Michael T. Vaughn

Introduction to Mathematical Physics

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Preface

Mathematics is an essential ingredient in the education of a professional physicist, indeed in the education of any professional scientist or engineer in the 21st century. Yet when it comes to the specifics of what is needed, and when and how it should be taught, there is no broad consensus among educators. The crowded curricula of undergraduates, especially in North America where broad general education requirements are the rule, leave little room for formal mathematics beyond the standard introductory courses in calculus, linear algebra, and differential equations, with perhaps one advanced specialized course in a mathematics department, or a one-semester survey course in a physics department.

The situation in (post)-graduate education is perhaps more encouraging—there are many institutes of theoretical physics, in some cases joined with applied mathematics, where modern courses in mathematical physics are taught. Even in large university physics departments there is room to teach advanced mathematical physics courses, even if only as electives for students specializing in theoretical physics. But in small and medium physics departments, the teaching of mathematical physics often is restricted to a one-semester survey course that can do little more than cover the gaps in the mathematical preparation of its graduate students, leaving many important topics to be discussed, if at all, in the standard physics courses in classical and quantum mechanics, and electromagnetic theory, to the detriment of the physics content of those courses.

The purpose of the present book is to provide a comprehensive survey of the mathematics underlying theoretical physics at the level of graduate students entering research, with enough depth to allow a student to read introductions to the higher level mathematics relevant to specialized fields such as the statistical physics of lattice models, complex dynamical systems, or string theory. It is also intended to serve the research scientist or engineer who needs a quick refresher course in the subject of one or more chapters in the book.

We review the standard theories of ordinary differential equations, linear vector spaces, functions of a complex variable, partial differential equations and Green functions, and the special functions that arise from the solutions of the standard partial differential equations of physics. Beyond that, we introduce at an early stage modern topics in differential geometry arising from the study of differentiable manifolds, spaces whose points are characterized by smoothly varying coordinates, emphasizing the properties of these manifolds that are independent of a particular choice of coordinates. The geometrical concepts that follow lead to helpful insights into topics ranging from thermodynamics to classical dynamical systems to Einstein's classical theory of gravity (general relativity). The usefulness of these ideas is, in my opinion, as significant as the clarity added to Maxwell's equations by the use of vector notation in place of the original expressions in terms of individual components, for example.

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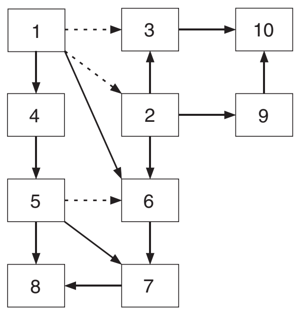
Thus I believe that it is important to introduce students of science to geometrical methods as early as possible in their education.

The material in Chapters 1–8 can form the basis of a one-semester graduate course on mathematical methods, omitting some of the mathematical details in the discussion of Hilbert spaces in Chapters 6 and 7 if necessary. There are many examples interspersed with the main discussion, and exercises that the student should work out as part of the reading. There are additional problems at the end of each chapter; these are generally more challenging, but provide possible homework assignments for a course. The remaining two chapters introduce the theory of finite groups and Lie groups—topics that are important for the understanding of systems with symmetry, especially in the realm of condensed matter, atoms, nuclei, and subnuclear physics. But these topics can often be developed as needed in the study of particular systems, and are thus less essential in a first course. Nevertheless, they have been included in part because of my own research interests, and in part because group theory can be fun!

Each chapter begins with an overview that summarizes the topics discussed in the chapter—the student should read this through in order to get an idea of what is coming in the chapter, without being too concerned with the details that will be developed later. The examples and exercises are intended to be studied together with the material as it is presented. The problems at the end of the chapter are either more difficult, or require integration of more than one local idea. The diagram at the right provides a flow chart for the chapters of the book.



Flow chart for chapters of the book.

I would like to thank many people for their encouragement and advice during the long course of this work. Ron Aaron, George Alverson, Tom Kephart, and Henry Smith have read significant parts of the manuscript and contributed many helpful suggestions. Tony Devaney and Tom Taylor have used parts of the book in their courses and provided useful feedback. Peter Kahn reviewed an early version of the manuscript and made several important comments. Of course none of these people are responsible for any shortcomings of the book.

I have benefited from many interesting discussions over the years with colleagues and friends on mathematical topics. In addition to the people previously mentioned, I recall especially Ken Barnes, Haim Goldberg, Marie Machacek, Jeff Mandula, Bob Markiewicz, Pran Nath, Richard Slansky, K C Wali, P K Williams, Ian Jack, Tim Jones, Brian Wybourne, and my thesis adviser, David C Peaslee.

Michael T Vaughn

Boston, Massachusetts
October 2006

1 Infinite Sequences and Series

In experimental science and engineering, as well as in everyday life, we deal with integers, or at most rational numbers. Yet in theoretical analysis, we use real and complex numbers, as well as far more abstract mathematical constructs, fully expecting that this analysis will eventually provide useful models of natural phenomena. Hence we proceed through the construction of the real and complex numbers starting from the positive integers[1]. Understanding this construction will help the reader appreciate many basic ideas of analysis.

