
NORMAL MODES AND LOCALIZATION IN NONLINEAR SYSTEMS

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PREFACE

The principal aim of this book is to introduce the reader to the concept and applications of a special class of nonlinear oscillations termed nonlinear normal modes (NNMs). These motions can be regarded as nonlinear analogs of the classical normal modes of linear vibration theory, although NNMs possess some distinctively nonlinear properties; first, the number of NNMs of a discrete nonlinear oscillator may exceed in number its degrees of freedom; second, in contrast to linear theory, a general transient nonlinear response cannot be expressed as a linear superposition of NNM responses; third, a subclass of NNMs is spatially localized and leads to nonlinear motion confinement phenomena. Hence, the study of NNMs and nonlinear mode localization in discrete and continuous oscillators reveals a variety of exclusively nonlinear phenomena that cannot be modeled by linear or even linearized methodologies. As shown in this book, these essentially nonlinear phenomena have direct applicability to the vibration and shock isolation of general classes of practical engineering structures. On a more theoretical level, the concept of NNMs will be shown to provide an excellent framework for understanding a variety of distinctively nonlinear phenomena such as mode bifurcations and standing or traveling solitary waves.

The material of this book is organized into ten chapters. In the first chapter a general discussion on the concept of NNMs and nonlinear mode localization is given. Lyapunov's and Rosenberg's definitions of NNMs are presented, along with a group-theoretic approach to nonlinear normal oscillations. A motivational example is included to demonstrate the concepts. In Chapter 2 general qualitative results on the existence of NNMs in a class of discrete conservative oscillators are presented and applications of the general theory are given for systems with convex or convex stiffness nonlinearities. In addition to general existence theorems, theorems regarding the nonlinear mode shapes of NNMs in discrete oscillators are also proved. In Chapter 3 quantitative analytical methodologies for computing NNMs of conservative and nonconservative discrete oscillators are discussed. NNMs are asymptotically studied by analyzing their trajectories in configuration space or by computing invariant normal mode manifolds in phase space; the later approach due to Shaw and Pierre provides an analytical framework for extending the concept of NNM in general classes of damped oscillators. In the same chapter, a group-theoretic approach for computing NNMs is presented, along with a discussion of

NNMs and nonlinear localization in vibro-impact oscillators. The stability and bifurcations of NNMs of discrete oscillators are discussed in Chapter 4. Linearized stability methodologies are considered, and the problem of stability of a NNM is converted to the equivalent problem of determining the stability of the zero solution of a set of variational equations with periodic coefficients. In many cases it is advantageous to transform this variational set to a set of equations with regular singular points. Analytical techniques for computing the instability zones of the transformed variational set are presented. In addition, conditions for the existence of finite numbers of instability zones in the variational equations are derived (finite-zoning instability). As a demonstrative example, the bifurcations of NNMs of a discrete oscillator in internal resonance are analyzed in more detail. In Chapter 5 forced resonances occurring in neighborhoods of NNMs are studied. It is shown that exact steady state motions of nonlinear systems occur close to NNMs of the corresponding unforced systems. Moreover, it is found that NNM bifurcations have profound effects on the topological structure of the nonlinear frequency response curves of the forced system. A new analytical methodology for studying nonlinear oscillations is formulated in Chapter 6, termed the method of nonsmooth temporal transformations (NSTTs). This method is based on nonsmooth (saw-tooth) transformations of the temporal variable and leads to asymptotic solutions that are valid even in strongly nonlinear regimes where conventional analytical methodologies are less accurate. An application of the NSTT methodology to the problem of computing NNMs in strongly nonlinear discrete systems is presented along with some additional strongly nonlinear (even nonlinearizable) applications. In Chapter 7 nonlinear mode localization in certain classes of periodic oscillators is discussed, and analytical studies of transitions from mode localization to nonlocalization are given; in addition, NNM bifurcations in a discrete system with cyclic symmetry are analyzed. In the same chapter a numerical example of nonlinear passive motion confinement of responses generated by impulsive loads in a cyclic system is presented. The extension of the concept of NNM in continuous oscillators is performed in Chapter 8. Several quantitative methodologies for studying continuous NNMs are discussed, based on discretization or on direct analysis of the governing partial differential equations of motion. It is shown that the concept of NNM can be employed to study nonlinear stationary waves in partial differential equations, or waves with decaying envelopes in attenuation zones of continuous periodic systems of infinite spatial extent. In Chapters 9 and 10 nonlinear localization and passive motion confinement in periodic assemblies of continuous oscillators is discussed, and three examples from mechanics are analyzed in detail: a system of coupled nonlinear beams, a multispan nonlinear beam, and a nonlinear periodic spring-mass chain. Experimental studies of nonlinear localization in systems of coupled nonlinear beams are also presented in Chapter 9, and a new design methodology based on the nonlinear motion confinement phenomenon is formulated. An interesting conclusion from the applications

