MULTIBODY DYNAMICS WITH UNILATERAL CONTACTS

FRIEDRICH PFEIFFER

CHRISTOPH CLOCKER

Wiley-VCH Verlag GmbH & Co. KGaA

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Everything should be made as simple as possible
but not simpler.

Albert Einstein

PREFACE

Machines and mechanisms continually become more complex and more perfect and, thus, are consistently accompanied by more mathematical modeling and simulation. Sophisticated machines require sophisticated methods, which, nevertheless, must relate to reality. Theories for rigid or elastic multibody systems and FEM/BEM-algorithms are typical examples which have considerably influenced progress in mechanical engineering.

Machines and mechanisms are systems of interconnected bodies in which the interconnections are often modeled by applying bilateral constraints. Although these models are sometimes correct, they often are not. Noise-generating mechanisms, fatigue, and wear in many cases are caused from relative motion between two bodies that is usually characterized by impulsive and stick-slip phenomena. Because machines are multibody configurations with many dependent contacts, adequate theories are required. Moreover, many technical systems rely on impulsive and stick-slip processes to perform their intended functions. The same type of theory is needed here as well. In this book we consider multibody systems with multiple dependent contacts and develop an adequate theory. In spite of the fact that the theory covers a huge, and still growing, number of applications, it was not available in a form accessible to engineers. This book tries to fill this gap. Our intention is not to give another version of multibody system theory, but focus on multibody systems with multiple, unilateral, and, often, undecoupled contacts.

The credit for establishing the mathematical foundation for nonsmooth mechanics belongs to a few European colleagues, especially Professor Moreau in Montpellier and Professor Panagiotopoulos in Thessaloniki. Their theories are mainly based on convex analysis and on accompanying fields such as linear and quadratic programming in optimization theory.

The Lehrstuhl B fur Mechanik (LBM) in Munich originally began in the 1980s with a series of practical problems, but then became more involved with the mathematical foundations. This book is the result of ten years' work with many dissertations and practical contributions concerning dependent contacts in multibody systems. The theory in Part 1 has reached a state which allows the treatment of very general problems of nonsmooth dynamics. The new ideas with respect to impacts with friction have been confirmed by many experiments, although additional research is necessary to improve the model. The level of confidence in the theory is very high. The number of successful industry applications, presented in Part **2,** confirms the relevancy of our modeling approach, which turns out to be quite general, including many classical methods as special cases. Its significance is increasing so quickly that we easily could fill an additional volume with sophisticated applications.

We have to thank many co-workers, associates, and friends for supporting us in writing this book. Dipl.-Ing. Markus Wosle supervised the printing and the computer generation of the figures. He also evaluated some of the examples and did some proofreading, as did Dipl.-Ing. Jurgen Braun. We are particularly indebted to Professor Ali Nayfeh of Virginia Tech, who invited us to contribute this book to his series on nonlinear dynamics. Many thanks are due to our editors at Wiley for their friendly assistance and cooperation. We apologize to those whose work was inadvertently omitted in the literature. We welcome all comments and corrections from readers.

München, im November 1995

Friedrich Pfeiffer
Christoph Glocker

CONTENTS

PART 2: Applications

PART 1

THEORY

1

INTRODUCTION

1.1 Modeling Mechanical Systems

Modeling mechanical systems such as machines and mechanisms is a matter of engineering intuition and of the relevant powers of practical imagination. Models of any area of mechanics, and of all physical fundamental subjects, necessarily include assumptions, usually in an approximate form: moreover, the realization of models is often accompanied by numerical problems. In technical applications models are limited by the data situation. Establishing a complete data set for a machine might take more time than creating a model.

The first step in considering models must be a very clear and precise elaboration of the goals of the model. Is it for simulating an object, or do I want to establish a plant model for control design, a parameter model for design improvements possibly in connection with optimization processes, or a system model replacing extensive laboratory and field tests? The different requirements will result in different model approaches. In any case the chances of establishing a good model depend very strongly on a deep understanding of the physical-technical processes of the object to be modeled. A good model means a good representation of mechanical properties and therefore a good correspondence to practice and its measurements.

One word on experiments and their modeling character: With the exception of direct field tests with complete machines or transportation systems, most experiments, even in industry, are models including all the properties mentioned. Therefore, measurements are not a dogma, but researchers must know what equipment was used, what sensors were employed, how they were applied, where they were used, how signal processing was performed, and so on. Good measurements are as rare as good theoretical models. But, on the other hand, an optimized combination of experiments and theory might accelerate considerably progress in research and development with respect to a problem. This seems to be noteworthy, although it is not the topic of this book.