We start with the positive integers and zero, and introduce negative integers to allow subtraction of integers. Then we introduce *rational numbers* to permit division by integers. From arithmetic we proceed to analysis, which begins with the concept of *convergence* of infinite sequences of (rational) numbers, as defined here by the Cauchy criterion. Then we define *irrational numbers* as limits of convergent (Cauchy) sequences of rational numbers.

In order to solve algebraic equations in general, we must introduce *complex numbers* and the representation of complex numbers as points in the *complex plane*. The fundamental theorem of algebra states that every polynomial has at least one root in the complex plane, from which it follows that every polynomial of degree n has exactly n roots in the complex plane when these roots are suitably counted. We leave the proof of this theorem until we study functions of a complex variable at length in Chapter 4.

Once we understand convergence of infinite sequences, we can deal with infinite series of the form

$$\sum_{n=1}^{\infty} x_n$$

and the closely related infinite products of the form

$$\prod_{n=1}^{\infty} x_n$$

Infinite series are central to the study of solutions, both exact and approximate, to the differential equations that arise in every branch of physics. Many functions that arise in physics are defined only through infinite series, and it is important to understand the convergence properties of these series, both for theoretical analysis and for approximate evaluation of the functions.

[1]To paraphrase a remark attributed to Leopold Kronecker: "God created the positive integers; all the rest is human invention."

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We review some of the standard tests (comparison test, ratio test, root test, integral test) for convergence of infinite series, and give some illustrative examples. We note that absolute convergence of an infinite series is necessary and sufficient to allow the terms of a series to be rearranged arbitrarily without changing the sum of the series.

Infinite sequences of functions have more subtle convergence properties. In addition to pointwise convergence of the sequence of values of the functions taken at a single point, there is a concept of *uniform convergence* on an interval of the real axis, or in a region of the complex plane. Uniform convergence guarantees that properties such as continuity and differentiability of the functions in the sequence are shared by the limit function. There is also a concept of *weak convergence*, defined in terms of the sequences of numbers generated by integrating each function of the sequence over a region with functions from a class of smooth functions (*test functions*). For example, the Dirac δ-function and its derivatives are defined in terms of weakly convergent sequences of well-behaved functions.

It is a short step from sequences of functions to consider infinite series of functions, especially *power series* of the form

$$\sum_{n=0}^{\infty} a_n z^n$$

in which the a_n are real or complex numbers and z is a complex variable. These series are central to the theory of functions of a complex variable. We show that a power series converges absolutely and uniformly inside a circle in the complex plane (the *circle of convergence*), with convergence *on* the circle of convergence an issue that must be decided separately for each particular series.

Even divergent series can be useful. We show some examples that illustrate the idea of a *semiconvergent*, or *asymptotic*, series. These can be used to determine the asymptotic behavior and approximate asymptotic values of a function, even though the series is actually divergent. We give a general description of the properties of such series, and explain Laplace's method for finding an asymptotic expansion of a function defined by an integral representation (Laplace integral) of the form

$$I(z) = \int_0^a f(t) e^{zh(t)}\, dt$$

Beyond the sequences and series generated by the mathematical functions that occur in solutions to differential equations of physics, there are sequences generated by dynamical systems themselves through the equations of motion of the system. These sequences can be viewed as *iterated maps* of the coordinate space of the system into itself; they arise in classical mechanics, for example, as successive intersections of a particle orbit with a fixed plane. They also arise naturally in population dynamics as a sequence of population counts at periodic intervals.

The asymptotic behavior of these sequences exhibits new phenomena beyond the simple convergence or divergence familiar from previous studies. In particular, there are sequences that converge, not to a single limit, but to a periodic limit cycle, or that diverge in such a way that the points in the sequence are dense in some region in a coordinate space.

An elementary prototype of such a sequence is the *logistic map* defined by

$$T_\lambda : \; x \to x_\lambda = \lambda x(1-x)$$

This map generates a sequence of points $\{x_n\}$ with

$$x_{n+1} = \lambda x_n(1-x_n)$$

($0 < \lambda < 4$) starting from a generic point x_0 in the interval $0 < x_0 < 1$. The behavior of this sequence as a function of the parameter λ as λ increases from 0 to 4 provides a simple illustration of the phenomena of *period doubling* and transition to *chaos* that have been an important focus of research in the past 30 years or so.

1.1 Real and Complex Numbers

1.1.1 Arithmetic

The construction of the real and complex number systems starting from the positive integers illustrates several of the structures studied extensively by mathematicians. The positive integers have the property that we can add, or we can multiply, two of them together and get a third. Each of these operations is *commutative*:

$$x \circ y = y \circ x \tag{1.1}$$

and *associative*:

$$x \circ (y \circ z) = (x \circ y) \circ z \tag{1.2}$$

(here $\circ$ denotes either addition or multiplication), but only for multiplication is there an *identity* element $\mathbf{e}$, with the property that

$$\mathbf{e} \circ x = x = x \circ \mathbf{e} \tag{1.3}$$

Of course the identity element for addition is the number zero, but zero is not a positive integer. Properties (1.2) and (1.3) are enough to characterize the positive integers as a *semigroup* under multiplication, denoted by $\mathbf{Z}_*$ or, with the inclusion of zero, a semigroup under addition, denoted by $\mathbf{Z}_+$.

