Boris M. Smirnov

Plasma Processes and Plasma Kinetics

586 Worked Out Problems for Science and Technology

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Preface

Various aspects of a low-temperature plasma are represented in the form of problems. Kinetics of this plasma is determined by elementary processes involving electrons, positive and negative ions, excited atoms, atoms, and molecules in the ground states. Along with collision processes involving these atomic particles, radiative processes are of importance for a low-temperature plasma and excited gas, both elementary radiative processes and transport of radiation through a gas that includes reabsorption processes. The collective processes, oscillation plasma properties, and nonlinear plasma processes are represented in the corresponding problems. Transport of particles in a plasma is of importance for a nonequilibrium plasma. A cluster plasma, an aerosol plasma, and other plasma forms with a dispersive phase are considered in the book. Because all these processes and phenomena are given in a specific form for each plasma, we consider separately two plasma types, a plasma of the atmosphere together with atmospheric phenomena due to this plasma and some types of gas-discharge plasma. Appendices contain information which is useful for the analysis of specific plasma types, and this information is represented in the convenient form for the user. The book is intended for students and professionals in the field of plasma physics, plasma chemistry, and plasma applications.

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1
Distributions and Equilibria for Particle Ensembles

1.1
Distributions of Identical Atomic Particles

▶ **Problem 1.1** An ensemble of n weakly interacting identical particles is located in a close space and does not interact with a surrounding environment. As a result of interactions, particles can change their state. Find the probability of a certain distribution of particles by states.

Distributing identical particles over states, if in each state many particles are found, we account for particles that can change their states, but an average number of particles in a given state almost conserve, and the better this is fulfilled, the more the number of particles found in this state on average. Let us distribute n identical particles over k states assuming that the probability for a test particle to be found in a given state, as well as the average number of particles in this state, is proportional to the number of versions which lead to this Gibbs principle.

Let us denote by $P(n_1, n_2, \ldots, n_i, \ldots)$ the number of ways to place n_1 particles in the first group of states, n_2 particles in the second group of states, n_i particles in the i-th group of states, etc. For determining this probability we use the character of distributions for the location of a particle in a certain group of states does not influence the character of distributions for other particles (*Boltzmann statistics*). Under these conditions, the probability of locating n_1 particles in the first state, n_2 particles in the second state, etc. is given by

$$P(n_1, n_2, \ldots, n_i, \ldots) = p(n_i)p(n_2)\cdots p(n_i)\cdots\,, \tag{1.1}$$

where $p(n_i)$ is the number of ways to distribute n_i particles in the i-th group of states. Evidently, the number of ways to place n_1 particles from the total number n of particles in the first group of states is

$$C_n^{n_1} = \frac{n!}{(n-n_1)!n_1!}\,.$$

Correspondingly, the number of ways to place n_2 particles from the remaining $n - n_1$ particles in the second group of states is $C_{n-n_1}^{n_2}$. Continuing this operation,

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we determine the probability of the indicated distribution of particles in states,

$$P(n_1, n_2, \ldots, n_i, \ldots) = \text{const}\frac{n!}{\prod\limits_i n_i!}, \tag{1.2}$$

where "const" is a normalization constant. The basis of this formula is the assumption that particles are free, so that the distribution of one particle does not influence the distribution of others.

▶ **Problem 1.2** Derive the Boltzmann distribution for an ensemble of weakly interacting particles.

The distribution under consideration relates to almost free classical particles when the number of states in a given state group is large compared to the number of particles which are found in states of a given group. Then the location of some particles in states of a given group does not influence the possibility of finding test particles in these states. Next, this distribution corresponds to conservation of the total number of particles in all the states,

$$n = \sum_i n_i, \tag{1.3}$$

and the total energy E for all the particles,

$$E = \sum_i \varepsilon_i n_i, \tag{1.4}$$

because particle's energy does not change with an environment. Here ε_i is the energy of a particle located in the i-th group of states.

For determining an average $\overline{n}_i$ or the most probable number of particles for a given i-th group of states, we account for any probable distribution that the variation of the number of particles in these states from the average value $\delta n_i = n_i - \overline{n}_i$ is relatively small. Next, according to formulas (1.3) and (1.4), these variations satisfy the relation

$$\sum_i \delta n_i = 0 \tag{1.5}$$

and

$$\sum_i \varepsilon_i \delta n_i = 0. \tag{1.6}$$

In addition, on the basis of the relation

$$\ln n! = \ln \prod_{m=1}^{n} m =\approx \int_0^n \ln x dx\ ,$$

we have $d \ln n!/dn = \ln n$. Using this relation with the expansion of formula (1.2) over a small parameter δn_i, we obtain

$$\ln P(n_1, n_2, \ldots, n_i, \ldots) = \ln P(\overline{n}_1, \overline{n}_2, \ldots, \overline{n}_i, \ldots) - \sum_i \ln \overline{n}_i \delta n_i - \sum_i \frac{\delta n_i^2}{2\overline{n}_i}. \tag{1.7}$$

Since the real distribution is near the maximum, the linear terms with respect to δn_i disappear, which corresponds to the relation

$$\sum_i \ln \overline{n}_i \delta n_i = 0. \tag{1.8}$$

In order to find the average number of particles in a given group of states, we use the formal operation of multiplying equation (1.5) by a constant $-\ln C$, equation (1.6) by the parameter $-1/T$, and adding these equations to (1.8). The resultant equation has the form

$$\sum_i \left(\ln \overline{n}_i - \ln C + \frac{\varepsilon_i}{T}\right) \delta n_i = 0. \tag{1.9}$$

Since variations δn_i are random, the expression in the parentheses is zero, which gives

$$\overline{n}_i = C \exp\left(-\frac{\varepsilon_i}{T}\right). \tag{1.10}$$

As a result, we obtain the Boltzmann formula. During its deduction we introduce two characteristic parameters, C and T. The first one is the normalization constant that follows from the relation

$$C \sum_i \exp(-\varepsilon_i/T) = n .$$

The energetic parameter T is the temperature of the system.

▶ **Problem 1.3** Represent the Boltzmann distribution by taking into account the statistical weight of a particle.

