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Heinz Georg Schuster

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The Editor

Heinz Georg Schuster
University of Kiel
schuster@theo-physik.uni-kiel.de

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Preface

Nonlinear behavior is ubiquitous in nature and ranges from fluid dynamics, via neural and cell dynamics to the dynamics of financial markets. The most prominent feature of nonlinear systems is that small external disturbances can induce large changes in their behavior. This can and has been used for effective feedback control in many systems, from Lasers to chemical reactions and the control of nerve cells and heartbeats. A new hot topic are nonlinear effects that appear on the nanoscale. Nonlinear control of the atomic force microscope has improved it accuracy by orders of magnitude. Nonlinear electromechanical oscillations of nano-tubes, turbulence and mixing of fluids in nano-arrays and nonlinear effects in quantum dots are further examples.

Complex systems consist of large networks of coupled nonlinear devices. The observation that scalefree networks describe the behavior of the internet, cell metabolisms, financial markets and economic and ecological systems has lead to new findings concerning their behavior, such as damage control, optimal spread of information or the detection of new functional modules, that are pivotal for their description and control.

This shows that the field of *Nonlinear Dynamics and Complexity* consists of a large body of theoretical and experimental work with many applications, which is nevertheless governed and held together by some very basic principles, such as control, networks and optimization. The individual topics are definitely interdisciplinary which makes it difficult for researchers to see what new solutions – which could be most relevant for them – have been found by their scientific neighbors. Therefore its seems quite urgent to provide *Reviews of Nonlinear Dynamics and Complexity* where researchers or newcomers to the field can find the most important recent results, described in a fashion which breaks the barriers between the disciplines.

In this first volume topics range from nano-mechanical oscillators, via random Boolean networks and control to extreme climatic and seismic events. I would like to thank all authors for their excellent contributions. If readers take from these interdisciplinary reviews some inspiration for their further research this volume would fully serve its purpose.

I am grateful to all members of the Editorial Board and the staff from Wiley-VCH for their excellent help and would like to invite my colleagues to contribute to the next volumes.

Kiel, February 2008 *Heinz Georg Schuster*

List of Contributors

Armin Bunde
Institut für Theoretische Physik III
Justus-Liebig-Universität Giessen
Heinrich-Buff-Ring 16
35392 Giessen
Germany
armin.bunde@physik.uni-giessen.de

M. C. Cross
Department of Physics
California Institute of Technology
Pasadena, CA 91125
USA
mcc@caltech.edu

Barbara Drossel
Institut für Festkörperphysik
Technische Universität Darmstadt
Hochschulstrasse 6
64289 Darmstadt
Germany
barbara.drossel@physik.tu-darmstadt.de

Jan F. Eichner
Institut für Theoretische Physik III
Justus-Liebig-Universität Giessen
Heinrich-Buff-Ring 16
35392 Giessen
Germany
jan.f.eichner@physik.uni-giessen.de

Bernold Fiedler
Institut für Mathematik I
FU Berlin
Arnimallee 2–6
14195 Berlin
Germany
fiedler@math.fu-berlin.de

Valentin Flunkert
Institut für Theoretische Physik
Technische Universität Berlin
Hardenbergstrasse 36
10623 Berlin
Germany
tino@physik.tu-berlin.de

Marc Georgi
Institut für Mathematik I
FU Berlin
Arnimallee 2–6
14195 Berlin
Germany
georgi@inf.fu-berlin.de

Bailin Hao
The Institute of Theoretical Physics
55 Zhongguancun East Road
Haidan District
Beijing 100080
China
and
The T-Life Research Center
Fudan University
220 Handan Road
Shanghai 200433
China
hao@mail.itp.ac.cn

Shlomo Havlin
Minerva Center and Department
of Physics
Bar-Ilan University
Ramat-Gan
Israel
havlin@ophir.ph.biu.ac.il

Philipp Hövel
Institut für Theoretische Physik
Technische Universität Berlin
Hardenbergstrasse 36
10623 Berlin
Germany
phoevel@physik.tu-berlin.de