Neither addition nor multiplication has an inverse defined within the positive integers. In order to define an inverse for addition, it is necessary to include zero and the negative integers. *Zero* is defined as the identity for addition, so that

$$x + 0 = x = 0 + x \tag{1.4}$$

and the *negative* integer $-x$ is defined as the *inverse* of x under addition,

$$x + (-x) = 0 = (-x) + x \tag{1.5}$$

With the inclusion of the negative integers, the equation

$$p + x = q \tag{1.6}$$

has a unique integer solution x ($\equiv q - p$) for every pair of integers p, q. Properties (1.2)–(1.5) characterize the integers as a *group* **Z** under addition, with 0 as an identity element. The fact that addition is commutative makes **Z** a *commutative*, or *Abelian*, group. The combined operations of addition with zero as identity, and multiplication satisfying Eqs. (1.2) and (1.3) with 1 as identity, characterize **Z** as a *ring*, a *commutative* ring since multiplication is also commutative. To proceed further, we need an inverse for multiplication, which leads to the introduction of *fractions* of the form p/q (with integers p, q). One important property of fractions is that they can always be reduced to a form in which the integers p, q have no common factors[2]. Numbers of this form are *rational*. With both addition and multiplication having well-defined inverses (except for division by zero, which is undefined), and the *distributive* law

$$a * (x + y) = a * x + a * c = y \tag{1.7}$$

satisfied, the rational numbers form a *field*, denoted by **Q**.

➔ **Exercise 1.1.** Let p be a prime number. Then $\sqrt{p}$ is not rational. □

Note. Here and throughout the book we use the convention that when a proposition is simply stated, the problem is to prove it, or to give a counterexample that shows it is false.

1.1.2 Algebraic Equations

The rational numbers are adequate for the usual operations of arithmetic, but to solve algebraic (polynomial) equations, or to carry out the limiting operations of calculus, we need more. For example, the quadratic equation

$$x^2 - 2 = 0 \tag{1.8}$$

has no rational solution, yet it makes sense to enlarge the rational number system to include the roots of this equation. The real *algebraic* numbers are introduced as the real roots of polynomials of any degree with integer coefficients. The algebraic numbers also form a field.

➔ **Exercise 1.2.** Show that the roots of a polynomial with rational coefficients can be expressed as roots of a polynomial with integer coefficients. □

Complex numbers are introduced in order to solve algebraic equations that would otherwise have no real roots. For example, the equation

$$x^2 + 1 = 0 \tag{1.9}$$

has no real solutions; it is "solved" by introducing the imaginary unit $i \equiv \sqrt{-1}$ so that the roots are given by $x = \pm i$. Complex numbers are then introduced as ordered pairs $(x, y) \sim$

[2]The study of properties of the positive integers, and their factorization into products of *prime* numbers, belongs to a fascinating branch of pure mathematics known as *number theory*, in which the reducibility of fractions is one of the elementary results.

$x + iy$, of real numbers; x, y can be restricted to be rational (algebraic) to define the complex rational (algebraic) numbers.

Complex numbers can be represented as points (x, y) in a plane (the *complex plane*) in a natural way, and the *magnitude* of the complex number $x + iy$ is defined by

$$|x + iy| \equiv \sqrt{x^2 + y^2} \tag{1.10}$$

In view of the identity

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{1.11}$$

we can also write

$$x + iy = re^{i\theta} \tag{1.12}$$

with $r = |x + iy|$ and $\tan\theta = y/x$. These relations have an obvious interpretation in terms of the polar coordinates of the point (x, y). We also define

$$\arg\, z \equiv \theta \tag{1.13}$$

for $z \neq 0$. The angle $\arg\, z$ is the *phase* of z. Evidently it can only be defined as $\bmod\ 2\pi$; adding any integer multiple of 2π to $\arg\, z$ does not change the complex number z, since

$$e^{2\pi i} = 1 \tag{1.14}$$

Equation (1.14) is one of the most remarkable equations of mathematics.

1.1.3 Infinite Sequences; Irrational Numbers

To complete the construction of the real and complex numbers, we need to look at some elementary properties of sequences, starting with the formal definitions:

Definition 1.1. A *sequence* of numbers (real or complex) is an ordered set of numbers in one-to-one correspondence with the positive integers; write $\{z_n\} \equiv \{z_1, z_2, \ldots\}$. ∎

Definition 1.2. The sequence $\{z_n\}$ is *bounded* if there is some positive number M such that $|z_n| < M$ for all positive integers n. ∎

Definition 1.3. The sequence $\{x_n\}$ of real numbers is *increasing* (*decreasing*) if $x_{n+1} > x_n$ ($x_{n+1} < x_n$) for every n. The sequence is *nondecreasing* (*nonincreasing*) if $x_{n+1} \geq x_n$ ($x_{n+1} \leq x_n$) for every n. A sequence belonging to one of these classes is *monotone* (or *monotonic*). ∎