Dividing the states of a particle ensemble in groups, we as above assume equal number of states for each group. If these are different, we introduce the statistical weight of the particle state g_I as the number of states per particle. For example, if this particle is a diatomic molecule, and we characterize its state by the rotational momentum J, the number of its projections $g_i = 2J + 1$ onto the molecular axis is the statistical weight of a given state. As is seen, the statistical weight is the number of states per particle of this particle ensemble.

By taking into account the statistical weight for a given state of a particle, the Boltzmann distribution (1.10) is transformed to the form

$$\overline{n}_i = C g_i \exp\left(-\frac{\varepsilon_i}{T}\right). \tag{1.11}$$

From this we have the following relation between the number densities of particles in two states when an ensemble consists of the infinite number of particles,

$$N_i = N_o \frac{g_i}{g_o} \exp\left(-\frac{\varepsilon_i}{T}\right), \tag{1.12}$$

where N_o, N_i are the number densities of particles in these states, and g_o, g_i are the statistical weights of these states.

▶ **Problem 1.4** Find the distribution of molecules over vibrational states considering vibrations of the molecule like harmonic oscillators.

In this approximation, the energy of excitation ε_v of the v-th vibrational state is

$$\varepsilon_v = \hbar\omega\, v\ ,$$

where $\hbar\omega$ is the difference of energies for neighboring vibrational states. On the basis of the Boltzmann formula (1.12) we obtain the number density of molecules located in the v-th vibrational state,

$$N_v = N_0 \exp\left(-\frac{\hbar\omega v}{T_v}\right)\ ,$$

where N_0 is the number density of molecules in the ground vibrational state, and T_v is the vibrational temperature. The total number density of molecules is

$$N = \sum_{v=0}^{\infty} N_v = N_0 \sum_{v}^{\infty} \exp\left(-\frac{\hbar\omega v}{T_v}\right) = \frac{N_0}{1-\exp\left(-\frac{\hbar\omega}{T_v}\right)}\ , \tag{1.13}$$

which allows us to express the number density of molecules in a given vibrational state, N_v, through the total number density N of molecules,

$$N_v = N\ \exp\left(-\frac{\hbar\omega v}{T_v}\right)\left[1-\exp\left(-\frac{\hbar\omega}{T_v}\right)\right]\ . \tag{1.14}$$

From this one can find the average vibrational excitation energy ε_{vib}

$$\varepsilon_{\text{vib}} = \hbar\omega\overline{v} = \frac{1}{N}\sum_{v=0}^{\infty} v\ N_v = \frac{\hbar\omega}{\exp\left(\frac{\hbar\omega}{T_v}\right)-1}\ . \tag{1.15}$$

▶ **Problem 1.5** Find the distribution of diatomic molecules over rotational states.

The energy of excitation of a rotational state with a rotational momentum J is

$$\varepsilon_J = BJ(J+1)\ ,$$

where B is the rotational constant. The statistical weight of the state with a momentum J, which is the number of momentum projections onto a given axis, equals $g_J = 2J+1$. On the basis of this and the Boltzmann formula (1.12), we obtain the number density of molecules with a given rotational momentum and vibrational state,

$$N_{vJ} = N_v\ (2J+1)\frac{B}{T}\ \exp\left[-\frac{BJ(J+1)}{T}\right]\ , \tag{1.16}$$

where we assume $B \ll T$, as it is usually, and N_v is the total number density of molecules in a given vibrational state. From this one can also find the average rotational energy of molecules,

$$\varepsilon_{\text{rot}} = \overline{BJ(J+1)} = T\ . \tag{1.17}$$

In this analysis we account for that typical vibrational energies exceed significantly typical rotational energies, which allows us to separate vibrational and rotational degrees of freedom. We also note that we assume a diatomic molecule to be consisting of other isotopes. Otherwise, because of the molecule symmetry, only certain values of the rotation momentum can be realized.

▶ **Problem 1.6** Determine the statistical weight for a free particle.

Let us place an ensemble of free particles in a rectangular box with the edge size L, so that particles are reflected from the box's walls and cannot penetrate outside wall boundaries. Each particle is free and moves freely inside the box. Hence the wave function of a particle moving inside the box in the axis x direction can be composed of two waves, $\exp(ip_x x/\hbar)$ and $\exp(-ip_x x/\hbar)$, propagated in opposite directions, where p_x is the particle momentum. Placing the origin at lower-left corner of the box cube and requiring the wave function to be zero at cube facets, we obtain from the first boundary condition $\psi(0) = 0$ for the particle wave function ψ,

$$\psi = \sin\frac{p_x x}{\hbar} .$$

The second boundary condition $\psi(L) = 0$ leads to quantization of the particle momentum

$$\frac{p_x L}{\hbar} = \pi k ,$$

where k is an integer. This relation gives the prohibited values of the particle momentum if it is moving in a rectangular box of size L.

This gives the number of states for a particle with a momentum in the range from p_x to $p_x + dp_x$, which is equal to $dg = L dp_x/(2\pi\hbar)$, where we take into account two directions of the particle momentum. Introducing a coordinate range to be dx, we find the number of states for a free particle to be

$$dg = \frac{dp_x dx}{2\pi\hbar} . \tag{1.18}$$

Generalization of this formula to the three-dimensional case leads to the following number of states for a free particle:

$$dg = \frac{dp_x dx}{2\pi\hbar}\frac{dp_y dy}{2\pi\hbar}\frac{dp_z dz}{2\pi\hbar} = \frac{d\boldsymbol{p}d\boldsymbol{r}}{(2\pi\hbar)^3} , \tag{1.19}$$

where the quantity $d\boldsymbol{p}d\boldsymbol{r}$ is an element of the phase space, and the notation is used for three-dimensional elements of space $d\boldsymbol{r} = dxdydz$ and momentum $d\boldsymbol{p} = dp_x dp_y dp_z$. Formula (1.19) gives the statistical weight of the continuous spectrum—the number of states per element of the phase space.

▶ **Problem 1.7** Find the velocity distribution function for free particles—the Maxwell distribution.