Jan W. Kantelhardt
Institut für Physik
Martin-Luther-Universität
Halle-Wittenberg
Germany
jan.kantelhardt@physik.uni-halle.de

Sabine Lennartz
Institut für Theoretische Physik III
Justus-Liebig-Universität Giessen
Heinrich-Buff-Ring 16
35392 Giessen
Germany
sabine.lennartz@physik.uni-giessen.de

Ron Lifshitz
Raymond and Beverly Sackler School of
Physics and Astronomy
Tel Aviv University
Tel Aviv 69978
Israel
lifshitz@caltech.edu

Arkady Pikovsky
Department of Physics
Potsdam University
Am Neuen Palais 10
14469 Potsdam
Germany
pikovsky@stat.physik.uni-potsdam.de

Michael Rosenblum
Department of Physics
Potsdam University
Am Neuen Palais 10
14469 Potsdam
Germany
mRos@agnld.uni-potsdam.de

Eckehard Schöll
Institut für Theoretische Physik
Technische Universität Berlin
Hardenbergstrasse 36
10623 Berlin
Germany
schoell@physik.tu-berlin.de

Huimin Xie
Department of Mathematics
Suzhou University
Suzhou 215006
China
szhmxie@yahoo.com.cn

1
Nonlinear Dynamics of Nanomechanical and Micromechanical Resonators

Ron Lifshitz and M. C. Cross

1.1
Nonlinearities in NEMS and MEMS Resonators

In the last decade we have witnessed exciting technological advances in the fabrication and control of microelectromechanical and nanoelectromechanical systems (MEMS & NEMS) [1–5]. Such systems are being developed for a host of nanotechnological applications, such as highly-sensitive mass [6–8], spin [9], and charge detectors [10,11], as well as for basic research in the mesoscopic physics of phonons [12], and the general study of the behavior of mechanical degrees of freedom at the interface between the quantum and the classical worlds [13,14]. Surprisingly, MEMS & NEMS have also opened up a whole new experimental window into the study of the nonlinear dynamics of discrete systems in the form of nonlinear micromechanical and nanomechanical oscillators and resonators.

The purpose of this review is to provide an introduction to the nonlinear dynamics of micromechanical and nanomechanical resonators that starts from the basics, but also touches upon some of the advanced topics that are relevant for current experiments with MEMS & NEMS devices. We begin in this section with a general motivation, explaining why nonlinearities are so often observed in NEMS & MEMS devices. In § 1.2 we describe the dynamics of one of the simplest nonlinear devices – the Duffing resonator – while giving a tutorial in secular perturbation theory as we calculate its response to an external drive. We continue to use the same analytical tools in § 1.3 to discuss the dynamics of a parametrically-excited Duffing resonator, building up to the description of the dynamics of an array of coupled parametrically-excited Duffing resonators in § 1.4. We conclude in § 1.5 by giving an amplitude equation description for the array of coupled Duffing resonators, allowing us to extend our analytic capabilities in predicting and explaining the nature of its dynamics.

1.1.1 Why Study Nonlinear NEMS and MEMS?

Interest in the nonlinear dynamics of microelectromechanical and nanoelectromechanical systems (MEMS & NEMS) has grown rapidly over the last few years, driven by a combination of practical needs as well as fundamental questions. Nonlinear behavior is readily observed in micro- and nano-scale mechanical devices [3, 15–32]. Consequently, there exists a practical need to understand this behavior in order to avoid it when it is unwanted, and exploit it efficiently when it is wanted. At the same time, advances in the fabrication, transduction, and detection of MEMS & NEMS resonators has opened up an exciting new experimental window into the study of fundamental questions in nonlinear dynamics. Typical nonlinear MEMS & NEMS resonators are characterized by extremely high frequencies – recently going beyond 1 GHz [33–35] – and relatively weak dissipation, with quality factors in the range of 10^2–10^4. For such devices the regime of physical interest is that of steady state motion as transients tend to disappear before they are detected. This, and the fact that weak dissipation can be treated as a small perturbation, provide a great advantage for quantitative theoretical study. Moreover, the ability to fabricate arrays of tens to hundreds of coupled resonators opens new possibilities in the study of nonlinear dynamics of intermediate numbers of degrees of freedom – much larger than one can study in macroscopic or table-top experiments, yet much smaller than one studies when considering nonlinear aspects of phonon dynamics in a crystal. The collective response of coupled arrays might be useful for signal enhancement and noise reduction [36], as well as for sophisticated mechanical signal processing applications. Such arrays have already exhibited interesting nonlinear dynamics ranging from the formation of extended patterns [37] – as one commonly observes in analogous continuous systems such as Faraday waves – to that of intrinsically localized modes [38–40]. Thus, nanomechanical resonator arrays are perfect for testing dynamical theories of discrete nonlinear systems with many degrees of freedom. At the same time, the theoretical understanding of such systems may prove useful for future nanotechnological applications.