Good models are economical models; they include everything to achieve the goals, but not more. For an example, the size of multibody models representing vibrational systems depends on the largest frequencies of interest. These frequencies also indicate if some bodies must be modeled as elastic bodies. Grabbing of the clutch in cars, for instance, usually is observed in a frequency range of 6–15 Hz. From this it is sufficient to represent that process by a **3-4** mass configuration and a realistic stick-slip model. Anything more would be not economical.

In this book we establish a unique theory on multibody systems with multiple contacts. Mechanically we deal with arrangements of an arbitrary number of rigid or elastic bodies which possess, in addition to their continuous constraints being represented by steady constraint equations, an arbitrary number of unilateral contacts characterized by noncontinuous constraint equations. Multibody systems with impact- and friction-driven processes are a typical example. To model systems of that kind we may think about quite a number of possibilities.

First, we might leave the concept of multibody theory and model all bodies and all couplings of a machine by a FEM system, which industry really does. This results in extremely large models that can be very helpful when correctly applied. Computing times will be very large, the correctness of meshsizing is not always ensured and the numerical results need interpretation. Existing FEM codes cannot deal with unilateral problems correctly.

As a second variant we may return to our multibody system approach but model all joints, linear and nonlinear couplings, and all contacts in a more detailed way, taking into consideration, for example, local deformation effects, including local nonlinear behavior. Again we would have large computing times, and, as in all cases, we must verify our local coupling models by experiments.

In the following we start with a multibody approach including arbitrary continuous joints and couplings, the last represented by any type of steady force law. With respect to unilateral contacts we shall consider impacts and friction and a combination of both. Classical contact laws are applied throughout but with a specific adaptation to multiple–contact situations. As we shall see, this leads to a formulation allowing for an application of a linear complementarity algorithm which can be interpreted as a modified form of the well-known simplex algorithm. Bringing the equations of motion into such a form requires that, at the very instant of change from static to sliding or from sliding to static friction, the coefficients of static and sliding friction be equal. This is neither a loss of generality nor a violation of our physical understanding of technical processes, for the following reasons. In technical applications we apply friction characteristics in the form of a friction force F_F as a function of relative velocity v_{rel} or as a friction torque dependent on the relative angular velocity (Fig. 1.1). Typically, such curves start at zero rela-

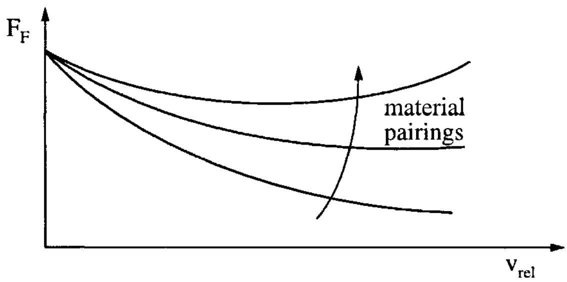

Figure 1.1: Friction Characteristics (Stribeck curves)

tive velocity with a negative slope, which, by the way, is the main reason for self-excited oscillations in frictional systems. Therefore, at the very instant of a change from stiction to sliding, or vice versa, the friction force (torque) and, thus, the friction coefficients remain approximately the same, justifying the above requirement. It excludes only a jump, which in no way is a good approximation to reality. For sliding contacts, of course, any frictional force law may be applied.

1.2 Single-Contact Dynamics

All classical textbooks on mechanics and most current research concentrate on mechanical systems with only one or two degrees of freedom and with one impulsive or frictional contact. Books and papers on chaotical properties very often use as mechanical examples impact or stick-slip systems. In the following we review the basic ideas [5,27, 35,43,49,53,59].

Two bodies will impact if their relative distance becomes zero. This event is then a starting point for a process, which usually is assumed to have an extremely short duration. Nevertheless, deformation of the two bodies occurs, being composed of compression and expansion phases (Fig. 1.2). The forces governing this deformation depend on the initial dynamics and kinematics of the contacting bodies. The impulsive process ends when the normal force of contact vanishes and changes sign. The condition of zero relative distance cannot be used as an indicator for the end of an impact.

In the general case of impact with friction we must also consider a possible change from sliding to sticking, or vice versa, which includes frictional aspects as treated later.

In the simple case of only normal velocities we sometimes can idealize impacts according to Newton's impact laws, which relate the relative velocity after an impact with that before an impact. Such an idealization can only be performed if the force budget allows it. In the case of impacts by hard

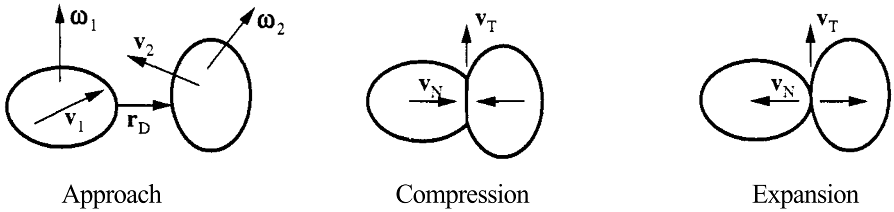

Figure 1.2: Details of an Impact

loaded bodies we must analyze the deformation in detail. Gear hammering taking place under heavy loads and gear rattling taking place under no load are typical examples.