We now use the Boltzmann formula (1.11) for the distribution of kinetic energies of free particles. In the one-dimensional case the kinetic energy of a particle whose velocity is v_x equals $\varepsilon_i = mv_x^2/2$, and the statistical weight of states when the particle velocity ranges from v_x to $v_x + dv_x$ is proportional to dv_x. Then formula (1.11) gives for the number of particles whose velocities are found in the range v_x to $v_x + dv_x$

$$f(v_x)dv_x = C\exp\left(-\frac{mv_x^2}{2T}\right)dv_x, \tag{1.20}$$

where C is the normalization factor. This is the Maxwell distribution for the one-dimensional case.

Transferring to the three-dimensional case by taking into account the independence of different directions of motion, we obtain

$$f(\boldsymbol{v})d\boldsymbol{v} = C\exp\left(-\frac{mv^2}{2T}\right)d\boldsymbol{v}. \tag{1.21}$$

Here the vector $\boldsymbol{v}$ has components v_x, v_y, v_z, and $d\boldsymbol{v} = dv_x dv_y dv_z$. The kinetic energy of a particle, $mv^2/2$, is the sum of the particle kinetic energies for all the directions of motion. Thus, the independence of different directions of particle motion results in the isotropy of the distribution function.

Let us rewrite the Maxwell distribution for the number density of particles of a given velocity, normalizing this distribution to the total number density N of particles. Then formula (1.21) gives

$$f(v) = N\left(\frac{m}{2\pi T}\right)^{3/2}\exp\left(-\frac{mv^2}{2T}\right). \tag{1.22}$$

It is convenient to rewrite the Maxwell distribution function (1.22) through one-dimensional distribution functions

$$f(v) = N\varphi(v_x)\,\varphi(v_y)\,\varphi(v_z), \tag{1.23}$$

where the functions $\varphi(v_i)$ are normalized to 1 and has the form

$$\varphi(v_x) = \sqrt{\frac{m}{2\pi T}}\exp\left(-\frac{mv_x^2}{2T}\right) \quad \int\limits_{-\infty}^{\infty}\varphi(v_x)dv_x = 1. \tag{1.24}$$

Figure 1.1 gives this dependence.

▶ **Problem 1.8** On the basis of the Maxwell distribution connect an average kinetic energy of a free particle with the temperature.

We introduce above the energetic parameter of the Boltzmann distribution function, T, which is the temperature of a given ensemble of particles and the temperature is expressed in energetic units. Below we find the average kinetic energy for an ensemble of free particles, and then the temperature of the particle ensemble will be expressed in terms of the average particle energy.

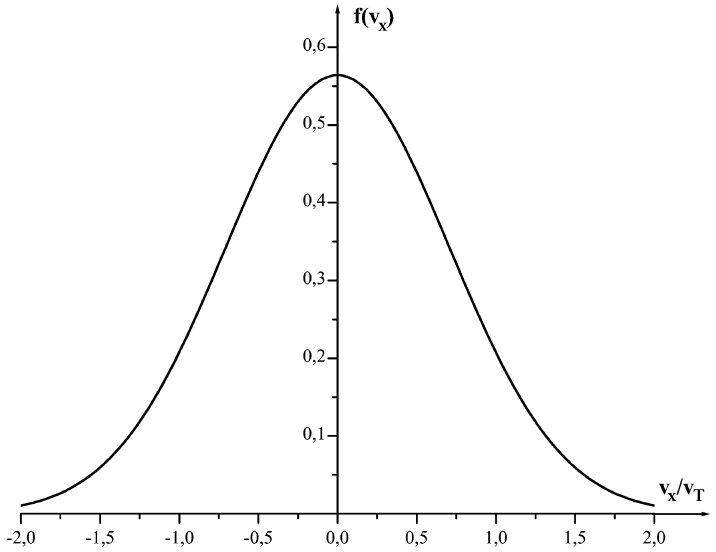

Fig. 1.1 The Maxwell distribution function $f(v_x)$ as a function of the reduced velocity $x = v_x\sqrt{m/(2T)}$, where v_x is the particle velocity in a given direction, m is the particle mass, T is the temperature.

Indeed, the average kinetic energy of free particles in a direction x is according to its definition

$$\frac{\overline{mv_x^2}}{2} = \frac{\int\limits_{-\infty}^{\infty} \frac{mv_x^2}{2} \exp\left(-\frac{mv_x^2}{2T}\right) dv_x}{\int\limits_{-\infty}^{\infty} \exp\left(-\frac{mv_x^2}{2T}\right) dv_x} = -\frac{d\ln \int\limits_{-\infty}^{\infty} \exp\left(-\frac{mv_x^2}{2T}\right) dv_x}{d(-1/T)}$$

$$= -\frac{d\ln\,(aT^{1/2})}{d(-1/T)} = \frac{T}{2}, \qquad (1.25)$$

where the bar means an average over particle velocities, and the constant a does not depend on the temperature. Thus, the particle kinetic energy per unit degree of freedom is equal to $T/2$.

Transferring to the three-dimensional case, we take into account the isotropy of particle motion, and the total kinetic energy of a particle is given by

$$\frac{\overline{mv^2}}{2} = \frac{\overline{mv_x^2}}{2} + \frac{\overline{mv_y^2}}{2} + \frac{\overline{mv_z^2}}{2} = \frac{3\,\overline{mv_x^2}}{2} = \frac{3T}{2}. \qquad (1.26)$$

Thus, the average particle kinetic energy in the three-dimensional space is $\overline{mv^2}/2 = 3T/2$. Formulas (1.25) and (1.26) may be used as the temperature definition.

▶ **Problem 1.9** Find the energy distribution function for free particles.

Our task is to rewrite the distribution function (1.22) in the energy space $\varepsilon = mv^2/2$, where we denote it by $f(\varepsilon)$. This distribution function is normalized according to the condition

$$\int_0^\infty f(\varepsilon)\varepsilon^{1/2} d\varepsilon = N, \tag{1.27}$$

where N is the number density of particles. In these terms the distribution function (1.22) takes the form

$$f(\varepsilon) = \frac{2\,N}{\sqrt{\pi} T^{3/2}} \exp\left(-\frac{\varepsilon}{T}\right) \tag{1.28}$$

and is represented in Fig. 1.2.