1.1.2 Origin of Nonlinearity in NEMS and MEMS Resonators

We are used to thinking about mechanical resonators as being simple harmonic oscillators, acted upon by linear elastic forces that obey Hooke's law. This is usually a very good approximation as most materials can sustain relatively large deformations before their intrinsic stress-strain relation breaks away from a simple linear description. Nevertheless, one commonly encounters nonlinear dynamics in micromechanical and nanomechanical resonators long before the intrinsic nonlinear regime is reached. Most evident are nonlinear effects that enter the equation of motion in the form of a force that is proportional to the cube of the displacement αx^3. These turn a simple harmonic resonator with a linear restoring force into a so-called Duffing resonator. The two main origins of the observed nonlinear effects are illustrated below with the help of two typical examples. These are due to the effect of external potentials that are often nonlinear, and geometric effects that introduce nonlinearities even though the individual forces that are involved are all linear. The Duffing nonlinearity αx^3 can be positive, assisting the linear restoring force, making the resonator stiffer, and increasing its resonance frequency. It can also be negative, working against the linear restoring force, making the resonator softer, and decreasing its resonance frequency. The two examples we give below illustrate how both of these situations can arise in realistic MEMS & NEMS devices.

Additional sources of nonlinearity may be found in experimental realizations of MEMS and NEMS resonators due to practical reasons. These may include nonlinearities in the actuation and in the detection mechanisms that are used for interacting with the resonators. There could also be nonlinearities that result from the manner in which the resonator is clamped by its boundaries to the surrounding material. These all introduce external factors that may contribute to the overall nonlinear behavior of the resonator.

Finally, nonlinearities often appear in the damping mechanisms that accompany every physical resonator. We shall avoid going into the detailed description of the variety of physical processes that govern the damping of a resonator. Suffice it to say that whenever it is reasonable to expand the forces acting on a resonator up to the cube of the displacement x^3, it should correspondingly be reasonable to add to the linear damping which is proportional to the velocity of the resonator $\dot{x}$, a nonlinear damping term of the form $x^2\dot{x}$, which increases with the amplitude of motion. Such nonlinear damping will be considered in our analysis below.

1.1.3
Nonlinearities Arising from External Potentials

As an example of the effect of an external potential, let us consider a typical situation, discussed for example by Cleland and Roukes [10,11], and depicted in Fig. 1.1, in which a harmonic oscillator is acted upon by an external electrostatic force. This could be implemented by placing a rigid electrically-charged base electrode near an oppositely-charged NEMS or MEMS resonator. If the equilibrium separation between the resonator and the base electrode in the absence of electric charge is d, the deviation away from this equilibrium position is denoted by X, the effective elastic spring constant of the resonator is K,

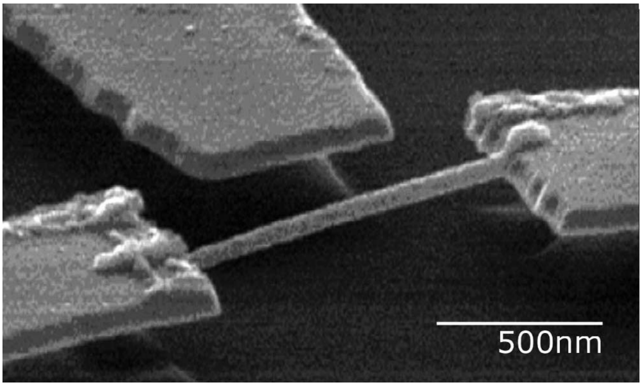

Fig. 1.1 A 43 nanometer thick doubly-clamped platinum nanowire with an external electrode that can be used to tune its natural frequency as well as its nonlinear properties. (Adapted with permission from [30]).