Remark. The preceding definition is restricted to real numbers because it is only for real numbers that we can define a “natural” ordering that is compatible with the standard measure of the distance between the numbers. □

Definition 1.4. The sequence $\{z_n\}$ is a *Cauchy sequence* if for every $\varepsilon > 0$ there is a positive integer N such that $|z_p - z_q| < \varepsilon$ whenever $p, q > N$. ∎

Definition 1.5. The sequence $\{z_n\}$ is *convergent* to the *limit* z (write $\{z_n\} \to z$) if for every $\varepsilon > 0$ there is a positive integer N such that $|z_n - z| < \varepsilon$ whenever $n > N$. ∎

There is no guarantee that a Cauchy sequence of rational numbers converges to a rational, or even algebraic, limit. For example, the sequence $\{x_n\}$ defined by

$$x_n \equiv \left(1 + \frac{1}{n}\right)^n \tag{1.15}$$

converges to the limit $e = 2.71828\ldots$, the base of natural logarithms. It is true, though nontrivial to prove, that e is not an algebraic number. A real number that is not algebraic is *transcendental*. Another famous transcendental number is π, which is related to e through Eq. (1.14).

If we want to insure that every Cauchy sequence of rational numbers converges to a limit, we must include the *irrational numbers*, which can be *defined* as limits of Cauchy sequences of rational numbers. As examples of such sequences, imagine the infinite, nonterminating, nonperiodic decimal expansions of transcendental numbers such as e or π, or algebraic numbers such as $\sqrt{2}$. Countless computer cycles have been used in calculating the digits in these expansions.

The set of *real* numbers, denoted by $\mathbf{R}$, can now be defined as the set containing rational numbers together with the limits of Cauchy sequences of rational numbers. The set of *complex* numbers, denoted by $\mathbf{C}$, is then introduced as the set of all ordered pairs $(x, y) \sim x + iy$ of real numbers. Once we know that every Cauchy sequence of real (or rational) numbers converges to a real number, it is a simple exercise to show that every Cauchy sequence of complex numbers converges to a complex number.

Monotonic sequences are especially important, since they appear as partial sums of infinite series of positive terms. The key property is contained in the

Theorem 1.1. A monotonic sequence $\{x_n\}$ is convergent if and only if it is bounded.

Proof. If the sequence is unbounded, it will diverge to $\pm\infty$, which simply means that for any positive number M, no matter how large, there is an integer N such that $x_n > M$ (or $x_n < -M$ if the sequence is monotonic nonincreasing) for any $n \geq N$. This is true, since for any positive number M, there is at least one member x_N of the sequence with $x_N > M$ (or $x_N < -M$)—otherwise M would be a bound for the sequence—and hence $x_n > M$ (or $x_n < -M$) for any $n \geq N$ in view of the monotonic nature of the sequence. ■

If the monotonic nondecreasing sequence $\{x_n\}$ is bounded from above, then in order to have a limit, there must be a bound that is smaller than any other bound (such a bound is the *least upper bound* of the sequence). If the sequence has a limit X, then X is certainly the least upper bound of the sequence, while if a least upper bound $\overline{X}$ exists, then it must be the limit of the sequence. For if there is some $\varepsilon > 0$ such that $\overline{X} - x_n > \varepsilon$ for all n, then $\overline{X} - \varepsilon$ will be an upper bound to the sequence smaller than $\overline{X}$.

The existence of a least upper bound is intuitively plausible, but its existence cannot be proven from the concepts we have introduced so far. There are alternative axiomatic formulations of the real number system that guarantee the existence of the least upper bound; the convergence of any bounded monotonic nondecreasing sequence is then a consequence as just explained. The same argument applies to bounded monotonic nonincreasing sequences, which must then have a *greatest lower bound* to which the sequence converges.

1.1.4 Sets of Real and Complex Numbers

We also need some elementary definitions and results about sets of real and complex numbers that are generalized later to other structures.

Definition 1.6. For real numbers, we can define an *open* interval:

$$(a, b) \equiv \{x|\, a < x < b\}$$

or a *closed* interval:

$$[a, b] \equiv \{x|\, a \leq x \leq b\}$$

as well as *semiopen* (or *semiclosed*) intervals:

$$(a, b] \equiv \{x|\, a < x \leq b\} \quad \text{and} \quad [a, b) \equiv \{x|\, a \leq x < b\}$$

A *neighborhood* of the real number x_0 is any open interval containing x_0. An ε-*neighborhood* of x_0 is the set of all points x such that

$$|x - x_0| < \varepsilon \tag{1.16}$$

This concept has an obvious extension to complex numbers: An (ε)-*neighborhood* of the complex number z_0, denoted by $N_\varepsilon(z_0)$, is the set of all points z such that

$$0 < |z - z_0| < \varepsilon \tag{1.17}$$

Note that for complex numbers, we exclude the point z_0 from the neighborhood $N_\varepsilon(z_0)$. ∎