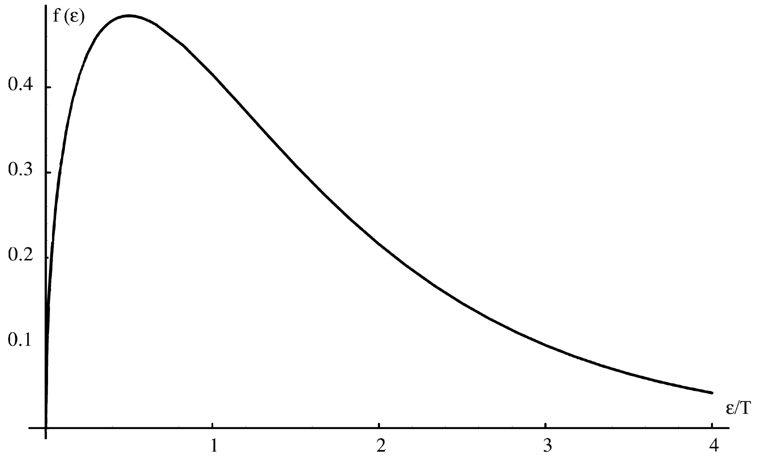

Fig. 1.2 The Maxwell distribution function $f(\varepsilon)$ as a function of the reduced particle energy (ε is the particle energy).

▶ **Problem 1.10** Show that the distribution function for two free particles can be expressed through the distribution function of their relative motion and the center-of-mass motion.

If the particles under consideration belong to two different groups, we have the product of their distribution functions

$$f(\boldsymbol{v}_1)f(\boldsymbol{v}_2)d\boldsymbol{v}_1 d\boldsymbol{v}_2 \,,$$

where $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ are the velocities of the corresponding particles, $f(\boldsymbol{v}_1)f(\boldsymbol{v}_2)$ are their distribution functions, and $d\boldsymbol{v}_1$, $d\boldsymbol{v}_2$ are the elements in the velocity space. Let us introduce the relative velocity $\boldsymbol{g}$ of the particles and the velocity $\boldsymbol{V}$ of their center of mass according to the relations

$$\boldsymbol{g} = \boldsymbol{v}_1 - \boldsymbol{v}_2 \,, \quad \boldsymbol{V} = \frac{m_1 \boldsymbol{v}_1 + m_2 \boldsymbol{v}_2}{m_1 + m_2} \,,$$

where $m_1,\ m_2$ are the masses of these particles. One can see from these formulas that

$$d\boldsymbol{v}_1 d\boldsymbol{v}_2 = d\boldsymbol{g} d\boldsymbol{V} \ .$$

Next, the total kinetic energy of the particles is

$$\frac{m_1}{2} v_1^2 + \frac{m_2}{2} v_2^2 = \frac{\mu g^2}{2} + \frac{M}{2} V^2 \ ,$$

where the reduced mass of the two particles is $\mu = m_1 m_2/(m_1 + m_2)$, and their total mass is $M = m_1 + m_2$. From this we obtain for the Maxwell distribution function $f(\boldsymbol{v})$, which is given by formula (1.21),

$$f(\boldsymbol{v}_1) f(\boldsymbol{v}_2) d\boldsymbol{v}_1 d\boldsymbol{v}_2 = f(\boldsymbol{g}) f(\boldsymbol{V}) d\boldsymbol{g} d\boldsymbol{V}, \tag{1.29}$$

and the relative particle motion is characterized by the reduced mass, whereas the motion of the center of mass is connected with the total particle mass.

1.2
Statistics of Bose–Einstein and Fermi–Dirac

▶ **Problem 1.11** Find the distribution function over states for an ensemble of particles in the case of the Bose–Einstein statistics if any number of particles can be located in one state.

The Bose–Einstein statistics relates to an ensemble of identical particles with a whole spin and permits us to find in the same state two and more particles. We take for this case the probability (1.11), w_i, of the location of a particle in a given state i reducing this formula to one state of this group

$$w_i = \exp\left(\frac{\mu - \varepsilon_i}{T}\right) . \tag{1.30}$$

We introduce here the chemical potential μ, which is determined by the normalization of the distribution function and therefore is expressed through the normalization constant of formula (1.11) as $C = \exp(\mu/T)$.

From this we find, for example, that the probability of the location of m particles in a given state is w_i^m, and therefore the average number of particles in this state is given by

$$\overline{n_i} = \sum_{m=1}^{\infty} m w_i^m = \frac{w_i}{1 - w_i} = \frac{1}{\exp\left(\frac{\varepsilon_i - \mu}{T}\right) - 1} . \tag{1.31}$$

One can derive the Bose–Einstein distribution function (1.31) in the other manner by placing particles over states, as was used for deduction of the Boltzmann distribution (1.11). Indeed, let us find the probability of placing n_i particles in g_i states when the position of one particle does not depend on the positions of others. To this end we take n_i particles and g_i states as elements of the same set and construct sequences from these elements such that the first place is occupied by a state, and other elements are arranged in a random order. Then we assume the number of particles which are found after the corresponding state and before the next ones which belong to this state, and this is the method of placing particles over states. Then the number of ways to obtain different distributions of particles over states is equal to $(g_i + n_i - 1)!$, and among them some are identical which can be obtained by permutation of states or particles. Hence, the total number of ways to distribute particles over states for the Bose–Einstein statistics is

$$p(n_i) = \frac{(n_i + g_i - 1)!}{n_i!(g_i - 1)!} . \tag{1.32}$$

The optimal number of particles $\overline{n_i}$ in a given state corresponds to the maximum of the function $p(n_i)$ or $\ln p(n_i)$. Hence from the condition that the derivative $\frac{d \ln p(n_i)}{dn_i}$ is zero at $n_i = \overline{n_i}$ in the limit $g_i \gg 1,\ n_i \gg 1$ we obtain formula (1.31) for the average number of particles in one state $\overline{n_i}/g_i$.

▶ **Problem 1.12** Find the distribution function over states for an ensemble of particles in the case of the Fermi–Dirac statistics if only one particle can be located in one state.