and the charge q on the resonator is assumed to be constant, then the potential energy of the resonator is given by

$$V(X) = \frac{1}{2}KX^2 - \frac{C}{d+X}\,. \tag{1.1}$$

In SI units $C = Aq^2/4\pi\epsilon_0$, where A is a numerical factor of order unity that takes into account the finite dimensions of the charged resonator and base electrode. The new equilibrium position X_0 in the presence of charge can be determined by solving the cubic equation

$$\frac{dV}{dX} = KX + \frac{C}{(d+X)^2} = 0. \tag{1.2}$$

If we now expand the potential acting on the resonator in a power series in the deviation $x = X - X_0$ from this new equilibrium we obtain

$$\begin{aligned} V(x) &\simeq V(X_0) + \frac{1}{2}\left(K - \frac{2C}{(d+X_0)^3}\right)x^2 + \frac{C}{(d+X_0)^4}x^3 - \frac{C}{(d+X_0)^5}x^4 \\ &= V(X_0) + \frac{1}{2}kx^2 + \frac{1}{3}\beta x^3 + \frac{1}{4}\alpha x^4. \end{aligned} \tag{1.3}$$

This gives rise, without any additional driving or damping, to an equation of motion of the form

$$m\ddot{x} + kx + \beta x^2 + \alpha x^3 = 0, \qquad \text{with} \quad \beta > 0,\ \alpha < 0, \tag{1.4}$$

where m is the effective mass of the resonator; k is the new effective spring constant, which is softened by the electrostatic attraction to the base electrode, but note that if $2C/(d+X_0)^3 > K$, the electrostatic force exceeds the elastic restoring force and the resonator is pulled onto the base electrode; β is a positive symmetry-breaking quadratic elastic constant that pulls the resonator towards the base electrode regardless of the sign of x; and α is the cubic, or Duffing, elastic constant that owing to its negative sign softens the effect of the linear restoring force. It should be sufficient to stop the expansion here, unless the amplitude of the motion is much larger than the size of the resonator, or if by some coincidence the effects of the quadratic and cubic nonlinearities happen to cancel each other out – a situation that will become clearer after reading § 1.2.3.

1.1.4
Nonlinearities Due to Geometry

As an illustration of how nonlinearities can emerge from linear forces due to geometric effects, consider a doubly-clamped thin elastic beam, which is one of the most commonly encountered NEMS resonators. Because of the clamps at both ends, as the beam deflects in its transverse motion it necessarily stretches. As long as the amplitude of the transverse motion is much smaller than the width of the beam this effect can be neglected. But with NEMS beams it is often the case that they are extremely thin, and are driven quite strongly, making it common for the amplitude of vibration to exceed the width. Let us consider this effect in some detail by starting with the Euler–Bernoulli equation, which is the commonly used approximate equation of motion for a thin beam [41]. For a transverse displacement $X(z,t)$ from equilibrium, which is much smaller than the length L of the beam, the equation is

$$\rho S \frac{\partial^2 X}{\partial t^2} = -EI \frac{\partial^4 X}{\partial z^4} + T \frac{\partial^2 X}{\partial z^2}, \tag{1.5}$$

where z is the coordinate along the length of the beam; ρ the mass density; S the area of the cross section of the beam; E the Young modulus; I the moment of inertia; and T the tension in the beam, which is composed of its inherent tension T_0 and the additional tension ΔT due to its bending that induces an extension ΔL in the length of the beam. Inherent tension results from the fact that in equilibrium in the doubly-clamped configuration, the actual length of the beam may differ form its rest length being either extended (positive T_0) or compressed (negative T_0). The additional tension ΔT is given by the strain, or relative extension of the beam $\Delta L/L$ multiplied by Young's modulus E and the area of the beam's cross section S. For small displacements the total length of the beam can be expanded as

$$L + \Delta L = \int_0^L dz \sqrt{1 + \left(\frac{\partial X}{\partial z}\right)^2} \simeq L + \frac{1}{2}\int_0^L dz \left(\frac{\partial X}{\partial z}\right)^2. \qquad (1.6)$$