As in all other contact dynamical problems, impacts possess complementarity properties. For ideal classical inelastic impacts either the relative velocity is zero and the accompanying normal constraint impulse is not zero, or vice versa. The scalar product of relative velocity and normal impulse is thus always zero. For the more complicated case of an impact with friction we shall find such a complementarity in each phase of the impact (Chapter 8). Friction in one contact only is characterized by a contact condition of vanishing relative distance and by two frictional conditions, either sliding or sticking (Fig. **1.3)**.

From the contact constraint $\boldsymbol{r}_D = 0$ we get a normal constraint force $\boldsymbol{F}_N$ which, according to Coulomb's laws, is proportional to the friction forces. For sliding $\boldsymbol{F}_{TS} = -\mu \boldsymbol{F}_N \operatorname{sgn}(\boldsymbol{v}_{\text{rel}})$, and for stiction $\boldsymbol{F}_{T0} = \mu_0 \boldsymbol{F}_N$, where μ and μ_0 are the coefficient of sliding and static friction, respectively. Stiction is indicated by $\boldsymbol{v}_{\text{rel}} = 0$ and by a surplus of the static friction force over the constraint force; i.e., $\mu_0|\boldsymbol{F}_N| - |\boldsymbol{F}_{TC}| \geq 0$. If this friction saturation becomes zero the stiction situation will end and sliding will start again with a nonzero relative acceleration $\boldsymbol{a}_{\text{rel}}$. Again we find here complementary behavior: Either the relative velocity (acceleration) is zero and the friction saturation is not zero, or vice versa. The product of relative acceleration and friction saturation is always zero.

1.3 Multiple-Contact Dynamics

We consider a multibody system with n bodies and f degrees of freedom. In addition we have n_G unilateral contacts where impacts and friction may occur. Each contact event is indicated by some indicator function — for example, the beginning by a relative distance or a relative velocity and the end by a relevant constraint force condition. The constraint equation itself is always a kinematical relationship. If a constraint is active it generates a constraint force; if it is passive no constraint force appears.

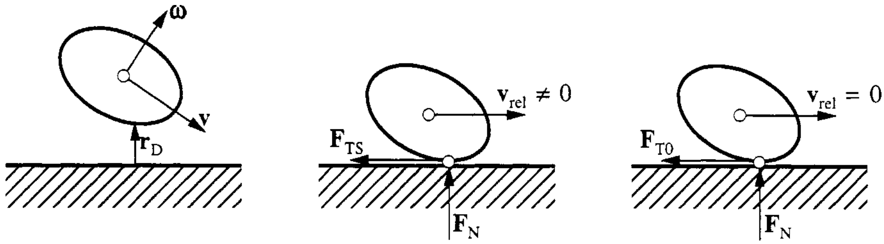

Figure 1.3: Sliding and Static Friction

In multibody systems with multiple contacts these contacts may be decoupled by springs or any other force law, or they may not. In the last case a change of the contact situation in only one contact results in a modified contact situation in the other contacts. If we characterize these situations by the combination of all active and passive constraint equations in all existing contacts, we get a combinatorial problem of considerable extent by any change in the unilateral and coupled contacts. Let us consider this problem in more detail.

Figure 1.4 shows ten masses which may stick or slide on each other. The little mass tower is excited by a periodically vibrating table. Gravity forces and friction forces act on each mass,and each mass can move to the left with v^-, to the right with v^+, or not move at all. Each type of motion is connected with some passive or active constraint situation. Combining all ten masses, each of which has three possibilities of motion, results in $3^{10} = 59{,}049$ possible combinations of constraints. But only one is the correct constraint configuration. To find this one configuration is a crucial task of combinatorial search or an elegant way of applying the complementarity idea. We shall focus on this way.

As pointed out all contact dynamical problems possess complementarity properties [34, 40, 41, 50, 60, 61, 67, 68]. For any unilateral contact the relative kinematics is zero and some constraint forces are not zero, or vice versa. The scalar product of magnitudes representing relative kinematics and constraint forces is always zero. This property possesses the character of a basic law in unilateral dynamics, the application of which makes multiple-contact problems solvable. Introducing these considerations into the equations of motion and into the active set of constraint equations allows a reduction of these equations to a standard complementarity problem, which is closely related to linear programming problems. The basic idea consists of the property that the complementary behavior of unilateral contact problems reduces the solution space for the constraint magnitudes drastically. Usually a unique solution can be found, and the combinatorial problem is solved.