The Fermi–Dirac statistics relates to particles with half-integer spin and does not permit two particles to be located in the same state. In order to find the distribution function of particles in this case, we place n_i particles over g_i states with the same energy ε_i ($n_i \ll g_i$). It can be done by $p(n_i)$ ways, and the number of such ways is

$$p(n_i) = C_{g_i}^{n_i} = \frac{g_i!}{n_i!(g_i - n_i)!} , \qquad n_i \leq g_i . \tag{1.33}$$

The optimal number of distributions follows from the condition

$$\frac{d \ln p(n_i)}{dn_i} = \ln \frac{n_i}{g_i - n_i} = 0 , \tag{1.34}$$

where we consider the limiting case $g_i \gg 1, n_i \gg 1$. Introducing the optimal number of particles in one state

$$\overline{n_i} = \frac{n_i}{g_i} ,$$

we obtain the average number of particles in one state for the Fermi–Dirac distribution

$$\overline{n_i} = \frac{1}{\exp\left(\frac{\varepsilon_i - \mu}{T}\right) + 1} . \tag{1.35}$$

▶ **Problem 1.13** Find the condition for the transition from the Bose–Einstein and Fermi–Dirac distributions to the Boltzmann distribution.

In the case of the Boltzmann distribution the probability of finding particles' location in each state is small. This holds true if

$$\varepsilon - \mu \gg T . \tag{1.36}$$

Using this criterion in formulas (1.32) and (1.35) for the Bose–Einstein and Fermi-Dirac distributions, we transfer them to the Boltzmann distribution

$$\overline{n_i} = \exp\left(\frac{\mu - \varepsilon_i}{T}\right) . \tag{1.37}$$

This coincides with the Boltzmann distribution (1.10).

▶ **Problem 1.14** Obtain the electron distribution over momenta in a dense electron gas at low temperature (a degenerate electron gas).

The distribution for a dense cold electron gas is governed by the Pauli principle according to which two electrons cannot be located in one state. We determine below the distribution of electrons over momenta p at zero temperature when formula (1.35) in a space of electron momenta takes the form

$$f(p) = f_o\, \eta(p - p_\mathrm{F}) .$$

This formula means that all electron states are occupied until $p \le p_\mathrm{F}$, where p_F is the Fermi momentum. Correspondingly, the maximum electron energy, Fermi energy ε_F, is equal to

$$\varepsilon_\mathrm{F} = \frac{p_\mathrm{F}^2}{2m_\mathrm{e}} .$$

This electron distribution corresponds to the location of electrons inside a ball that is restricted by the Fermi sphere. One can connect the parameters p_F and ε_F of this distribution with the electron density N_e. Indeed, the number of electrons in an element of the phase space is given by

$$n = 2 \int\limits_{p \le p_\mathrm{F}} \frac{d\boldsymbol{p}d\boldsymbol{r}}{(2\pi\hbar)^3} ,$$

where the factor 2 accounts for two directions of the electron spin, and $d\boldsymbol{p}$ and $d\boldsymbol{r}$ are elements of the electron momentum and volume. Taking the electron number density as $N_\mathrm{e} = n/\int d\boldsymbol{r}$, we obtain the relation between the electron number density and the maximum electron momentum and maximum electron energy,

$$p_\mathrm{F} = (3\pi^2\hbar^3 N_\mathrm{e})^{1/3} , \qquad \varepsilon_\mathrm{F} = \frac{p_\mathrm{F}^2}{2m_\mathrm{e}} = \frac{(3\pi^2 N_\mathrm{e})^{2/3}\hbar^2}{2m_\mathrm{e}} . \tag{1.38}$$

Note that the chemical potential of electrons in the Fermi–Dirac formula (1.35) for this distribution is

$$\mu = \varepsilon_\mathrm{F} .$$

▶ **Problem 1.15** Determine the total energy per unit volume of a degenerate electron gas at low temperatures.

At zero temperature the total energy per unit volume of a degenerate electron gas is equal to

$$E_o = \int_0^{\varepsilon_F} \frac{\varepsilon \cdot 2d\mathbf{p}}{(2\pi\hbar)^3} = \frac{2\sqrt{2}}{5\pi^2} \frac{m_e^{3/2} \varepsilon_F^{5/2}}{\hbar^3} . \tag{1.39}$$

At low temperatures the distribution of a degenerate electron gas is determined by the Fermi–Dirac formula (1.35) and is characterized by a small parameter

$$\eta = \frac{T}{\varepsilon_F} . \tag{1.40}$$

We find below the next term of formula (1.39) for the expansion of the total electron energy over the small parameter above.

We consider a general formula for the electron energy per unit volume that at low temperatures has the form

$$E = \int_0^{\infty} \frac{\varepsilon \cdot 2d\mathbf{p}}{(2\pi\hbar)^3} \frac{1}{\exp\left(\frac{\varepsilon-\mu}{T}\right)+1} , \tag{1.41}$$

where we use formula (1.35) for the electron distribution with the chemical potential $\mu = \varepsilon_F$, which corresponds to zero temperature, and the energy of an individual electron is $\varepsilon = p^2/(2m_e)$. Note that under the condition $T \ll \varepsilon_F = \mu$ the integral

$$E - E_o = \frac{m_e^{3/2}\sqrt{2}}{\pi^2\hbar^3} \left(\int_0^{\infty} \varepsilon^{3/2} d\varepsilon \frac{1}{\exp\left(\frac{\varepsilon-\mu}{T}\right)+1} - \int_0^{\varepsilon_F} \varepsilon^{3/2} d\varepsilon \right) \tag{1.42}$$

converges near $\varepsilon = \varepsilon_F$. Introducing a new variable $x = (\varepsilon - \mu)/T$, we transform this expression to the form

$$E - E_o = \frac{m_e^{3/2}\sqrt{2}T^{5/2}}{\pi^2\hbar^3} \left(\int_0^{\infty} \left(x + \frac{\mu}{T}\right)^{3/2} dx \frac{1}{1+\exp x} - \int_{-\mu/T}^{0} \left(x + \frac{\mu}{T}\right)^{3/2} dx \frac{\exp x}{1+\exp x} \right) .$$