Definition 1.7. The set $\mathcal{S}$ of real or complex numbers is *open* if for every x in $\mathcal{S}$, there is a neighborhood of x lying entirely in $\mathcal{S}$. $\mathcal{S}$ is *closed* if its complement is open. $\mathcal{S}$ is *bounded* if there is some positive M such that $x < M$ for every x in $\mathcal{S}$ (M is then a *bound* of $\mathcal{S}$). ∎

Definition 1.8. x is a *limit point* of the set $\mathcal{S}$ if every neighborhood of x contains at least one point of $\mathcal{S}$. ∎

While x itself need not be a member of the set $\mathcal{S}$, this definition implies that every neighborhood of x in fact contains an infinite number of points of $\mathcal{S}$. An alternative definition of a closed set can be given in terms of limit points, and one of the important results of analysis is that every bounded infinite set contains at least one limit point.

➔ **Exercise 1.3.** Show that the set $\mathcal{S}$ of real or complex numbers is closed if and only if every limit point of $\mathcal{S}$ is an element of $\mathcal{S}$. □

➔ **Exercise 1.4.** (Bolzano–Weierstrass theorem) Every bounded infinite set of real or complex numbers contains at least one limit point. □

Definition 1.9. The set $\mathcal{S}$ is *everywhere dense*, or simply *dense*, in a region $\mathcal{R}$ if there is at least one point of $\mathcal{S}$ in any neighborhood of every point in $\mathcal{R}$. ∎

❑ **Example 1.1.** The set of rational numbers is everywhere dense on the real axis. ∎

1.2 Convergence of Infinite Series and Products

1.2.1 Convergence and Divergence; Absolute Convergence

If $\{z_k\}$ is a sequence of numbers (real or complex), the formal sum

$$S \equiv \sum_{k=1}^{\infty} z_k \tag{1.18}$$

is an *infinite series*, whose *partial sums* are defined by

$$s_n \equiv \sum_{k=1}^{n} z_k \tag{1.19}$$

The series $\sum z_k$ is *convergent* (to the *value* s) if the sequence $\{s_n\}$ of partial sums converges to s, otherwise *divergent*. The series is *absolutely convergent* if the series $\sum |z_k|$ is convergent; a series that is convergent but not absolutely convergent is *conditionally convergent*. Absolute convergence is an important property of a series, since it allows us to rearrange terms of the series without altering its value, while the sum of a conditionally convergent series can be changed by reordering it (this is proved later on).

➔ **Exercise 1.5.** If the series $\sum z_k$ is convergent, then the sequence $\{z_k\} \to 0$. □

➔ **Exercise 1.6.** If the series $\sum z_k$ is absolutely convergent, then it is convergent. □

To study absolute convergence, we need only consider a series $\sum x_k$ of positive real numbers ($\sum |z_k|$ is such a series). The sequence of partial sums of a series of positive real numbers is obviously nondecreasing. From the theorem on monotonic sequences in the previous section then follows

Theorem 1.2. The series $\sum x_k$ of positive real numbers is convergent if and only if the sequence of its partial sums is bounded.

❑ **Example 1.2.** Consider the *geometric series*

$$S(x) \equiv \sum_{k=0}^{\infty} x^k \tag{1.20}$$

for which the partial sums are given by

$$s_n = \sum_{k=0}^{n} x^k = \frac{1-x}{1-x^{n+1}} \tag{1.21}$$

These partial sums are bounded if $0 \le x < 1$, in which case

$$\{s_n\} \;\to\; \frac{1}{1-x} \tag{1.22}$$

The series diverges for $x \geq 1$. The corresponding series

$$S(z) \equiv \sum_{k=0}^{\infty} z^k \tag{1.23}$$

for complex z is then absolutely convergent for $|z| < 1$, divergent for $|z| > 1$. The behavior on the *unit circle* $|z| = 1$ in the complex plane must be determined separately (the series actually diverges everywhere on the circle since the sequence $\{z^k\} \not\to 0$; see Exercise 1.5). ∎

Remark. We will see that the function $S(z)$ defined by the series (1.23) for $|z| < 1$ can be defined to be $1/(1-z)$ for complex $z \neq 1$, even outside the region of convergence of the series, using the properties of $S(z)$ as a function of the complex variable z. This is an example of a procedure known as *analytic continuation*, to be explained in Chapter 4. □

❑ **Example 1.3.** The *Riemann ζ-function* is defined by

$$\zeta(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1.24}$$

The series for $\zeta(s)$ with $s = \sigma + i\tau$ is absolutely convergent if and only if the series for $\zeta(\sigma)$ is convergent. Denote the partial sums of the latter series by

$$s_N(\sigma) = \sum_{n=1}^{N} \frac{1}{n^\sigma} \tag{1.25}$$

Then for $\sigma \leq 1$ and $N \geq 2^m$ (m integer), we have

$$s_N(\sigma) \geq s_N(1) \geq s_{2^m}(1) > s_{2^{m-1}}(1) + \frac{1}{2} > \cdots > \frac{m}{2} \tag{1.26}$$