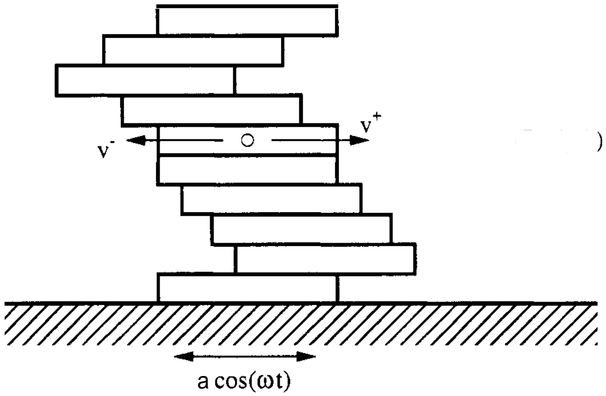

Figure 1.4: A Combinatorial Problem

Changes of the contact situation, and thus the constraint configuration, depend on the evolution of the state and, therefore, on the motion itself. They generate a discontinuously varying structure of the equations of motion. Such systems are often called systems with time-variant structure or with time-variant topology. It is a typical property of all mechanical systems with impacts and friction in unilateral contacts.

2

MULTIBODY KINEMATICS

Kinematics is geometry of motion. Applied to multibody systems it describes the linear and angular positions of all bodies within the system and provides methods for calculating their velocities and accelerations. It also takes into account the directions of unconstrained and constrained motion which might occur when bodies are linked together by certain joints.

2.1 Geometry and Definitions

Multibody kinematics requires a precise and unique definition of coordinate frames and the transformations between them [11, 75]. In the following we shall use the inertial base $\mathbf{I}$, the body-fixed base B or B_i and some arbitrary reference frame R or R_i for convenience (Fig. 2.1). We say that a vector $\boldsymbol{v} \in \mathrm{V}$ is a component of vector space V . It can be represented in any of the mentioned coordinate frames.

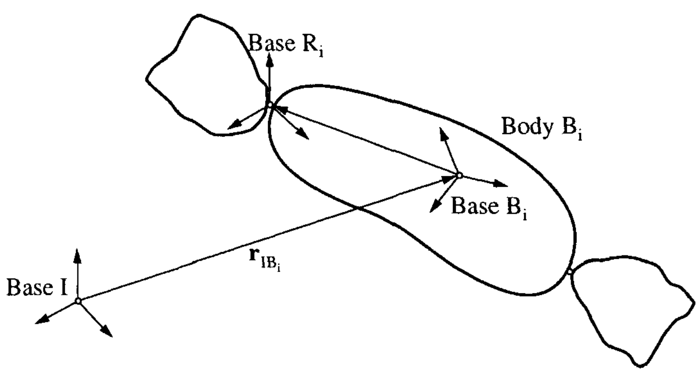

Figure 2.1: Coordinate Frames

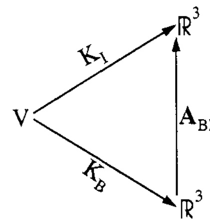

Figure 2.2: Transformation Triangle

For such a representation we apply the convention

$$\begin{aligned} K_B(\boldsymbol{v}) &:= {}_B\boldsymbol{v} \in \mathbb{R}^3 , \\ K_I(\boldsymbol{v}) &:= {}_I\boldsymbol{v} \in \mathbb{R}^3 , \\ K_R(\boldsymbol{v}) &:= {}_R\boldsymbol{v} \in \mathbb{R}^3 , \end{aligned} \tag{2.1}$$

which says that the components of the vector v are written in the coordinate frames B ,I, R, respectively. Furthermore, we define the composition

$$K_I = \boldsymbol{A}_{IB} \circ K_B , \tag{2.2}$$

which has to be applied to any of such transformations in an adequate form (Fig. **2.2**) [83]. Figure **2.2** nicely gives a direct geometrical interpretation of $\boldsymbol{A}_{BI}\boldsymbol{A}_{IB} = \boldsymbol{E}$ resulting from eq. **(2.2)** and $K_B = \boldsymbol{A}_{BI} \circ K_I$. We may derive this important result following another route.