Changing the variable in the second integral $x \to -x$ and the lower limit of integration $-\mu/T$ by $-\infty$, we obtain the expansion over a small parameter T/ε_F,

$$E - E_o = \frac{m_e^{3/2}\sqrt{2}T^{5/2}}{\pi^2\hbar^3} \int_0^{\infty} \frac{dx}{1+\exp x} \left[\left(\frac{\mu}{T} + x\right)^{3/2} - \left(\frac{\mu}{T} - x\right)^{3/2} \right]$$

$$= \frac{3m_e^{3/2}\sqrt{2}T^2\sqrt{\mu}}{\pi^2\hbar^3} \int_0^{\infty} \frac{xdx}{1+\exp x} = \frac{m_e^{3/2}T^2\sqrt{\mu}}{\sqrt{2}\hbar^3} ,$$

and the expansion of the total electron energy over a small parameter $\eta = \frac{T}{\varepsilon_F}$ takes the form

$$E = E_o \left(1 + \frac{5\pi^2}{4} \frac{T^2}{\varepsilon_F^2}\right). \tag{1.43}$$

In particular, this gives the heat capacity of a degenerate electron gas per unit volume,

$$C = \frac{dE}{dT} = \frac{5\pi^2}{2} \frac{T}{\varepsilon_F^2} E_o = \frac{m_e^{3/2} T \sqrt{2\varepsilon_F}}{\hbar^3}. \tag{1.44}$$

1.3
Distribution of Particle Density in External Fields

▶ **Problem 1.16** Derive the barometric formula for the distribution of particles in the gravitation field of the Earth.

Let us use the Boltzmann formula (1.12) and use the particle potential energy U in an external field as the particle energy ε_i in this formula. For particles located in the gravitational field of the Earth we have $U = mgh$, where m is the particle mass, g is the free fall acceleration and h is the altitude above the Earth surface. Hence the Boltzmann formula (1.12) takes the form

$$N(h) = N(0) \exp\left(-\frac{mgh}{T}\right), \tag{1.45}$$

where $N(z)$ is the molecule number density at an altitude z. This is the barometric formula.

From this formula it follows that a typical altitude where the number density of particles varies noticeably is $\sim (mg)^{-1}$. In particular, for air molecule we have $mg = 0.11\,\mathrm{km}^{-1}$, which tells us that a significant drop of the atmospheric pressure proceeds at altitudes of several kilometers.

▶ **Problem 1.17** Find the relation between the drift velocity of particles in a gas in a weak external field and the diffusion coefficient of particles in a gas.

If a weak external field acts on particles located in a gas, it causes a flux $\boldsymbol{j}$ of these particles that is proportional to the number density N of these particles and a force $\boldsymbol{F}$ that acts on an individual particle. So, we have

$$\boldsymbol{j} = N\boldsymbol{w} = Nb\boldsymbol{F}, \tag{1.46}$$

where $\boldsymbol{w}$ is the drift velocity of a particle, which is the definition of the mobility b for a neutral particle. For a charged particle in a gas the following mobility definition is used:

$$\boldsymbol{w} = K\boldsymbol{E}, \tag{1.47}$$

where $\boldsymbol{E}$ is the electric field strength and K is the mobility of a charged particle in a gas.

If the distribution of admixture particles in a gas is nonuniform, the diffusion flux $\boldsymbol{j}_{\mathrm{dif}}$ arises,

$$\boldsymbol{j}_{\mathrm{dif}} = -D\nabla N \,, \tag{1.48}$$

which tends to remove the gradient. Here D is the diffusion coefficient for test particles in a gas. When a gas is located in an external field that acts on its particles, a nonuniform distribution of particles occurs. But since it is a stationary distribution, the flux due to the external field is compensated by the diffusion flux, and we have

$$\boldsymbol{j} = Nb\boldsymbol{F} - D\nabla N = 0 \,.$$

Since the test particles are found in thermodynamic equilibrium with the gas, the distribution for the number density of the test particles is given by the Boltzmann formula $N = N_0 \exp(-U/T)$, where U is the potential due to the external field, and T is the temperature of the gas. Using it in the above equation and accounting for the force acting on the test particle is $\boldsymbol{F} = -\nabla U$, we find from the last equation

$$b = \frac{D}{T}. \tag{1.49}$$

This expression is known as the Einstein relation. It is valid for small fields that do not disturb the thermodynamic equilibrium between the test and gaseous particles. For the mobility of a charged particle K in a gas this gives

$$K = \frac{eD}{T} \,. \tag{1.50}$$

▶ **Problem 1.18** Find the character of distributions of positively charged particles located in a weakly ionized gas.

The electric potential of a particle of charge e in vacuum is

$$\varphi = \frac{e}{r}. \tag{1.51}$$

However, if this particle is surrounded by charge particles of a quasineutral plasma, this field is screened since negatively charged particles are attracted to a test particle and positively charged particles are repulsed from it. In order to ascertain the result of this interaction, we analyze the Poisson equation for the field of a positively charged test particle that has the form

$$\Delta\varphi = 4\pi e(N_{\mathrm{e}} - N_{\mathrm{i}}) \,,$$

where we consider a plasma to be consisted of electrons and ions of a charge e, and their number densities are denoted as N_{e} and N_{i}, respectively. The distributions of electrons and ions near a test particle are determined by the Boltzmann formula (1.12) and are given by

$$N_- = N_0 \exp\left(\frac{e\varphi}{T}\right), \quad N_+ = N_0 \exp\left(-\frac{e\varphi}{T}\right), \tag{1.52}$$

where N_0 is the average number density of electrons and ions in the plasma (its average charge equals zero), and T is the temperature of electrons and ions. Substituting this in the Poisson equation, reduce it to the form

$$\Delta\varphi = 8\pi e N_0 \sinh\left(\frac{e\varphi}{T}\right). \tag{1.53}$$

This equation is valid at distances from a test particle where electrons and ions are located, i. e., at distances larger than $N_0^{-1/3}$, while at small distances from a test particle, where the pressure of electrons and ions is negligible on average, the right-hand side of the Poisson equation is zero, and the electric potential of a test particle is given by formula (1.51).