presented in Chapter 10 is that the concept of localized NNM can be used to analyze solitary waves or solitons in certain classes of nonlinear partial differential equations. In that context, localized NNMs in discrete oscillators can be regarded as discrete analogs of spatially localized solitary waves and solitons encountered in nonlinear partial differential equations on infinite domains.

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CHAPTER 1

INTRODUCTION

1.1 CONCEPTS OF NONLINEAR NORMAL MODE AND NONLINEAR LOCALIZATION

It is well established that normal modes are of fundamental importance in the theory of linear conservative and nonconservative dynamical systems. This is so because linear normal modes can be used to uncouple the governing equations of motion, and to analytically evaluate the free or forced dynamic response for arbitrary sets of initial conditions. This is performed by employing modal analysis and making use of the principle of linear superposition to express the system response as a superposition of modal responses. In classical vibration theory, the problem of computing the normal vibrations of discrete or continuous oscillators is reduced to the equivalent problem of computing the eigensolutions (natural frequencies and corresponding eigenvectors or eigenfunctions) of linear transformations. Clearly, such an approach as well as the principle of linear superposition are generally inapplicable in nonlinear theory. So, the obvious question arises: *Is there a reason to extend the concept of normal modes in the nonlinear case?*

Naturally, one can define nonlinear normal modes (NNMs) merely as synchronous periodic particular solutions of the nonlinear equations of motion without seeking any connection of such motions to the (linear) superposition principle. In the framework of such a restricted definition, a nonlinear generalization of the concept of normal mode is possible, and beginning with the works of Lyapunov several attempts were undertaken in this direction. Lyapunov's theorem (1907) proves the existence of n synchronous periodic solutions (NNMs) in neighborhoods of stable equilibrium points of n degree-of-freedom (DOF) hamiltonian systems whose linearized eigenfrequencies are not integrably related. Weinstein (1973) and Moser (1976) extended Lyapunov's result to systems with integrably related linearized eigenfrequencies (systems in "internal resonance"). Kauderer (1958) studied analytically (and graphically) the free periodic oscillations of a two-DOF system, thus becoming a forerunner in the conceivement of quantitative methods for analyzing NNMs. The

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formulation and development of the theory of NNMs can be attributed to Rosenberg and his co-workers who developed general qualitative (Pak and Rosenberg, 1968), and quantitative (Rosenberg, 1960, 1961, 1962, 1963, 1966; Rosenberg and Hsu, 1961; Rosenberg and Kuo, 1964) techniques for analyzing NNMs in discrete conservative oscillators. *Rosenberg considered n DOF conservative oscillators and defined NNMs as "vibrations in unison," i.e., synchronous periodic motions during which all coordinates of the system vibrate equiperiodically, reaching their maximum and minimum values at the same instant of time.* Some additional representative quantitative techniques based on the previous formal definition of NNMs were performed in (Magiros, 1961; Rand, 1971a,b, 1973, 1974; Rand and Vito, 1972; Manevitch and Mikhlin, 1972; Manevitch and Pilipchuk, 1981; Mikhlin, 1985; Vakakis, 1990; Caughey and Vakakis, 1991; Shaw and Pierre, 1991, 1992, 1993, 1994; Boivin et al., 1993; Nayfeh and Nayfeh, 1993, 1994; Nayfeh et al., 1992; Pakdemirli and Nayfeh, 1993). Application of the concept of NNM to control theory is studied by Slater (1993). General reviews of analytical and numerical methods for computing NNMs in discrete and continuous oscillators can be found in King (1994) and Slater (1993, 1994).