Figure **2.3** shows two reference frames B and R and an arbitrary vector v with given coordinates with respect to frame R, ${}_R\boldsymbol{v} = ({}_Rv_x, {}_Rv_y, {}_Rv_z)^T$. In order to get its coordinates ${}_B\boldsymbol{v} = ({}_Bv_x, {}_Bv_y, {}_Bv_z)^T$ we only have to write it as a linear combination of the basis vectors of R, but using frame B instead. Hence,

$$\begin{aligned} {}_B\boldsymbol{v} &= {}_B\boldsymbol{e}_x^R \, {}_Rv_x + {}_B\boldsymbol{e}_y^R \, {}_Rv_y + {}_B\boldsymbol{e}_z^R \, {}_Rv_z \\ &= \left({}_B\boldsymbol{e}_x^R, {}_B\boldsymbol{e}_y^R, {}_B\boldsymbol{e}_z^R\right) \begin{pmatrix} {}_Rv_x \\ {}_Rv_y \\ {}_Rv_z \end{pmatrix} \\ &= \boldsymbol{A}_{BR} \cdot {}_R\boldsymbol{v} , \quad \boldsymbol{A}_{BR} = \left({}_B\boldsymbol{e}_x^R, {}_B\boldsymbol{e}_y^R, {}_B\boldsymbol{e}_z^R\right) , \end{aligned} \tag{2.3}$$

which expresses the well-known fact that the transformation matrix $\boldsymbol{A}_{BR}$ from R to B is composed of the unit basis vectors of frame R written down in frame B. From eq. **(2.3)** we get immediately

$$\boldsymbol{A}_{BR}^T \boldsymbol{A}_{BR} = \boldsymbol{E} , \quad \boldsymbol{A}_{BR}^T = \boldsymbol{A}_{BR}^{-1} = \boldsymbol{A}_{RB} . \tag{2.4}$$

The evaluation of the transformation matrices $\boldsymbol{A}_{IB}$ or $\boldsymbol{A}_{BI}$ follows well-known standard methods of rigid body kinematics. To rotate a coordinate frame B

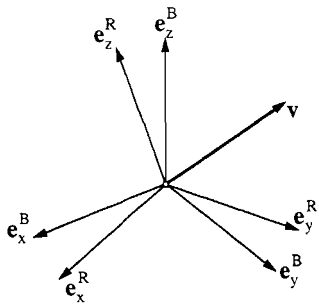

Figure 2.3: Transformation of Vectors

into a frame R we may use Euler or Kardan angles or any other set of angles which is convenient for our problem (Fig. **2.4**).

Every mapping can be composed of elementary rotations around some known axes. In the example of Kardan angles the three elementary rotations are given by

$$\boldsymbol{A}_\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} ;$$

$$\boldsymbol{A}_\beta = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} ; \qquad (2.5)$$

$$\boldsymbol{A}_\gamma = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

From this the complete transformation from B to R is simply

$$\boldsymbol{A}_{RB} = \boldsymbol{A}_\gamma \boldsymbol{A}_\beta \boldsymbol{A}_\alpha ,$$

where the transformation sequence

$$B \xrightarrow{\boldsymbol{A}_\alpha} B_1 \xrightarrow{\boldsymbol{A}_\beta} B_2 \xrightarrow{\boldsymbol{A}_\gamma} R$$

consisting of only elementary rotations has been used. The frames B_1 and B_2 are intermediate systems which result from the first two elementary rotations. In the case of Euler angles the evaluation is similar and leads, with respect to Fig. **2.4**, to an overall transformation matrix $\boldsymbol{A}_{RB} = \boldsymbol{A}_\gamma(\psi)\boldsymbol{A}_\alpha(\vartheta)\boldsymbol{A}_\gamma(\varphi)$.

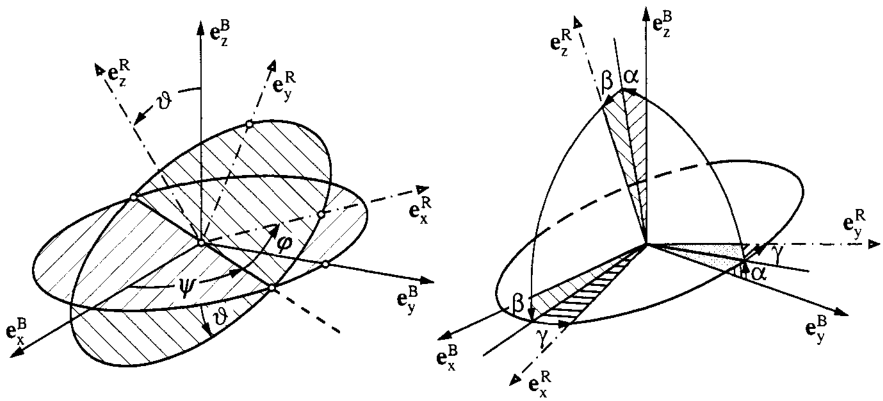

Figure 2.4: Euler and Kardan Angles [55]

From the equality $\boldsymbol{A}_{RB} = \boldsymbol{A}_{\gamma}\boldsymbol{A}_{\beta}\boldsymbol{A}_{\alpha}$ and the structure of the elementary rotations we immediately get $\det(\boldsymbol{A}_{RB}) = 1$, which is a general characteristic feature of any rotational mapping. For practical calculations these are helpful formulas, where in many cases the structure of existing machines and mechanisms allows a simple connection of the components by one degree of freedom only and thus by only one elementary rotation.