Because of the problem symmetry, the particle electric potential is spherically symmetric on average, which gives at large distances where $e\varphi \ll T$

$$\frac{1}{r}\frac{d^2}{dr^2}(r\varphi) = \frac{8\pi N_0 e^2}{T}\varphi. \tag{1.54}$$

We consider the case when the solution of this equation is substituted into (1.51) at small distances r from a test particle. Then this solution has the form

$$\varphi = \frac{e}{r}\exp\left(-\frac{r}{r_D}\right), \qquad r_D = \sqrt{\frac{T}{8\pi N_0 e^2}}. \tag{1.55}$$

The value r_D is the Debye–Hückel radius that characterizes the character of screening of electric fields in the plasma. The solution obtained is valid if the Debye–Hückel radius for this plasma is large compared to an average distance between the charged particles of the plasma, $N_0^{-1/3}$, which corresponds to the criterion

$$\frac{e^2 N_0^{1/3}}{T} \ll 1. \tag{1.56}$$

If this criterion holds true, many electrons and ions take part in the shielding process that corresponds to the physical nature of this phenomenon. A plasma that satisfies to the criterion (1.56) is the ideal plasma. In this plasma a typical energy of the interaction of charged particles or the interaction energy of two charged particles at an average distance between charged particles, $e^2 N_0^{1/3}$, is small compared to a typical thermal energy of particles ($\sim T$). This criterion tells that the main part of time charged particles of the plasma is free. The same criterion is fulfilled for neutral atomic particles of a gas.

▶ **Problem 1.19** Obtain the expression for the Debye–Hückel radius r_D if the electron T_e and ion T_i temperatures are different.

In this case formula (1.52) for number densities of electrons and ions has the form

$$N_- = N_o \exp\left(\frac{e\varphi}{T_e}\right), \qquad N_+ = N_o \exp\left(-\frac{e\varphi}{T_i}\right) .$$

Repeating the steps of the previous problem, we finally obtain formula (1.55) for the potential of a positively charged test particle, but the expression for the Debye–Hückel radius now takes the form

$$r_D = \sqrt{\frac{\left(\frac{1}{T_e} + \frac{1}{T_i}\right)^{-1}}{4\pi N_o e^2}} . \tag{1.57}$$

▶ **Problem 1.20** A dense plasma propagates in a buffer gas and conserves its quasineutrality, i. e., electrons as more mobile particles come off the ions and create in this way a field that breaks electrons and accelerates ions. As a result, electrons and ions propagate in a buffer gas together and a plasma almost conserves its quasineutrality. Find the diffusion coefficient for this plasma in a buffer gas.

One more general plasma property relates to the character of its propagation in a neutral gas. Electrons as light particles move faster in a gas than ions, which violates the plasma quasineutrality, and electric fields are generated. Only these fields make the plasma almost quasineutral. As a result, the plasma propagates in a buffer gas as a whole, and we consider below such a process.

In the regime under consideration, when electrons and ions propagate in a buffer gas and the mean free paths of electrons and ions in the gas are relatively small, we have the following expressions for the electron flux $\boldsymbol{j}_e$ and the ion flux $\boldsymbol{j}_i$,

$$\boldsymbol{j}_e = -D_e \nabla N_e - K_e \boldsymbol{E} N_e \ ; \ \boldsymbol{j}_i = -D_i \nabla N_i + K_i \boldsymbol{E} N_i .$$

Here N_e, N_i are the number densities of electrons and ions, respectively, D_e, D_i are their diffusion coefficients, and K_e, K_i are their mobilities. Because the electric field acts on electrons and ions in opposite directions, the field enters into the flux expressions with different signs. The electric field strength $\boldsymbol{E}$ satisfies the Poisson equation

$$\operatorname{div} \boldsymbol{E} = 4\pi e(N_i - N_e) .$$

In the case of propagation of a dense plasma, the plasma converges to quasineutrality during evolution, i. e., the charge difference is small $\Delta N = |N_i - N_e| \ll N_e$, which gives $N_e \approx N_i \approx N$. Hence the plasma motion is self-consistent, and we have $\boldsymbol{j}_e \approx \boldsymbol{j}_i$. Next, because of the higher mobility of electrons, the electron flux is zero $\boldsymbol{j}_e = 0$ on the scale of electron quantities. This means $D_e \nabla N_e \gg j_i$ and $eEK_e N_e \gg j_i$, so that in terms of the magnitudes of electron parameters we have $\boldsymbol{j}_e = 0$. This gives for the electric field strength that arises due to plasma motion as a single whole,

$$\boldsymbol{E} = -\frac{D_e}{eK_e}\frac{\nabla N}{N} .$$

From this we have for the flux of charged particles

$$\boldsymbol{j} = \boldsymbol{j}_i = -\left(D_i + D_e\frac{K_i}{K_e}\right)\nabla N = -D_a\nabla N .$$

In this way we define D_a, the coefficient of ambipolar diffusion. Thus the plasma evolution has a diffusive character with a self-consistent diffusion coefficient. In particular, when electrons and ions are found in thermodynamic equilibrium that allows us to define the electron T_e and ion T_i temperatures for each subsystem on the basis of the Einstein relation (1.50), we have for the diffusion coefficient of a collective plasma motion

$$D_a = D_i\left(1 + \frac{T_e}{T_i}\right) . \tag{1.58}$$

One can see that in the regime under consideration a plasma propagates in a buffer gas with the speed of the ions rather than that of the electrons.

In this regime of plasma motion, the plasma remains almost quasineutral, so the relation $\Delta N = |N_i - N_e| \ll N$ holds true. Then the Poisson equation gives $\Delta N \sim E/(4\pi eL)$, where L is a typical plasma dimension. The above equation for the electric field strength that follows from $j_e = 0$ together with the Einstein relation (1.50) gives $E \sim T/(e^2 L)$, where for simplicity we assume the electron and ion temperatures to be equal. From this we have $\Delta N \sim Nr_D^2/L^2$. Thus the criterion of the regime of ambipolar diffusion corresponds to the plasma criterion $L \gg r_D$.

► **Problem 1.21** Find the distribution function over the potential energies for ions in a quasineutral ideal plasma.

In considering a quasineutral ideal plasma, we assume that neutral atomic particles (atoms or molecules) whose number density exceeds that of electrons and ions provide the stability of this plasma. But because they do not influence the electric properties of such a plasma, they will not be considered below.

