

L. C. Woods

Physics of Plasmas



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Author

Prof. Dr. Leslie C. Woods

University of Oxford and Balliol College

leslie.woods@balliol.oxford.ac.uk

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Cover Picture

Left: "Lightbulb" CME. A coronal mass ejection

Credit: NASA

Upper right: Heating coronal loops

Credit: M. Aschwanden et al. (LMSAL), TRACE, NASA

Lower right: A soft X-ray image of the sun

Credit: ESA, NASA

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Preface

This text gives an account of the principal properties of a tenuous gas, hot enough for some of the molecules to shed electrons and become ionized. In general a macroscopic volume of such a gas consists of a mixture of free electrons and the ions and neutrals of several molecular species and is called a *plasma*. If the temperature is high enough, e.g. $\sim 10\,000$ K at a pressure of 1 Pascal, a hydrogen plasma will be fully ionized, which is the case of most interest in this book. If there is also a magnetic field present, the ions and electrons will gyrate about the field lines, producing an anisotropic medium with some very interesting properties. Because of the orbiting motions, it is more difficult for the plasma to flow *across* the magnetic field lines than along them and with very strong fields both the plasma and its energy are said to be ‘confined’ by the field, although some leakage across the field lines does occur.

Examples of naturally occurring magnetoplasmas are found in the Sun’s corona, the solar wind and comet’s tails; laboratory examples include the plasma created in the fusion research machines known as tokamaks and in the application of what is termed ‘plasma processing’ to the manufacture of semiconductor devices. Although molten metal is not a plasma, it is a conductor of electricity and therefore subject to magnetic forces; its behaviour is described by the equations of magnetohydrodynamics (MHD), which are a limiting case of the magnetoplasma equations. Electric currents are used in industry to heat metals to the liquid state, when these metals can be stirred, levitated and pumped with magnetic fields. New applications of plasma physics arise from time to time; however, in a short book such as this there is space for little more than the basic principles of the subject.

One of the attractions of plasma physics is the range of subjects required for its understanding; these include fluid mechanics, electricity and magnetism, kinetic theory and thermodynamics, although for this text relatively little experience in these topics is assumed. There are many equations, so some effort has been made to cross-reference them at each stage of developing the theory. To help the reader with mathematical points, I have included ‘mathematical notes’ at appropriate stages in the chapters, and I have also added some appendices covering standard analyses. With a subject like plasma theory, subscripts are essential to distinguish between the properties of the several fluid components, so to avoid doubling up on subscripts, I have followed the common practice of employing the dyadic notation for tensors and where the vector and tensor analysis is complicated, I have filled in the steps involved.

Many texts on plasma theory begin with a description of the collisionless motion of individual charged particles known as particle orbit theory. Particles at a given time t and at a point \mathbf{r} in physical space are then grouped according to their velocity \mathbf{w} and a ‘kinetic’ equation describing the evolution of the number density of particles at a point $\mathbf{P} = \mathbf{P}(\mathbf{r}, \mathbf{w}, t)$ in phase-space is found. It is at this stage that particle collisions enter the model via a collision operator \mathbb{C} , which removes particles from \mathbf{P} or introduces particles into \mathbf{P} by collisional scattering. Finally, integrals of the kinetic equation over velocity space yield the fluid or MHD equations. However, these moments representing the conservation of mass, momentum and

energy, are independent of \mathbf{C} , the term containing which vanishes in each integration. Hence \mathbf{C} could in fact be zero. The standard account thus precedes from a microscopic description to what purports to be a collisional macroscopic model, without collisions playing any role at all. Terms corresponding to pressure and temperature appear in the moment equations and yet these properties are essentially continuum concepts that require the existence of local thermodynamic equilibrium, a state for which particle collisions are essential.

To avoid the confusion and occasional errors that the standard approach has introduced into plasma theory, in this text the subject is developed in the reverse order from that described above, that is we start with collision-dominated classical fluid mechanics in Chapter 1, adding the effects of electromagnetic fields in Chapter 2. At this stage we only need sufficient knowledge of particle orbit theory to determine the length and time scales below which a fluid or continuum description is not valid.