In linearizable systems with weak nonlinearities it is natural to suppose that NNMs are particular periodic solutions that, as the nonlinearities tend to zero, approach in limit the classical normal modes of the corresponding linearized systems. Evidently the number of these NNMs must be less or equal to the number of DOF of the systems considered. Moreover, when weak periodic forcing is applied, NNMs can be used to study the structure of the system's nonlinear resonances (Malkin, 1956; Hsu, 1959, 1960; Kinney, 1965; Kinney and Rosenberg, 1966; Manevitch and Cherevatzky, 1972; Mikhlin, 1974; Vakakis and Caughey, 1992; Vakakis, 1992b). *Here rests a first practical application of defining NNMs: Although the principle of superposition does not hold in the nonlinear case, forced resonances of nonlinear systems occur in neighborhoods of NNMs, in direct analogy to linear theory.* Hence, understanding the structure of NNMs of discrete or continuous oscillators enables one to better study the forced responses of these systems to external periodic inputs.

In addition, in some of the aforementioned works particular attention was devoted to "homogeneous" systems, i.e., to nonlinearizable (essentially nonlinear) systems whose stiffness nonlinearities are proportional to the

same power of the displacement. It was shown that the NNMs of homogeneous systems can exceed in number their DOF, a feature with no counterpart in linear theory (with the exception of the case of multiple natural frequencies). This is due to NNM bifurcations, which become exceedingly more complicated as the number of DOF of the systems increase. *Thus, not all NNMs can be regarded as nonlinear analytic continuations of normal modes of linearized systems; indeed, an accurate computation of NNMs can reveal dynamic behavior that cannot be modeled by conventional linear or linearized approaches.* Bifurcations of NNMs in discrete systems were first studied in works by Rand and co-workers (Rand, 1971a; Rand and Vito, 1972; Month and Rand, 1977; Johnson and Rand, 1979; Month, 1979; Rand et al., 1992), and in (Zhupiev and Mikhlin, 1981; Manevitch et al., 1989; Caughey et al., 1990). In these works it was found that bifurcating NNMs are typically localized in a small portion of the dynamical system. *It will be shown that such localized NNMs lead to nonlinear spatial confinement of motions generated by external inputs, a feature which is one of the most interesting and important applications of the theory of NNMs. Nonlinear mode localization can be studied in the framework of NNMs* and gives rise to a variety of nonlinear dynamic phenomena that can be used to develop robust shock and vibration isolation designs for certain classes of engineering systems.

Some alternative ways of viewing nonlinear normal oscillations are formulated in the following exposition. It is known that linear conservative systems possess certain symmetries that reflect on the properties of their normal modes. Every such property can be associated with a specific symmetry of the governing equations of motion, and in classical vibration theory the normal modes of linear conservative systems can be computed by imposing an invariance of the equations of motions with respect to arbitrary temporal shifts (temporal invariance). In particular, for oscillations on a normal mode all position coordinates of a linear system are proportional to the same exponential function, $e^{j\omega t}$, where $j=(-1)^{1/2}$, ω is the frequency of the normal mode oscillation, and t is the temporal variable. Part of the properties of linear normal modes can be extended to the nonlinear case. More specifically, *for a certain class of nonlinear systems it is possible to define NNMs as special periodic solutions with exponential temporal dependence of all positional variables.* The simplest system representative of this class is a system composed of two weakly coupled particles that are