Hence the sequence $\{s_N(\sigma)\}$ is unbounded and the series diverges. Note that for $s = 1$, Eq. (1.24) is the *harmonic series*, which is shown to diverge in elementary calculus courses. On the other hand, for $\sigma > 1$ and $N \leq 2^m$ with m integer, we have

$$\begin{aligned} s_N(\sigma) &< s_{2^m}(\sigma) < s_{2^{m-1}}(\sigma) + \left(\frac{1}{2}\right)^{(m-1)\,(\sigma-1)} < \cdots \\ &< \sum_{k=0}^{m-1} \left(\frac{1}{2}\right)^{k(\sigma-1)} < \frac{1}{1-2^{(1-\sigma)}} \end{aligned} \tag{1.27}$$

Thus the sequence $\{s_N(\sigma)\}$ is bounded and hence converges, so that the series (1.24) for $\zeta(s)$ is absolutely convergent for $\sigma = \text{Re } s > 1$. Again, we will see in Chapter 4 that $\zeta(s)$ can be defined for complex s beyond the range of convergence of the series (1.24) by analytic continuation. ∎

1.2.2 Tests for Convergence of an Infinite Series of Positive Terms

There are several standard tests for convergence of a series of positive terms:

Comparison test. Let $\sum x_k$ and $\sum y_k$ be two series of positive numbers, and suppose that for some integer $N > 0$ we have $y_k \leq x_k$ for all $k > N$. Then

(i) if $\sum x_k$ is convergent, $\sum y_k$ is also convergent, and

(ii) if $\sum y_k$ is divergent, $\sum x_k$ is also divergent.

This is fairly obvious, but to give a formal proof, let $\{s_n\}$ and $\{t_n\}$ denote the sequences of partial sums of $\sum x_k$ and $\sum y_k$, respectively. If $y_k \leq x_k$ for all $k > N$, then

$$t_n - t_N \leq s_n - s_N$$

for all $n > N$. Thus if $\{s_n\}$ is bounded, then $\{t_n\}$ is bounded, and if $\{t_n\}$ is unbounded, then $\{s_n\}$ is unbounded.

Remark. The comparison test has been used implicitly in the discussion of the ζ-function to show the absolute convergence of the series 1.24 for $\sigma = \mathrm{Re}\, s > 1$. □

Ratio test. Let $\sum x_k$ be a series of positive numbers, and let $r_k \equiv x_{k+1}/x_k$ be the ratios of successive terms. Then

(i) if only a finite number of $r_k > a$ for some a with $0 < a < 1$, then the series converges, and

(ii) if only a finite number of $r_k < 1$, then the series diverges.

In case (i), only a finite number of the r_k are larger than a, so there is some positive M such that $x_k < Ma^k$ for all k, and the series converges by comparison with the geometric series. In case (ii), the series diverges since the individual terms of the series do not tend to zero.

Remark. The ratio test works if the largest limit point of the sequence $\{r_k\}$ is either greater than 1 or smaller than 1. If the largest limit point is exactly equal to 1, then the ratio test does not answer the question of convergence, as seen by the example of the ζ-function series (1.24). □

Root test. Let $\sum x_k$ be a series of positive numbers, and let $\varrho_k \equiv \sqrt[k]{x_k}$. Then

(i) if only a finite number of $\varrho_k > a$ for some positive $a < 1$, then the series converges, and

(ii) if infinitely many $\varrho_k > 1$, the series diverges.

As with the ratio test, we can construct a comparison with the geometric series. In case (i), only a finite number of roots ϱ_k are bigger than a, so there is some positive M such that $x_k < Ma^k$ for all k, and the series converges by comparison with the geometric series. In case (ii), the series diverges since the individual terms of the series do not tend to zero.

Remark. The root test, like the ratio test, works if the largest limit point of the sequence $\{\varrho_k\}$ is either greater than 1 or smaller than 1, but fails to decide convergence if the largest limit point is exactly equal to 1. □

Integral test. Let $f(t)$ be a continuous, positive, and nonincreasing function for $t \geq 1$, and let $x_k \equiv f(k)$ $(k = 1, 2, \ldots)$. Then $\sum x_k$ converges if and only if the integral

$$I \equiv \int_1^\infty f(t)\, dt < \infty \tag{1.28}$$

also converges. To show this, note that

$$\int_{k}^{k+1} f(t)\,dt \le x_k \le \int_{k-1}^{k} f(t)\,dt \tag{1.29}$$

which is easy to see by drawing a graph. The partial sums s_n of the series then satisfy

$$\int_{1}^{n+1} f(t)\,dt \le s_n = \sum_{k=1}^{n} x_k \le x_1 + \int_{1}^{n} f(t)\,dt \tag{1.30}$$

and are bounded if and only if the integral (1.28) converges.