An important process in considering multibody kinematics consists of the evaluation of many successive coordinate frames (Fig. 2.1), which has been used in the composition of the transformation matrix $\boldsymbol{A}_{RB}$.
With respect to Fig. *2.5* we get, for example,

$$
\begin{aligned}
{}_I\boldsymbol{v} &= \boldsymbol{A}_{IB} \cdot {}_B\boldsymbol{v}\,, \\
{}_R\boldsymbol{v} &= \boldsymbol{A}_{RB} \cdot {}_B\boldsymbol{v}\,, \\
{}_I\boldsymbol{v} &= \boldsymbol{A}_{IR} \cdot {}_R\boldsymbol{v} \;=\; \boldsymbol{A}_{IR}\boldsymbol{A}_{RB}\,{}_B\boldsymbol{v}\,.
\end{aligned}
$$

Comparing the first and last equations results in the important relationship

$$\boldsymbol{A}_{IB} = \boldsymbol{A}_{IR} \cdot \boldsymbol{A}_{RB}, \tag{2.6}$$

which says that the transformation matrix from B to I can be composed by the transformation matrix from B to R and by the one from R to I.

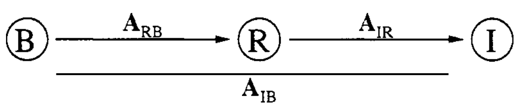

Figure 2.5: Successive Coordinate Frames

2.2 Time Derivations

One tedious task in multibody kinematics is the evaluation of the above transformations. The other tedious task consists of determining all velocities and accelerations in any of the chosen coordinate frames, mostly in the inertial and body-fixed frames. Therefore, a crucial factor in establishing multibody kinematics is time derivatives with respect to moving coordinate systems.

Let us start again with some vector $\boldsymbol{r} \in \mathrm{V}$ as a component of a vector space V. Applying strictly the definitions of eq. (2.1), we remember that $K_B(\dot{\boldsymbol{r}}) := {}_B(\dot{\boldsymbol{r}})$ means that the components of the time derivative r are given in frame B. In contrast, $({}_B\boldsymbol{r})^{\bullet}$ denotes the time derivatives of the components of a vector $\boldsymbol{r}$ given in frame B which we abbreviate ${}_B\dot{\boldsymbol{r}} := ({}_B\boldsymbol{r})^{\bullet}$. This is a necessary formal definition with respect to the reference for the $(\dot{\boldsymbol{r}})$-components. Realizing a time derivation needs some additional considerations. We know from basic mechanics that a vector given in an inertial coordinate frame can be derived with respect to time directly. An inertial system is the only one where mapping and time differentiation can be interchanged. Taken in our form

$$K_I(\dot{\boldsymbol{r}}) := {}_I(\dot{\boldsymbol{r}}) = ({}_I\boldsymbol{r})^{\bullet} = {}_I\dot{\boldsymbol{r}}\,. \tag{2.7}$$

Consequently, and wherever we want to perform time derivation, we have to go back to an inertial form and transform the result to the desired frame. Let us apply this idea to the time derivative of some vector with respect to a moving reference B:

$$\begin{aligned} {}_I\boldsymbol{r} &= \boldsymbol{A}_{IB} \cdot {}_B\boldsymbol{r}\,, \\ {}_I(\dot{\boldsymbol{r}}) &= \dot{\boldsymbol{A}}_{IB} \cdot {}_B\boldsymbol{r} + \boldsymbol{A}_{IB} \cdot {}_B\dot{\boldsymbol{r}} \end{aligned}$$

Multiplying the last equation from the left by $\boldsymbol{A}_{BI}$ results in

$$\boldsymbol{A}_{BI} \cdot {}_I(\dot{\boldsymbol{r}}) = \boldsymbol{A}_{BI}\dot{\boldsymbol{A}}_{IB} \cdot {}_B\boldsymbol{r} + \boldsymbol{A}_{BI}\boldsymbol{A}_{IB} \cdot {}_B\dot{\boldsymbol{r}}\,. \tag{2.8}$$

We then derive the important formula of all relative kinematics (sometimes called the Coriolis-equation)

$${}_B(\dot{\boldsymbol{r}}) = {}_B\dot{\boldsymbol{r}} + {}_B\tilde{\boldsymbol{\omega}}_{IB} \cdot {}_B\boldsymbol{r}\,. \tag{2.9}$$

In words: The time derivative of $\boldsymbol{r}$ represented in the coordinates of the moving frame B is equal to the time derivative of the $\boldsymbol{r}$-components as given in B and the vector product of the angular velocity between B and I (written in B) and the vector ${}_B\boldsymbol{r}$ with its components in B. We now have to explain the last term of eq. (2.9).