The equation of motion (1.5) then clearly becomes nonlinear

$$\rho S \frac{\partial^2 X}{\partial t^2} = -EI\frac{\partial^4 X}{\partial z^4} + \left[T_0 + \frac{ES}{2L}\int_0^L dz \left(\frac{\partial X}{\partial z}\right)^2\right]\frac{\partial^2 X}{\partial z^2}. \qquad (1.7)$$

We can treat this equation perturbatively [42,43]. We consider first the linear part of the equation, which has the form of Eq. (1.5) with T_0 in place of T, separate the variables,

$$X_n(z,t) = x_n(t)\phi_n(z), \qquad (1.8)$$

and find its spatial eigenmodes $\phi_n(z)$, where we use the convention that the local maximum of the eigenmode $\phi_n(z)$ that is nearest to the center of the beam is scaled to 1. Thus $x_n(t)$ measures the actual deflection at the point nearest to its center that extends the furthest. Next, we assume that the beam is vibrating predominantly in one of these eigenmodes and use this assumption to evaluate the effective Duffing parameter α_n, multiplying the x_n^3 term in the equation of motion for this mode. Corrections to this approximation will appear only at higher orders of x_n. We multiply Eq. (1.7) by the chosen eigenmode $\phi_n(z)$, and integrate over z to get, after some integration by parts, a Duffing equation of motion for the amplitude of the n^{th} mode $x_n(t)$,

$$\ddot{x}_n + \left[\frac{EI}{\rho S}\frac{\int \phi_n''^2 dz}{\int \phi_n^2 dz} + \frac{T_0}{\rho S}\frac{\int \phi_n'^2 dz}{\int \phi_n^2 dz}\right] x_n + \left[\frac{E}{2\rho L}\frac{\left(\int \phi_n'^2 dz\right)^2}{\int \phi_n^2 dz}\right] x_n^3 = 0, \qquad (1.9)$$

where primes denote derivatives with respect to z, and all the integrals are from 0 to L. Note that we have obtained a positive Duffing term, indicating a stiffening nonlinearity, as opposed to the softening nonlinearity that we saw in the previous section. Also note that the effective spring constant can be made negative by compressing the equilibrium beam, thus making T_0 large and negative. This may lead to the so-called Euler instability, which is a buckling instability of the beam.

To evaluate the effective Duffing nonlinearity α_n for the n^{th} mode, we introduce a dimensionless parameter $\hat{\alpha}_n$ by rearranging the equation of motion (1.9), to have the form

$$\ddot{x}_n + \omega_n^2 x_n \left[1 + \hat{\alpha}_n \frac{x_n^2}{d^2}\right] = 0, \qquad (1.10)$$

where ω_n is the normal frequency of the n^{th} mode, d is the width, or diameter, of the beam in the direction of the vibration, and x_n is the maximum displacement of the beam near its center. This parameter can then be evaluated regardless of the actual dimension of the beam.

In the limit of small residual tension T_0 the eigenmodes are those dominated by bending given by [41]

$$\phi_n(z) = \frac{1}{a_n}\Big[(\sin k_n L - \sinh k_n L)(\cos k_n z - \cosh k_n z) \\ - (\cos k_n L - \cosh k_n L)(\sin k_n z - \sinh k_n z)\Big], \tag{1.11}$$

where a_n is the value of the function in the square brackets at its local maximum that is closest to $z = 0.5$, and the wave vectors k_n are solutions of the transcendental equation $\cos k_n L \cosh k_n L = 1$. The first few values are

$$\{k_n L\} \simeq \{4.7300, 7.8532, 10.9956, 14.1372, 17.2788, 20.4204 \ldots\}, \tag{1.12}$$

and the remaining ones tend towards odd-integer multiples of $\pi/2$, as n increases. Using these eigenfunction we can obtain explicit values for the dimensionless Duffing parameters for the different modes, by calculating

$$\hat{\alpha}_n = \frac{Sd^2}{2I} \frac{\left(\frac{1}{L}\int \phi_n'^2 dz\right)^2}{\frac{1}{L}\int \phi_n''^2 dz} \equiv \frac{Sd^2}{2I}\hat{\beta}_n. \tag{1.13}$$