Chapter 3 presents the theory of small amplitude plasma waves and shock waves, and finishes with a brief introduction to magneto-ionic theory, required in studying the reflection and scattering of radio waves in the ionosphere. Stability of plasmas is treated in Chapter 4, covering the usual macroscopic instabilities of ideal plasmas, and also an important instability that depends on the electrical resistivity. Finally we remove collisions entirely from the model and introduce the Vlasov theory of plasma waves, applying it to Landau damping and the ion-acoustic instability, which has important applications in solar physics. Chapter 5, which is concerned with transport in magnetoplasmas, starts from the Fokker-Planck equation and gives an account of the theory of electron-ion collision intervals and several other relaxation times of important in the transport of particle energy and momentum.

The final chapter collects a miscellany of important topics, including second-order transport theory, thermal instabilities, particle orbit theory, magnetic mirrors, partially ionized plasmas and a brief introduction to some important applications of plasma physics. By second-order transport is meant, for example, the transport of heat in the presence of strong fluid shear, when the heat flux vector depends not only on the temperature gradient as in Fourier's law, but also on the rate of strain of the fluid. This proves to be very important in the presence of magnetic fields and leads to the thermal instabilities next described in the chapter. Particle orbits in the presence of magnetic field gradients is a particularly important phenomenon in near-collisionless plasmas, with applications to transport in tokamaks. Partially ionized plasmas add the complexity of a third fluid comprised of the neutral particles, to the model, so a brief introduction to Saha's equation for the dependence of the degree of ionization on the temperature and pressure is included. The final section briefly describes a few important applications of the theory – fusion research, solar physics, metallurgy, MHD direct generation of electricity and dusty plasmas.

The treatment is pitched at a level suitable for graduate students in mathematics, engineering and physics who need an introductory account of plasma physics. It is recommended that the reader should aim to get a clear physical picture of the *mechanisms* at each stage before checking through the analysis. Most of the exercises are straightforward extensions of the theory and therefore worthy of attention.

L. C. Woods

Oxford, 1st August, 2003

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Lists of physical constants, plasma parameters and frequently used symbols

In SI units, the constants required in plasma theory are:

Physical Quantity	Symbol	Value	units
Electron mass	m_e	9.1095×10^{-31}	kg
Proton mass	m_p	1.6726×10^{-27}	kg
Electron charge	e	1.6022×10^{-19}	C
Boltzmann constant	k_B	1.3807×10^{-23}	J K^{-1}
Permittivity (Free Space)	ϵ_0	8.8542×10^{-12}	F m^{-1}
Permeability (Free Space)	μ_0	$4\pi \times 10^{-7}$	H m^{-1}
Speed of light (Vacuum)	c	2.9979×10^8	m s^{-1}
Proton/electron mass ratio	m_p/m_e	1.8362×10^3	
Temperature at 1 eV	e/k_B	1.1605×10^4	K
Planck constant	h	6.6262×10^{-34}	J s
Stefan-Boltzmann constant	σ	5.6703×10^{-8}	$\text{W m}^{-2} \text{K}^{-4}$
Gas constant	R	8.3144	$\text{J K}^{-1} \text{mol}^{-1}$

The important plasma parameters are:

Parameter	Symbol	Formula	see pages
Resistivity	η	$\alpha m_e / (e^2 n_e \tau_e)$	43, 152
Cyclotron frequency (electrons)	ω_{ce}	eB/m_e	39
Thermal speed	C	$\sqrt{2k_B T/m}$	26
Larmor radius	r_L	C/ω_c	40
Coulomb logarithm	$\ln \Lambda$		145
Collision intervals	τ_e, τ_i		156
Thermal conductivity ($B = 0$)	κ	$\gamma k_B p T/m$	153
Magnetic diffusivity	ξ	η/μ_0	52
Magnetic Reynolds number	R_m	UL/ξ	53
Plasma frequency	ω_{pe}	$\sqrt{n_e e^2 / \epsilon_0 m_e}$	41
Collisionless skin-depth	δ_e	c/ω_{pe}	62
Debye length	λ_D	$\sqrt{\epsilon_0 k_B T_e / n_e e^2}$	42

Commonly used symbols are defined on the pages indicated below:

a 72	B 33	b 71	c 9	C_s 124	C 26	C 29
D 21	D 29	E 33	e 39	e 9	f 5	f_0 25
h 84	j 34	k_B 11	k_N 17	m 5	M 184	n 5
p 12	p 13	P 111	Q 34	q 19	\mathcal{R} 12	r_L 40
R_e 43	T 11	T 48	S 48	s 22	u 11	v 5
v_g 80	w 5	W 47				
β 74	γ 22	δ_e 62	δ_i 72	δ 43	ϵ_0 34	ζ 6
η 43	η_{\perp} 153	η 52	κ 46	λ_D 42	Λ 144	μ_0 34
ν_e 43	ξ 52	π 14	ρ 5	σ 34	τ_e 43	τ_i 156
ϕ 55	Ω 6	ω_c 40	ω_{pe} 41			