We first show that $\boldsymbol{A}_{BI}\dot{\boldsymbol{A}}_{IB} = {}_B\tilde{\boldsymbol{\omega}}_{IB}$. For this purpose we consider the rotation of a body with respect to I (Fig. 2.6), where B is a body-fixed frame.

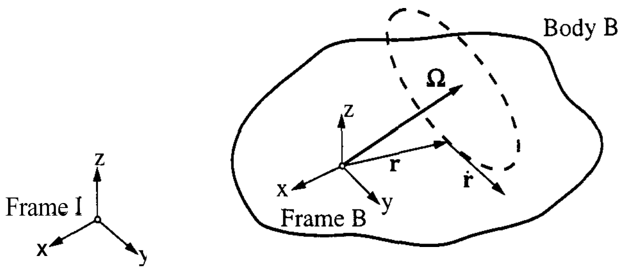

Figure 2.6: Rotation of a Body B

Thus, the angular velocity $\boldsymbol{\Omega}$ of the body is the same as the angular velocity between the frames B and I, $\boldsymbol{\omega}_{IB}$. Next we connect the origin of frame B and an arbitrary point of the rigid body by a vector $\boldsymbol{r}$. Then we can derive in one step [83]

$$\begin{aligned} K_B(\dot{\boldsymbol{r}}) &= K_B(\boldsymbol{\Omega} \times \boldsymbol{r}) \\ &= K_B(\boldsymbol{\omega}_{IB} \times \boldsymbol{r}) \\ &= {}_B\boldsymbol{\omega}_{IB} \times {}_B\boldsymbol{r} \\ &= {}_B\tilde{\boldsymbol{\omega}}_{IB}\, {}_B\boldsymbol{r} . \end{aligned} \tag{2.10}$$

In a second step we argue in the following way (eq. 2.2):

$$\begin{aligned} K_B(\dot{\boldsymbol{r}}) &= \boldsymbol{A}_{BI}\, K_I(\dot{\boldsymbol{r}}) \\ &= \boldsymbol{A}_{BI}\, ({}_I\boldsymbol{r})^{\bullet} \\ &= \boldsymbol{A}_{BI}\, (\boldsymbol{A}_{IB}\, {}_B\boldsymbol{r})^{\bullet} \\ &= \boldsymbol{A}_{BI}\, (\dot{\boldsymbol{A}}_{IB}\, {}_B\boldsymbol{r} + \boldsymbol{A}_{IB}\, {}_B\dot{\boldsymbol{r}}) \\ &= (\boldsymbol{A}_{BI}\dot{\boldsymbol{A}}_{IB})\, {}_B\boldsymbol{r} , \end{aligned} \tag{2.11}$$

where ${}_B\dot{\boldsymbol{r}} = 0$ in the body-fixed frame. A comparison of eqs. (2.10) and **(2.11)** yields

$$ {}_B\tilde{\boldsymbol{\omega}}_{IB} = \boldsymbol{A}_{BI}\dot{\boldsymbol{A}}_{IB} , \tag{2.12}$$

which is the first term of eq. (2.8). Transforming this expression into the I-frame and noting the transformation necessities of a tensor give

$$ {}_I\tilde{\boldsymbol{\omega}}_{IB} = \boldsymbol{A}_{IB}(\boldsymbol{A}_{BI}\dot{\boldsymbol{A}}_{IB})\boldsymbol{A}_{BI} = \dot{\boldsymbol{A}}_{IB}\boldsymbol{A}_{BI} . \tag{2.13}$$

The skew-symmetry of $\tilde{\boldsymbol{\omega}}$ follows from eq. (2.4) with $\mathrm{R} \equiv \mathrm{I}$, which we differentiate with respect to time to get

$$\dot{\boldsymbol{A}}_{IB}\boldsymbol{A}_{BI} + \boldsymbol{A}_{IB}\dot{\boldsymbol{A}}_{BI} = 0 .$$

We can then write (eq. **2.4)**

$$\left(\dot{A}_{IB}A_{BI}\right) = -\left(\dot{A}_{BI}^{T}A_{IB}^{T}\right)^{T} = -\left(\dot{A}_{IB}A_{BI}\right)^{T}, \tag{2.14}$$

because we may easily prove that $\dot{A}_{BI}^{T} = \left(A_{BI}^{T}\right)^{\bullet} = (A_{IB})^{\bullet} = \dot{A}_{IB}$. Thus the tensor $\mathbf{3}$ is skew-symmetric and possesses the principal form

$$\tilde{\omega} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}, \tag{2.15}$$

from which we derive the correspondence

$$\mathbf{3}\ .\ \boldsymbol{r} \equiv \boldsymbol{\omega} \times \boldsymbol{r}\,. \tag{2.16}$$