The first few values are

$$\left\{\hat{\beta}_n\right\} \simeq \{0.1199, 0.2448, 0.3385, 0.3706, 0.3908, 0.4068, 0.4187, \ldots\}, \tag{1.14}$$

tending to an asymptotic value of $1/2$ as $n \to \infty$. For beams with rectangular or circular cross sections, the geometric prefactor evaluates to

$$\frac{Sd^2}{2I} = \begin{cases} 16 & \text{Circular cross section} \\ 6 & \text{Rectangular cross section} \end{cases} \tag{1.15}$$

Thus the dimensionless Duffing parameters are of order 1, and therefore the significance of the nonlinear behavior is solely determined by the ratio of the deflection to the width of the beam.

In the limit of large equilibrium tension, the beam essentially behaves as a string with relatively negligible resistance to bending. The eigenmodes are those of a string,

$$\phi_n(z) = \sin\left(\frac{n\pi}{L}z\right), \quad n = 1, 2, 3, \ldots, \tag{1.16}$$

and, if we denote the equilibrium extension of the beam as $\Delta L_0 = LT_0/ES$, the dimensionless Duffing parameters are exactly given by

$$\hat{\alpha}_n = \frac{d^2}{2\Delta L_0}\int \phi_n'^2 dz = \frac{(n\pi d)^2}{4L\Delta L_0}. \tag{1.17}$$

In the large tension limit, as in the case of a string, the dimensionless Duffing parameters are proportional to the inverse aspect ratio of the beam d/L times the ratio between its width and the extension from its rest length $d/\Delta L_0$, at least one of which can be a very small parameter. For this reason nonlinear effects are relatively negligible in these systems.

1.2 The Directly-driven Damped Duffing Resonator

1.2.1 The Scaled Duffing Equation of Motion

Let us begin by considering a single nanomechanical Duffing resonator with linear and nonlinear damping that is driven by an external sinusoidal force. We shall start with the common situation where there is symmetry between x and $-x$, and consider later the changes that are introduced by adding symmetry-breaking terms. Such a resonator is described by the equation of motion

$$m\frac{d^2\tilde{x}}{d\tilde{t}^2} + \Gamma\frac{d\tilde{x}}{d\tilde{t}} + m\omega_0^2\tilde{x} + \tilde{\alpha}\tilde{x}^3 + \tilde{\eta}\tilde{x}^2\frac{d\tilde{x}}{d\tilde{t}} = \tilde{G}\cos\tilde{\omega}\tilde{t}, \tag{1.18}$$

where m is its effective mass, $k = m\omega_0^2$ is its effective spring constant, $\tilde{\alpha}$ is the cubic spring constant, or Duffing parameter, Γ is the linear damping rate, and $\tilde{\eta}$ is the coefficient of nonlinear damping – damping that increases with the amplitude of oscillation. We follow the convention that physical parameters that are immediately rescaled appear with twidles, as the first step in dealing with such an equation is to scale away as many unnecessary parameters as possible, leaving only those that are physically-significant, thus removing all of the twidles. We do so by: (1) Measuring time in units of ω_0^{-1} so that the dimensionless time variable is $t = \omega_0\tilde{t}$. (2) Measuring amplitudes of motion in units of length for which a unit-amplitude oscillation doubles the frequency of the resonator. This is achieved by taking the dimensionless length variable to be $x = \tilde{x}\sqrt{\tilde{\alpha}/m\omega_0^2}$. For the doubly-clamped beam of width or diameter d, discussed in § 1.1.4, this length is $x = \tilde{x}\sqrt{\hat{\alpha}_n}/d$. (3) Dividing the equation by an overall factor of $\omega_0^3\sqrt{m^3/\tilde{\alpha}}$. This yields a scaled Duffing equation of the