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1 The Equations of Gas Dynamics

1.1 Molecular models and fluids

A plasma is a mixture of positive ions, electrons and neutral particles, electrically neutral over macroscopic volumes, and usually permeated by macroscopic electrical and magnetic fields. In addition to these ‘smoothed’ or averaged electromagnetic fields, which with laboratory plasmas are often imposed from outside the plasma volume, there are the localized micro-fields due to the individual particles. The trajectories of the charged particles are thus continuously modified by a range of electromagnetic forces, the average fields acting like body forces and the micro-fields like collisional forces. The micro-fields are responsible for the transmission of pressure and viscous forces, for the conduction of particle energy, and for the friction forces between diffusing components of the plasma. Some care is needed in dividing the continuum of electromagnetic forces into their macroscopic and microscopic components, but with this achieved, there is little formal distinction between the theory of the macroscopic behaviour of neutral gases and that of magnetoplasmas.

The aim of this book is to describe the various physical processes that underpin plasma theory and the equations representing these processes. The distinction between what we shall term the ‘mechanisms’ and the equations based on them – *symbolisms* – is particularly important in a complex subject like plasma physics. Definitions of physical properties can be taken either from the mechanisms or the symbolisms, but one must take care not to mix the two, e.g. to adopt a purely mathematical definition of a property and then to assume that this automatically entails the usual physical attributes of that property.

Our method is to commence with the macroscopic description of the individual components of the plasma, that is we shall treat the collections of electrons, ions and neutrals as comprising separate fluids, and their fluid properties developed. In the next chapter they will be combined to make a plasma, an approach with the merit of making a clear distinction between the fluid and electrical properties of a plasma. Readers already familiar with fluid mechanics might skip to Chapter 2, although in §1.5 there is an introduction to kinetic theory that will be required in later chapters.

1.1.1 Introduction

Except for the basic concept of a ‘mean-free-path’, the trajectories of individual particles will be described in a later chapter. In this chapter we shall introduce the standard macroscopic variables of gas dynamics, such as pressure, temperature, fluid velocity and entropy, and derive the equations relating them. Excepting entropy, these physical properties are best defined

in terms of mechanisms although sometimes synthetic definitions have a place. Consider temperature for example; either it is defined physically via thermometers and the mechanism of *thermal equilibrium*, which requires close physical contact through molecular *particle collisions*, or it may be defined symbolically as a *kinetic temperature*, which is a property of the distribution of molecular velocities and collisions are not explicitly involved. The danger of employing the second definition is that it is too easy to adopt properties of temperature that really depend on the first definition. For example the conduction of heat, which depends on the temperature gradient, is a collisional process in which the gradient of the kinetic temperature would be misplaced without the additional constraint that the medium is collision-dominated, the precise meaning of which will be discussed later in §1.3.4.

1.1.2 Microscopic particles

A substance in the gaseous state consists of an assembly of a vast number of microscopic particles that, excepting when they collide with each other, move freely and independently through the region of physical space available to them. The nature of the particles depends largely on the temperature of the assembly. At low temperatures, but above the critical value at which liquefaction can occur, they are molecules. At higher temperatures the molecules dissociate into atoms, and at still higher temperatures the atoms become ions by shedding some of their electrons. The resulting assembly is termed a ‘plasma’. Partially ionized plasmas consist of a mixture of neutral atoms, electrons, and ions, requiring at least three distinct species of microscopic particles to be included in a complete mathematical representation of their collective behaviour.

The simplest model of a microscopic particle is a small featureless sphere, possessing a spherically-symmetric force field. For neutral particles this field has a very short range, and the particles can be pictured as being almost rigid ‘billiard-balls’, with an effective diameter equal to the range of the force field. As they have no structure, these particles have only energy of translation. The gas is usually assumed to be sufficiently tenuous for collisions involving more than two particles at a time to be ignored, i.e. only *binary* collisions are considered. The model is appropriate for monatomic uncharged molecules.