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connected to the ground by weakly nonlinear springs. This system is governed by the following equations of motion:

$$\begin{aligned} \ddot{u} + a u + \varepsilon e (u - v) - \varepsilon b u^3 &= 0 \\ \ddot{v} + a v + \varepsilon e (v - u) - \varepsilon b v^3 &= 0 \end{aligned} \quad (1.1.1)$$

where a , e , and b are real quantities, and $|\varepsilon| \ll 1$. It is well known (Nayfeh and Mook, 1984) that the free response of this system is governed by two time scales, t and εt , which exist due to the weak nonlinearity and the weak coupling. Moreover, weak coupling leads to energy redistribution (beating) between the two particles of the system. Hence, a two-scales asymptotic analysis seems to be the natural way to proceed in computing an approximation to the free oscillation (Rand et al., 1992). In this case normal modes correspond to single-frequency motions, which to the leading order of approximation possess exponential temporal dependence. However, one can consider an alternative method for solving the problem. Instead of computing an approximate asymptotic solution of system (1.1.1), it is possible to replace it by the following equivalent nonlinear system which admits exact (closed-form) solutions (Kosevich and Kovalev, 1989):

$$\begin{aligned} j\dot{\psi}_1 &= \omega_0 \psi_1 + (\Omega/2) (\psi_1 - \psi_2) - \alpha |\psi_1|^2 \psi_1 \\ j\dot{\psi}_2 &= \omega_0 \psi_2 + (\Omega/2) (\psi_2 - \psi_1) - \alpha |\psi_2|^2 \psi_2 \end{aligned} \quad (1.1.2)$$

where $j = (-1)^{1/2}$ and Ω characterizes the coupling term. Note that if $\alpha=0$ system (1.1.2) becomes linear. It is easy to show that both linear and nonlinear systems admit single-frequency exponential solutions of the form:

$$\psi_i = a_i \exp(-j\omega t), \quad i = 1, 2 \quad (1.1.3)$$

where the complex amplitudes a_i are expressed as:

$$\begin{aligned} a_1 &= N^{1/2} \cos\theta, \quad a_2 = N^{1/2} \sin\theta e^{j\phi}, \\ N &= |\psi_1|^2 + |\psi_2|^2, \quad \phi = 0 \text{ or } \pi \end{aligned} \quad (1.1.4)$$

If $\theta = \pi/4$ one obtains at most two single-frequency solutions. When $\phi = 0$ the system vibrates in an inphase NNM and the oscillators undergo in-phase vibrations with frequency equal to

$$\omega = \omega_0 - \alpha N/2$$

Similarly, when $\phi = \pi$ the system vibrates in an antiphase NNM and the motions of the oscillators are in antiphase, with frequency

$$\omega = \omega_0 + \Omega - \alpha N/2$$

Hence, one obtains a nonlinear generalization of linear normal modes from the viewpoint considered. The previous example illustrates an additional dynamical feature of the nonlinear system. It is easily proven that, if

$$N > N_0 = \Omega/\alpha$$

two additional NNMs exist, corresponding to

$$\phi = 0, \quad \theta = \pi/4 \pm \cos^{-1}(\Omega/\alpha N)$$

These additional modes bifurcate from the in-phase NNM, which for $N > N_0$ becomes orbitally unstable. The two bifurcating modes are stable (Kosevich and Kovalev, 1989), and *spatially localized*, since the energy of each bifurcating mode is found to be predominantly confined to only one of two particles of the system. So one observes two essentially nonlinear features of system (1.1.1), namely, that its NNMs can exceed in number the DOF of the oscillator, and that some of its NNMs are spatially localized.

Spatial nonlinear localization is one of the most important properties encountered in NNMs and provides a link between NNMs and solitary solutions (solitary waves and solitons) in the theory of nonlinear waves. To demonstrate this link one must consider a generalization of system (1.1.1) for arbitrarily large or infinite degrees of freedom (Scott et al., 1985). The analysis then shows that in the limit of weak nonlinearity the n DOF system possesses NNMs in direct analogy to the linear case. Moreover, when the coupling terms become of the same order as the nonlinear terms, there exist numerous mode bifurcations, and the system possesses $(3^n - 1)/2$ NNMs, the majority of which are spatially localized; this is in contrast to the corresponding linear n DOF system which only possesses n normal modes. Hence, nonlinear mode localization is a general property of a wide class of weakly coupled oscillators. An additional interesting feature of the n DOF generalization of system (1.1.1), is that as $n \rightarrow \infty$ the system reduces to the discrete approximation of the continuous nonlinear Schrödinger's equation (NSE) with periodic boundary conditions [for an application of Schrödinger's equation to model a linear disordered lattice, see Kuske et al.