Remark. If the integral (1.28) converges, it provides a (very) rough estimate of the value of the infinite series, since

$$\int_{N+1}^{\infty} f(t)\,dt \;\le\; s - s_N \;=\; \sum_{k=N+1}^{\infty} x_k \;\le\; \int_{N}^{\infty} f(t)\,dt \tag{1.31}$$

1.2.3 Alternating Series and Rearrangements

In addition to a series of positive terms, we consider an *alternating series* of the form

$$S \equiv \sum_{k=0}^{\infty} (-1)^k x_k \tag{1.32}$$

with $x_k > 0$ for all k. Here there is a simple criterion (due to Leibnitz) for convergence: if the sequence $\{x_k\}$ is nonincreasing, then the series S converges if and only if $\{x_k\} \to 0$, and if S converges, its value lies between any two successive partial sums. This follows from the observation that for any n the partial sums s_n of the series (1.32) satisfy

$$s_{2n+1} < s_{2n+3} < \cdots < s_{2n+2} < s_{2n} \tag{1.33}$$

❑ **Example 1.4.** The alternating harmonic series

$$A \equiv 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \tag{1.34}$$

is convergent according to this criterion, even though it is not absolutely convergent (the series of absolute values is the harmonic series we have just seen to be divergent). In fact, evaluating the logarithmic series (Eq. (1.69) below) for $z = 1$ shows that $A = \ln 2$. ∎

Is there any significance of the ordering of terms in an infinite series? The short answer is that terms can be rearranged at will in an absolutely convergent series without changing the value of the sum, while changing the order of terms in a conditionally convergent series can change its value, or even make it diverge.

Definition 1.10. If $\{n_1, n_2, \ldots\}$ is a permutation of $\{1, 2, \ldots\}$, then the sequence $\{\zeta_k\}$ is a *rearrangement* of $\{z_k\}$ if

$$\zeta_k = z_{n_k} \tag{1.35}$$

for every k. Then also the series $\sum \zeta_k$ is a rearrangement of $\sum z_k$. ∎

❑ **Example 1.5.** The alternating harmonic series (1.34) can be rearranged in the form

$$A' = \left(1 + \frac{1}{3} - \frac{1}{2}\right) + \left(\frac{1}{5} + \frac{1}{7} - \frac{1}{4}\right) + \cdots \tag{1.36}$$

which is still a convergent series, but its value is not the same as that of A (see below). ∎

Theorem 1.3. If the series $\sum z_k$ is absolutely convergent, and $\sum \zeta_k$ is a rearrangement of $\sum z_k$, then $\sum \zeta_k$ is absolutely convergent.

Proof. Let $\{s_n\}$ and $\{\sigma_n\}$ denote the sequences of partial sums of $\sum z_k$ and $\sum \zeta_k$, respectively. If $\varepsilon > 0$, choose N such that $|s_n - s_m| < \varepsilon$ for all $n, m > N$, and let $Q \equiv \max\{n_1, \ldots, n_N\}$. Then $|\sigma_n - \sigma_m| < \varepsilon$ for all $n, m > Q$. ■

On the other hand, if a series in not absolutely convergent, then its value can be changed (almost at will) by rearrangement of its terms. For example, the alternating series in its original form (1.34) can be expressed as

$$A = \sum_{n=0}^{\infty} \left(\frac{1}{2n+1} - \frac{1}{2n+2}\right) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+2)} \tag{1.37}$$

This is an absolutely convergent series of positive terms whose value is $\ln 2 = 0.693\ldots$, as already noted. On the other hand, the rearranged series (1.36) can be expressed as

$$A' = \sum_{n=0}^{\infty} \left(\frac{1}{4n+1} + \frac{1}{4n+3} - \frac{1}{2n+2}\right) = \sum_{n=0}^{\infty} \frac{8n+5}{2(n+1)(4n+1)(4n+3)} \tag{1.38}$$

which is another absolutely convergent series of positive terms. Including just the first term of this series shows that

$$A' > \frac{5}{6} > \ln 2 = A \tag{1.39}$$

In fact, any series that is not absolutely convergent can be rearranged into a divergent series.

Theorem 1.4. If the series $\sum x_k$ of real terms is conditionally convergent, then there is a divergent rearrangement of $\sum x_k$.

Proof. Let $\{\xi_1, \xi_2, \ldots\}$ be the sequence of positive terms in $\{x_k\}$, and $\{-\eta_1, -\eta_2, \ldots\}$ be the sequence of negative terms. Then at least one of the series $\sum \xi_k$, $\sum \eta_k$ is divergent (otherwise the series would be absolutely convergent). Suppose $\sum \xi_k$ is divergent. Then we can choose a sequence $n_1, n_2, \ldots$ such that

$$\sum_{k=n_m}^{n_{m+1}-1} \xi_k > 1 + \eta_m \tag{1.40}$$

($m = 1, 2, \ldots$), and the rearranged series

$$S' \equiv \sum_{k=n_1}^{n_2-1} \xi_k - \eta_1 + \sum_{k=n_2}^{n_3-1} \xi_k - \eta_2 + \cdots \tag{1.41}$$

is divergent. ■

Remark. It follows as well that a conditionally convergent series $\sum z_k$ of complex terms must have a divergent rearrangement. For if $z_k = x_k + iy_k$, then either $\sum x_k$ or $\sum y_k$ is conditionally convergent, and hence has a divergent rearrangement. □