Diatomic and more complex molecules do not have symmetric force fields, but for many purposes they are also well represented by the billiard-ball model. Their relative orientations at collisions may be assumed to be randomly distributed, so that averages taken over a large number of encounters will have values independent of orientation, just as with symmetric force fields. It is the internal vibratory energies possessed by multi-atomic particles that give rise to the largest discrepancies between the predictions of the simple billiard-ball model and observation, a phenomenon that is easily included in kinetic theory by adding an average internal energy to the translatory energy of each molecule.

Kinetic theory is concerned mainly with the connection between the motions and interactions of microscopic particles comprising a gas and the transport of macroscopic properties like fluid momentum and energy through that gas. The oldest example relating macroscopic properties to microscopic behaviour is provided by the pressure force acting on the walls of a gas container. That it is due to the near-continuous bombardment of the walls by the vast number of neighbouring molecules, is a concept dating back to Boyle and Newton. The more subtle relationship between heat and the energy of molecular agitation required more than an-

other century before it was revealed with increasing detail in the works of Waterston, Clausius, and Maxwell¹. Clausius' main contribution to kinetic theory was the concept of the mean free path, which is the average distance travelled by a molecule between successive collisions, and which led to Maxwell's introduction of the velocity distribution function, to be discussed in §1.5.

The intermolecular force law plays a central role in kinetic theory and classical kinetic theory proceeds on the assumption that this law has been separately established, either empirically or from quantum theory, except with charged particles, when the well-known Coulomb force law applies. We shall return to this topic in Chapter 5; for the present it is sufficient to understand the concept of the mean-free-path.

1.1.3 The mean free path

Two microscopic parameters play a leading role in our account of the collective behaviour of an assembly of particles. These are the mean free path λ , which is the average distance moved by a particle between successive encounters with other particles, and the collision interval τ , which is the average time taken by a particle to move this distance. The reciprocal of τ is known as the 'collision' frequency, $\nu = \tau^{-1}$. The terminology is particularly fitting for 'hard' molecules, i.e. those with force fields abruptly falling to zero outside a molecular diameter σ , say.

An approximate formula for λ can be found as follows. Suppose there are n molecules per unit volume, and we assume that all are stationary, save one that has a velocity v_r relative to the others. In a tenuous or dilute gas, $\lambda \gg \sigma$ and hence $\pi\sigma^2v_r$ is a good approximation to the volume swept in one second by the sphere of influence of the moving particle. Those molecules with centres lying within this volume will experience a collision and therefore the collision frequency per molecule is $\tau^{-1} = \pi\sigma^2nv_r$. Replacing v_r by the average molecular speed \bar{c} relative to the centre of mass of all the similar molecules within a macroscopic volume element, and writing $\lambda = \tau\bar{c}$, we arrive at the estimate

$$\lambda \approx 1/(\pi\sigma^2n) \quad (\tau = \lambda/\bar{c}). \quad (1.1)$$

The accurate formula for λ is $2^{-\frac{1}{2}}$ times this value.

'Soft' molecules have extended force fields that make only slight changes in the momentum and energy of most passing molecules, so many such 'grazing' collisions are required to accumulate significant changes in these properties for a given test particle. However, by modifying σ to denote an 'effective' diameter, we can extend (1.1) to the case of soft molecules. Then λ becomes the average distance that a sequence of small-angle collisions takes to stop a test particle moving in a given direction, i.e. to give a 90° deflection, and τ is the time it takes for this change to happen. Even with hard molecules, a small sequence of collisions is required to 'stop' a particle. Another consequence of this cascade process is that momentum and energy require related but slightly different times to be transported in a specified direction.

The Coulomb force fields of electrons and ions have ranges extensive enough to influence great numbers of nearby particles, so that purely binary collisions are very rare. The billiard-ball model and the associated concept of a mean free path are not strictly relevant, although

¹See *The kind of motion we call heat* by Stephan Brush, North-Holland Publishing Company, 1976.

it is usual to describe the distance required for a 90° deflection of a test particle as being a ‘mean free path’. More precisely, λ is defined to be the distance over which this particle loses its momentum along its initial direction of motion.

1.1.4 Fluid particles

A fluid is sometimes described as being a ‘continuum’, that is a substance that has a continuous rather than a discrete structure, but since nature is particulate, obviously this is an approximate model, valid only on a length scale so large that the mean free path appears negligible². Familiar properties of a fluid are density, pressure, temperature and velocity, which we shall discuss in detail shortly. However, there are other important properties of fluids which depend on λ having a non-zero magnitude and that would vanish in a genuine continuum, e.g. fluid viscosity, electrical conductivity and thermal conductivity, all of which are proportional to relevant mean free paths.