An electric potential of this plasma is determined by charged particles, and according to formula (1.55) the average potential energy for a test ion with other electrons and ions is equal to

$$\overline{U} = e\overline{\varphi} = \int_0^\infty \frac{e^2}{r}\exp\left(-\frac{r}{r_D}\right)\left[N_o\exp\left(-\frac{e\varphi}{T}\right) - N_o\exp\left(\frac{e\varphi}{T}\right)\right]4\pi r^2 dr$$
$$= \frac{e^2}{4r_D} , \qquad e\varphi \ll T . \tag{1.59}$$

As is seen, for an ideal plasma that submits to the criterion (1.56) the average potential energy of ions or electrons is small compared to its thermal kinetic energy

$$\overline{U} \ll T .$$

In addition, this average potential energy is identical for positively charged ions and electrons.

In the same manner we have for the mean square of the potential energy for an ion or electron

$$\begin{aligned}\overline{U^2} &= \int_0^\infty \frac{e^4}{r^2} \exp\left(-\frac{2r}{r_\mathrm{D}}\right) \left[N_\mathrm{o} \exp\left(-\frac{e\varphi}{T}\right) + N_\mathrm{o} \exp\left(\frac{e\varphi}{T}\right)\right] 4\pi r^2 dr \\ &= 4\pi N_\mathrm{o} e^4 r_\mathrm{D} = \frac{T}{2}\frac{e^2}{r_\mathrm{D}}, \qquad e\varphi \ll T\,. \end{aligned} \tag{1.60}$$

One can see that this value is small compared to the square of the thermal energy,

$$\frac{\overline{U^2}}{T^2} \sim \frac{e^2}{r_\mathrm{D} T} \ll 1\,,$$

for an ideal plasma, but is large compared to the square of the average potential energy,

$$\frac{\overline{U^2}}{\left(\overline{U}\right)^2} = 16\pi N_\mathrm{o} r_\mathrm{D}^3 \gg 1\,.$$

The last relation allows us to neglect the divergence of the average ion potential energy from zero, and the distribution function over ion potential energies has the form of the Gauss distribution

$$f(U)dU = \frac{1}{\sqrt{2\pi\Delta U^2}} \exp\left[-\frac{U^2}{2\Delta U^2}\right]\,,$$

where $\Delta U = \sqrt{\overline{U^2}/2}$ is the fluctuation of the ion potential energy.

Substituting in this formula the average square of the ion potential energy (1.60), we find the distribution over ion potential energies,

$$f(U)dU = \frac{1}{\sqrt{2\pi\Delta U^2}} \exp\left(-\frac{U^2}{2\Delta U^2}\right) = \frac{dU}{2\pi e^2\sqrt{2N_\mathrm{o} r_\mathrm{D}}} \exp\left(-\frac{U^2}{8\pi N_\mathrm{o} e^4}\right)\,. \tag{1.61}$$

This value $f(U)dU$ is the probability that the potential energy for a test ion and an electron located inside a plasma ranges from U up to $U + dU$.

1.4
Laws of Black Body Radiation

▶ **Problem 1.22** Determine the distribution of thermal photons over frequencies.

Let us consider thermal or black body radiation that is characterized by a temperature T. Such a radiation is located inside a vessel with the wall temperature T, and an equilibrium for photons of different frequencies there results from absorption and radiation of the walls. The number of photons is not fixed, and the relative probability that n photons of an energy $\hbar\omega$ are found in a given state according to the Boltzmann formula (1.11) is equal to $\exp(-n\hbar\omega/T)$. This gives the average number of photons $\overline{n_\omega}$ in this state,

$$\overline{n_\omega} = \frac{\sum\limits_n n\exp(-\frac{\hbar\omega n}{T})}{\sum\limits_n \exp(-\frac{\hbar\omega n}{T})} = \frac{1}{\exp(\hbar\omega/T) - 1}. \tag{1.62}$$

This is the Planck formula. As is seen, it corresponds to the Bose–Einstein distribution (1.31) with zero chemical potential. Thus, black body radiation as an ensemble of photons is characterized by zero chemical potential.

▶ **Problem 1.23** Determine the spectral density of black body radiation, i. e., the energy of this radiation per unit time, unit volume, and unit frequency.

On the basis of the definition of the spectral density of radiation, the radiation energy per unit time and unit volume with a range of frequencies from ω up to $\omega + d\omega$ is $\Omega U_\omega d\omega$. On the other hand, on the basis of the statistical weight of continuous spectrum (1.19), one can represent this value as $2\hbar\omega n_\omega \Omega d\boldsymbol{k}/(2\pi)^3$, where $\boldsymbol{k}$ is the photon wave number, $d\boldsymbol{k}/(2\pi)^3$ is the number of states per unit volume and for a given element of the wave numbers, the factor 2 accounts for two polarizations of an electromagnetic wave, and n_ω is the number of photons located in one state. We take into account that the electromagnetic wave is the transversal one, i. e., its electric field strength $\boldsymbol{E}$ is directed perpendicular to the propagation direction that is determined by the vector $\boldsymbol{k}$. On the basis of the dispersion relation for photons $\omega = ck$ between the photon frequency ω and its wave number k (c is the light velocity), we obtain the Planck radiation formula

$$U_\omega = \frac{\hbar\omega^3}{\pi^2 c^3} n_\omega. \tag{1.63}$$

On the basis of formula (1.62) one can rewrite formula (1.63) in the form

$$U_\omega = \frac{\hbar\omega^3}{\pi^2 c^3 \left[\exp(\hbar\omega/T) - 1\right]} . \tag{1.64}$$

Let us consider the limiting case of this formula. In the case of small frequencies $\hbar\omega \ll T$ this formula is converted into the Rayleigh–Jeans formula

$$U_\omega = \frac{\omega^2 T}{\pi^2 c^3} , \quad \hbar\omega \ll T . \tag{1.65}$$

Because this formula corresponds to the classical limit, it does not contain the Planck constant.

In the other limiting case of high frequencies $\hbar\omega \gg T$, the above formula is transformed to the Wien formula

$$U_\omega = \frac{\hbar\omega^3}{\pi^2 c^3} \exp\left(-\frac{\hbar\omega}{T}\right) . \tag{1.66}$$