1.2.4 Infinite Products

Closely related to infinite series are *infinite products* of the form

$$\prod_{m=1}^{\infty} (1 + z_m) \tag{1.42}$$

($\{z_m\}$ is a sequence of complex numbers), with *partial products*

$$p_n \equiv \prod_{m=1}^{n} (1 + z_k) \tag{1.43}$$

The product $\prod (1 + z_m)$ is *convergent* (to the *value* p) if the sequence $\{p_n\}$ of partial products converges to $p \neq 0$, *convergent to zero* if a finite number of factors are 0, *divergent to zero* if $\{p_n\} \to 0$ with no vanishing p_n, and *divergent* if $\{p_n\}$ is divergent. The product is *absolutely convergent* if $\prod (1 + |z_m|)$ is convergent; a product that is convergent but not absolutely convergent is *conditionally convergent*.

The absolute convergence of a product is simply related to the absolute convergence of a related series: if $\{x_m\}$ is a sequence of positive real numbers, then the product $\prod (1 + x_m)$ is convergent if and only if the series $\sum x_m$ is convergent. This follows directly from the observation

$$\sum_{m=1}^{n} x_m < \prod_{m=1}^{n} (1 + x_m) < \exp\left(\sum_{m=1}^{n} x_m\right) \tag{1.44}$$

Also, the product $\prod (1 - x_m)$ is convergent if and only if the series $\sum x_m$ is convergent (show this).

❑ **Example 1.6.** Consider the infinite product

$$P \equiv \prod_{m=2}^{\infty} \left(\frac{m^3 - 1}{m^3 + 1}\right) < \prod_{m=2}^{\infty} \left(1 - \frac{1}{m^3}\right) \tag{1.45}$$

The product is (absolutely) convergent, since the series

$$\sum_{m=1}^{\infty} \frac{1}{m^3} = \zeta(3)$$

is convergent. Evaluation of the product is left as a problem. ■

1.3 Sequences and Series of Functions

1.3.1 Pointwise Convergence and Uniform Convergence of Sequences of Functions

Questions of convergence of sequences and series of functions in some domain of variables can be answered at each point by the methods of the preceding section. However, the issues of continuity and differentiability of the limit function require more care, since the limiting procedures involved approaching a point in the domain need not be interchangeable with passing to the limit of the sequence or series (convergence of an infinite series of functions is defined in the usual way in terms of the convergence of the sequence of partial sums of the series). Thus we introduce

Definition 1.11. The sequence $\{f_n(z)\}$ of functions of the variable z (real or complex) is (*pointwise*) *convergent* to the function $f(z)$ in the region $\mathcal{R}$:

$$\{f_n(z)\} \to f(z) \text{ in } \mathcal{S}$$

if the sequence $\{f_n(z_0)\} \to f(z_0)$ at every point z_0 in $\mathcal{R}$.

Definition 1.12. $\{f_n(z)\}$ is *uniformly convergent* to $f(z)$ in the closed, bounded $\mathcal{R}$:

$$\{f_n(z)\} \Rightarrow f(z) \text{ in } \mathcal{S}$$

if for every $\varepsilon > 0$ there is a positive integer N such that $|f_n(z) - f(z)| < \varepsilon$ for every $n > N$ and *every* point z in $\mathcal{R}$. ∎

Remark. Note the use of different arrow symbols ($\to$ and $\Rightarrow$) to denote strong and uniform convergence, as well as the symbol ($\rightharpoonup$) introduced below to denote weak convergence. □

❑ **Example 1.7.** Consider the sequence $\{x^n\}$. Evidently $\{x^n\} \to 0$ for $0 \leq x < 1$. Also, the sequence $\{x^n\} \Rightarrow 0$ on any closed interval $0 \leq x \leq 1 - \delta$ ($0 < \delta < 1$), since for any such x, we have $|x^n| < \varepsilon$ for all $n > N$ if N is chosen so that $|1-\delta|^N < \varepsilon$. However, we cannot say that the sequence is uniformly convergent on the open interval $0 < x < 1$, since if $0 < \varepsilon < 1$ and n is any positive integer, we can find some x in (0, 1) such that $x^n > \varepsilon$. The point here is that to discuss uniform convergence, we need to consider a region that is closed and bounded, with no limit point at which the series is divergent. ∎

It is one of the standard theorems of advanced calculus that properties of continuity of the elements of a uniformly convergent sequence are shared by the limit of the sequence. Thus if $\{f_n(z)\} \Rightarrow f(z)$ in the region $\mathcal{R}$, and if each of the $f_n(z)$ is continuous in the closed bounded region $\mathcal{R}$, then the limit function $f(z)$ is also continuous in $\mathcal{R}$. Differentiability requires a separate check that the sequence of derivative functions $\{f_n'(z)\}$ is convergent, since it may not be. If the sequence of derivatives actually is uniformly convergent, then it converges to the derivative of the limit function $f(z)$.

❑ **Example 1.8.** Consider the function $f(z)$ defined by the series

$$f(z) \equiv \sum_{n=1}^{\infty} \frac{1}{n^2} \sin n^2 \pi z \tag{1.46}$$