The mechanism that qualifies an assembly of particles to be described as being a *fluid* is the frequency of collisions between the particles, which in turn depends on the size \mathcal{L} of the assembly. Evidently if \mathcal{L} is smaller than the mean free path λ , there will be few collisions and the assembly will lack the continuum properties required of a fluid. If $\mathcal{L} \ll \lambda$, the assembly is certainly not a fluid and is described as being a *ballistic* system; in plasmas the word ‘collisionless’ is adopted with the same meaning. The smallest physical element that can be called a fluid is termed a ‘fluid particle’ and such elements have the usual properties of thermodynamic systems. Neighbouring fluid particles interact with each other via collisions, e.g. by exerting a force on each other and by exchanging particle energy, whereas adjacent ballistic elements can not do this. It follows that a key dimensionless parameter in fluid mechanics is the ratio of a microscopic length (or time) to a macroscopic scale length (or time), a ratio known as the Knudsen number, k_N . We shall return to this basic concept in §1.3.4.

1.2 Macroscopic variables

1.2.1 Number density

The first task in the application of thermodynamics to continua is to specify precisely what is meant by the ‘thermodynamic system’ at a *point* (\mathbf{r}, t) in the medium. A ‘point’ in fluid mechanics is not a mathematical point, i.e. an object in space that has position but no magnitude; by the point $P(\mathbf{r}, t)$ is meant an infinitesimal volume element $d\mathbf{r}$ ($= dx dy dz$), centred at the mathematical point $M(\mathbf{r}, t)$, and with sufficient extension so that $d\mathbf{r}$ contains a vast number of microscopic particles of the species under consideration. This is termed a *macroscopic* point. Let the number of particles in $d\mathbf{r}$ be $n d\mathbf{r}$, then the fluctuations in n due to particles entering or leaving $d\mathbf{r}$ merely due to the discreteness of matter, will be negligible. We may therefore introduce the *number density* $n(\mathbf{r}, t)$ as a continuum variable provided the size $d\ell \equiv (d\mathbf{r})^{1/3}$ of the fluid particle is very much larger than the inter-particle distance, $n^{-1/3}$. On the other hand, we must not take $d\ell$ to be too large, otherwise significant variations

²A ‘fluid’ is to be distinguished from a ‘liquid’, which is a special case of a fluid.

in $n(\mathbf{r}, t)$ may be smothered. Thus $d\ell$ should satisfy

$$n^{-1/3} \ll d\ell \ll \mathcal{L}_n \quad (\mathcal{L}_n \equiv (d \ln n / dx)^{-1}), \quad (1.2)$$

where \mathcal{L}_n is a typical distance over which n changes by a significant fraction of its local value³.

The *fluid density* is the macroscopic variable

$$\rho(\mathbf{r}, t) = mn(\mathbf{r}, t), \quad (1.3)$$

where m is the particle mass, and for the present we are assuming that only one molecular species is present.

Let \mathbf{w} be the velocity of a typical particle at P , measured relative to a frame L (the ‘laboratory’ frame), and use $\langle \dots \rangle$ to denote average values taken over the particles in $d\mathbf{r}$. The *fluid velocity* at P is the macroscopic variable $\mathbf{v}(\mathbf{r}, t) = \langle \mathbf{w} \rangle$. It is the velocity of the centre of mass of the particles at P . For a more precise account, we need a statistical treatment.

The basic variable for statistical mechanics was introduced by Maxwell at the end of the eighteenth fifties; this is the number density $f(\mathbf{r}, \mathbf{w}, t)$ in six-dimensional phase space, usually termed the ‘velocity distribution function’. Thus $f(\mathbf{r}, \mathbf{w}, t) d\mathbf{r} d\mathbf{w}$ is the number of molecules that at time t have positions lying in a volume element $d\mathbf{r}$ ($= dx dy dz$) about the position \mathbf{r} , and velocities (of translation) lying within the velocity-space element $d\mathbf{w}$ ($= dv_x dv_y dv_z$) about \mathbf{w} .