With the above properties we can easily derive a well-known formula with respect to Fig. **2.3** and the unit vectors used in eq. **(2.3).** Multiplying eq. **(2.13)** from the right by A_{IB} gives

$$\dot{A}_{IB} = {}_{I}\tilde{\omega}_{IB}A_{IB} \qquad \text{with } A_{IB} = \left({}_{I}e_{x}^{B}, {}_{I}e_{y}^{B}, {}_{I}e_{z}^{B}\right). \tag{2.17}$$

Differentiating every component results in

$${}_{I}\dot{e}_{i}^{B} = {}_{I}\omega_{IB}\ x\ {}_{I}e_{i}^{B},$$

which expresses the physical fact that the time derivative of a unit vector can only result in a change in its direction but not in a change in its magnitude.

2.3 Velocities and Accelerations

With the preceding chapters we have a sound basis for the evaluation of velocities and accelerations in various coordinate frames.

A typical situation of relative kinematics is shown in Fig. **2.7.** We frequently apply a number of reference points (e.g., P, R, O) and various coordinate systems (e.g., I, C, B), where C might be some frame convenient for the problem under consideration. Point O has velocity v_O, and point P has velocity v_P. The body is exhibited to some angular velocity $\boldsymbol{\Omega}$ which we define later in detail. Our convenience reference C might have an angular velocity ω_{IC} between C and I. Point R is fixed in the inertial frame. As seen from the point R and according to eq. (2.9), the absolute velocity of point P represented in the moving frame C can be written as

$${}_{C}v_{P} = {}_{C}(\dot{r}_{RP}) = {}_{C}\dot{r}_{RP} + {}_{C}\omega_{IC} \times {}_{C}r_{RP}\,. \tag{2.18}$$

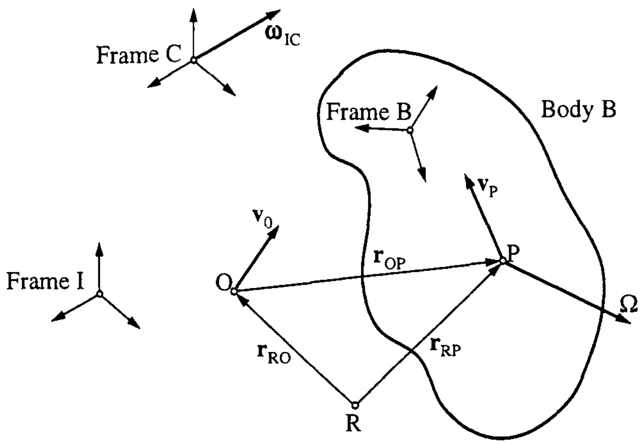

Figure 2.7: A Typical Configuration of Relative Kinematics

With the geometrical relationship $r_{RP} = r_{RO} + r_{OP}$ (Fig. 2.7) we obtain the well-known relation

$$
\begin{aligned}
{}_C v_P &= {}_C v_O + {}_C \dot{r}_{OP} + {}_C \omega_{IC} \times {}_C r_{OP} \\
&\quad \text{with} \quad {}_C v_O = {}_C \dot{r}_{RO} + {}_C \omega_{IC} \times {}_C r_{RO} \,.
\end{aligned}
\tag{2.19}
$$

The physical interpretation is straightforward and easy.

In many cases, though, our moving coordinate frame will be some body-fixed system B. If, for example, in eq. (2.19) we replace point O by a new body-fixed point P and the old point P by a second body-fixed point Q (Fig. 2.8), we obtain

$$
\begin{aligned}
{}_B v_Q &= {}_B v_P + {}_B \dot{r}_{PQ} + {}_B \omega_{IB} \times {}_B r_{PQ} \\
&\quad \text{with} \quad {}_B v_P = {}_B \dot{r}_{RP} + {}_B \omega_{IB} \times {}_B r_{RP} \,.
\end{aligned}
\tag{2.20}
$$

In frame B we have ${}_B \dot{r}_{PQ} = 0$, which is the rigid body definition. Therefore,

$$
{}_B v_Q = {}_B v_P + {}_B \Omega \times {}_B r_{PQ} \tag{2.21}
$$

where ${}_B v_P$ is given by the second equation in (2.20). Equation (2.21) can be easily written down in a body-fixed frame and then, if needed, transformed in any base C:

$$
\begin{aligned}
A_{CB}\, {}_B v_Q &= A_{CB}\, {}_B v_P + (A_{CB}\, {}_B \tilde{\Omega} A_{BC})(A_{CB}\, {}_B r_{PQ}) \,, \\
{}_C v_Q &= {}_C v_P + {}_C \tilde{\Omega}\, {}_C r_{PQ}
\end{aligned}
\tag{2.22}
$$