereas our earlier work followed the exposition of Morse and Feshbach and that of Moon and Spencer closely, we have since incorporated a more general differential-geometric approach. Both approaches are featured in this book. As mathematical physicists, it was a pleasure to put together a book that blends together knowledge in mathematics and physics going back 100 years.

The research and book writing has received generous financial support over the years. The work of M.W. has been supported by Syddansk Universitet and Sønderborg Kommune. The work of L.C.L.Y.V. has been funded by the National Science Foundation (USA), the Balslev Foundation, and Sønderborg Kommune. L.C.L.Y.V. would also like to thank the College of Science and Mathematics at Wright State University for release from duties to write this book and the hospitality of the Mads Clausen Institute at Syddansk Universitet, where most of the writing took place. Two individuals have contributed to some parts of this work. First, Prof. Jens Gravesen was an indispensable collaborator in our work on the differential-geometric formulation and this is obvious from his coauthorship of many of our joint papers in this area. Second, we would like to thank Lars Duggen for his help in making some of the figures in the book.

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October 2010

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Part One Preliminaries

1

Introduction

This is a textbook about how to solve boundary-value problems in physics using the method of separation of variables which goes beyond the few simple coordinate systems presented in most textbook discussions. Our goal is to present an application-oriented approach to the study of the general theory of the method of separation of variables, whereby the variety of separable orthogonal coordinate systems is included to illustrate various aspects of the theory (e.g., lesser known coordinate systems, the coupling of separation constants, and solving for the boundary-value problem particularly for many-parameter surfaces) and also to discuss the variety of special functions that can result (e.g., from transcendental to Lamé functions). We will add, right upfront, that this is not a text about special functions, though sufficient results about the latter are included to make the text as self-contained as possible.

In numerous areas of science and engineering, one has to solve a partial differential equation (PDE) for some fairly regular shape. Examples include Newtonian gravity for an ellipsoidal meteorite [1], the temperature distribution over a paraboloidal aircraft cone [2], the electric field in the vicinity of the brain modeled as an ellipsoid [3], and the electronic structure of spherical quantum dots [4]. A very powerful method is the method of separation of variables, whereby the PDE is separated into ordinary differential equations (ODEs). The latter then need to be solved, often in the form of power series, leading to special functions such as the Legendre functions and the Bessel functions, and, finally, boundary conditions are applied. Even when the shape deviates from the ideal regular shape, a preliminary investigation using the regular shape is often useful both as a validation technique for some other, more numerical approach and as a first step in a, for example, perturbative approach to the exact solution. Indeed, according to Morse and Feshbach [5], the method of separation of variables is only one of two generally practical methods of solution, the other being the integral solution. Furthermore, practically all mathematical physics texts discuss the method heuristically applied to one or more of the following coordinate systems: rectangular, circular cylindrical, and spherical polar. Nevertheless, the restriction to a few coordinate systems hides a number of features of the method as well as, of course, its range of applicability. Discussion of more advanced features of the method has been reserved to a few texts [5–9]. Thus, the separability of the Helmholtz equation in 11 orthogonal coordinate systems is not generally known in spite of the utility of many of these coordinate systems for

applications. Even the formal definition of “separation of variables” is rarely given. It has been argued that such a definition is needed before general results can be demonstrated [10, 11].

In this book, the problem of separating the Laplacian in various orthogonal coordinate systems in Euclidean 3-space is presented and the resulting ODEs for a number of PDEs of physical interest are given. Explicit solutions in terms of special functions are then described. Various physical problems are discussed in detail, including in acoustics, in heat conduction, in electrostatics, and in quantum mechanics, as the corresponding PDEs represent three general forms to which many other differential equations reduce (Laplace, Helmholtz, and Schrödinger). Furthermore, they represent two classes of differential equations (elliptic and hyperbolic) and different types of boundary conditions. A unique feature of our book is the part devoted to the differential geometric formulation of PDEs and their solutions for various kinds of confined geometries and boundary conditions. Such a treatment, though not entirely new, has recently been extended by a few authors, including us, and has mostly only appeared in the research literature.

There are obviously many applications of the method of separation of variables, particularly for the common rectangular, circular cylindrical, and spherical polar coordinate systems. The general theory has also been worked out and discussed in the mathematical physics literature. Our treatment follows closely the books by Morse and Feshbach [5] and Moon and Spencer [6] in covering more than just the standard coordinate systems. The former gives an exposition of the method as applied to the Laplace, Helmholtz, and Schrödinger equations, whereas the latter lists the coordinate systems, resulting ODEs, and series solutions in a very compact and formal form, leading occasionally to less practical solutions (see, e.g., the “corrections” in [12]). We extend their treatments by giving many examples of boundary-value problems and include some more recent results mostly in the field of nanotechnology. Our book is not a comprehensive review of all the special-function literature, nor is the formal mathematical theory presented. The former is done in the many books on special functions, whereas the latter is presented in a nice book by Miller [8]. It is also worthwhile pointing out that the method of separation of variables has been applied to other PDEs such as the Dirac equation and the Klein–Gordon equation. One of the foci of the book is to emphasize that there are three distinct separability problems: that of the differential equations, that of the separation constants, and that of the boundary conditions. The separability of the differential equations is addressed by presenting the results in 11 coordinate systems (even though there can be separability in additional coordinate systems for special cases such as the Laplace equation).

The consequence of a varying degree of separability of the separation constants is made clear in connection with the boundary-value problem; this is an aspect that is missing in Moon and Spencer’s treatment. Finally, the separability of the boundary conditions relates to the choice of the coordinate system. Last but not least, we present a variety of computational algorithms for the more difficult boundary-value problems that should be of practical help to readers for a complete solution to such problems. In this respect, we show the limited practical value of the series

solutions in the book by Moon and Spencer and the usefulness but also restricted applicability of the algorithms given by Zhang and Jin [13]. This aspect is also not covered in the book by Morse and Feshbach.

The book is divided into four parts. The first part deals with the general theory of the method of separation of variables and also has a brief summary of the areas of physical applications discussed in the book. Part Two presents the technique in two dimensions. The solutions of the resulting ODEs are discussed in some detail, particularly when a special function appears for the first time. Part Three considers the three-dimensional coordinate systems, which include the simple three-dimensional extension of the two-dimensional systems of Part Two (rectangular and cylindrical systems) and of systems with rotational symmetry, and also the lesser known conical, ellipsoidal, and paraboloidal systems. Part Four provides an alternative formulation of the method of separation of variables in terms of differential geometry. Illustrations are provided for problems with nanowire structures and a recent perturbative theory is discussed in detail. Finally, a few key results on special functions are included in the appendices. Functions that appear directly as solutions to the separated ODEs are described in separate appendices (except for Appendix I on elliptic functions) and other useful functions which show up occasionally are collected in Appendix A on the hypergeometric function.

In summary, it is intended that this book not only contains the standard introductory topics to the study of separation of variables but will also provide a bridge to the more advanced research literature and monographs on the subject. The fundamental material presented and a few of the coordinate systems can serve as a textbook for a one-semester course on PDEs either at the senior undergraduate level or at the graduate level. It is also expected to complement the many books that have already been published on boundary-value problems and special functions (e.g., [5–9, 14–19]), particularly in the treatment of the Helmholtz problem. The chapters not covered in a course would be appropriate for self-study and even serve as sources of ideas for both undergraduate- and graduate-level research projects.