The physical number density is

$$n(\mathbf{r}, t) = \int_{\text{all } \mathbf{w}} f(\mathbf{r}, \mathbf{w}, t) d\mathbf{w}, \quad (1.4)$$

where the integral is over the whole of velocity space. It follows that $f d\mathbf{w}/n$ is the probability that a given molecule at the (macroscopic) point (\mathbf{r}, t) has a velocity in the element $d\mathbf{w}$ at \mathbf{w} . We shall sometimes adopt the notation

$$\langle \phi \rangle(\mathbf{r}, t) \equiv \int \phi(\mathbf{r}, \mathbf{w}, t) f(\mathbf{r}, \mathbf{w}, t) / n d\mathbf{w}, \quad (1.5)$$

where the average on the left-hand side is the macroscopic variable corresponding to the phase-space function ϕ .

1.2.2 Fluid velocity

The fluid velocity \mathbf{v} is defined by the average

$$n\mathbf{v}(\mathbf{r}, t) = \int \mathbf{w} f(\mathbf{r}, \mathbf{w}, t) d\mathbf{w}, \quad (1.6)$$

³Let δn be the root mean square fluctuation in n , then it is a standard result in fluctuation theory that $\delta n/n \sim N^{-1/2}$, where N is the total number of particles in the system. For a system of volume $(\delta\ell)^3$, the condition that $\delta n/n \ll 1$ requires that $n(\delta\ell)^3 \gg 1$, as in (1.2).

and similarly the acceleration of the fluid at $P(\mathbf{r}, t)$ is

$$\frac{d\mathbf{v}}{dt} = \int \dot{\mathbf{w}} f(\mathbf{r}, \mathbf{w}, t)/n d\mathbf{w}, \quad (1.7)$$

where $\dot{\mathbf{w}}$ is the acceleration of the typical particle relative to L . When there are several distinct components present, the i -th component of which is a fluid with a velocity \mathbf{v}_i and density ρ_i , the velocity of the fluid as a whole is

$$\rho \mathbf{v}(\mathbf{r}, t) = \sum_i \rho_i \mathbf{v}_i(\mathbf{r}, t), \quad (1.8)$$

and we can therefore interpret \mathbf{v} as being the velocity of mass flux. An obvious property of \mathbf{v} is its dependence on the choice of the laboratory frame L , and as the equations describing the behaviour of fluids must be independent of this choice, \mathbf{v} can not appear alone in these equations. A 'frame-indifference' molecular velocity will be introduced shortly.

There is another fluid motion of considerable importance in the theory. This is the 'spin' Ω of the fluid at $P(\mathbf{r}, t)$, the meaning of which is that the fluid circulates around the mathematical point $M(\mathbf{r}, t)$ with an angular velocity Ω . Below we shall show that it is related to the fluid vorticity $\zeta \equiv \nabla \times \mathbf{v}$ by⁴

$$\Omega = \frac{1}{2} \nabla \times \mathbf{v}. \quad (1.9)$$

A macroscopic point that coincides momentarily with $P(\mathbf{r}, t)$, that has the same velocity, acceleration, rate of change of acceleration and so on, as the fluid, is called a *convected point*, and its locus is termed a *path line*. If axes are fixed relative to the fluid at this point and allowed to rotate with the local fluid spin Ω , then the point thus augmented, say P_c , is a *convected frame*. Viewed from this frame, the fluid near P_c will appear to be almost stationary, and without spin. This is a frame in which the ambient fluid is stationary and therefore it is the appropriate frame in which to specify the local thermodynamic system. By referring to a fluid property φ 'at P_c ', we shall mean that value of φ as observed in a convected, spinning macroscopic point at \mathbf{r}, t . The spin and acceleration of P_c are important when time derivatives of vectors and tensors at P_c are required in the theory.

Spatial changes in the fluid velocity $\mathbf{v}(\mathbf{r}, t)$ influence the transport of properties between adjacent fluid particles. To calculate the effects on transport we need the following analysis. Suppose that a convected point $P_c(\mathbf{r}, t)$ moves with a velocity $\mathbf{v}(\mathbf{r}, t)$, then a neighbouring convected point $Q_c(\mathbf{r} + \mathbf{R}, t)$ has the fluid velocity

$$\mathbf{v}'(\mathbf{r} + \mathbf{R}, t) = \mathbf{v}(\mathbf{r}, t) + \mathbf{R} \cdot \nabla \mathbf{v}(\mathbf{r}, t) + O(R^2), \quad (1.10)$$

⁴In Cartesian coordinates, with unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$,

$$\begin{aligned} \nabla \times \mathbf{v} &= \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \times (v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}) \\ &= \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \mathbf{i} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \mathbf{j} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \mathbf{k}. \end{aligned}$$