(1993)]. The important work of Ford (1961) and Waters and Ford (1966) must be mentioned here. They studied recurrence phenomena in the Ulam-Fermi-Pasta (1955) problem, and showed that lack of equipartition of energy in an infinite nonlinear lattice with periodic boundary conditions is partly due to the existence of stable nonlinear normal modes in this system. The NSE equation is well known (Lamb, 1980; Novikov et al., 1984) to describe a fully integrable dynamical system and to possess soliton solutions of different types in the form of spatially localized waves. *Hence, there is a relationship between the localized NNMs of certain weakly coupled mechanical systems and the soliton solutions of the NSE.* As shown in (Vedenova et al., 1985; Vedenova and Manevitch, 1981; King and Vakakis, 1994), in the context of NNM theory, stationary periodic solitary waves can be regarded as NNMs of infinite-dimensional systems defined on unbounded domains. A note of caution is appropriate, however, here. If the NSE is regarded as the continuous approximation of an infinite nonlinear lattice of weakly coupled particles, the continuous approximation is only applicable for waves whose wavelengths exceed the distance between adjacent particles. In contrast to such solutions, certain (strongly) localized NNMs of the discrete infinite system are localized predominantly to single particles. Taking this observation into account one notes that *the concepts of localized NNMs and solitons mutually complement each other.*

A third distinct formulation of NNMs can be performed by considering symmetries in the configuration space of a nonlinear oscillator. If one expresses the equations of motion of a linear conservative system in Jacobi's form (geometric formulation), one finds that these equations are invariant with respect to a *continuous* group of extensions or compressions in the configuration space. Linear normal modes (which correspond to straight lines in the configuration space) turn out to be the only possible solutions that are invariant with respect to this group of transformations. Taking this property of normal modes into account, it is possible to construct a systematic analytic methodology for computing normal modes, by reducing the problem to an algebraic eigenvalue one. From this viewpoint, this group-invariance method is equivalent to the previous approach for computing NNMs based on temporal invariance. However, in contrast to the latter, the former approach provides the eigenvectors or eigenfunctions, but does not compute directly the eigenfrequencies of the normal modes. Invariance with respect to extensions or compressions in the configuration space is not a

distinctive property of linear systems. Considering a nonlinear conservative discrete oscillator with homogeneous potential function of even degree, it can be shown that its equations of motion can be made invariant to extensions or compressions in the configuration space. *Hence, for a homogeneous system it is possible to seek NNMs that correspond to straight lines in the configuration space and possess the group-invariance properties of the linear modes. This formulation provides an alternative nonlinear generalization of the concept of linear normal modes.* What distinguishes the nonlinear from the linear case is the fact that a nonlinear homogeneous system may possess more straight-line NNMs than its DOF. This feature was also noticed in the previous definitions of NNMs, where it was noted that the majority of the additional NNMs are spatially localized. So, one finds that homogeneous systems (i.e., systems with essential nonlinearities) exhibit nonlinear mode localization. As shown in the following chapters this is not an exclusive feature of homogeneous systems, since localized NNMs will be detected in a wider class of nonlinear oscillators.

A last generalization of the concept of normal mode to the nonlinear case can be carried out by noting that the equations of the motion of linear systems possess an additional *discrete* symmetry group in the configurational space: After transforming to normal coordinates, any Cartesian transformation of coordinates is equivalent to mere inversions of normal coordinates. This reveals that linear normal modes are invariant solutions with respect to the group in Cartesian transformations in the configuration space. This viewpoint turns out to provide a very efficient way of computing normal modes of linear systems with geometric symmetries. In the linear case, there exists a linear vector space that is formed by the linearly independent normal modes; certainly, this is not the case in nonlinear theory. A first attempt was undertaken by Yang (1968) to employ discrete symmetries of certain nonlinear systems for computing NNMs, without resorting to group theoretic techniques. As discussed by Manevitch and Pinsky (1972a), NNMs can be determined in the framework of the theory of invariant-group solutions. In that context, one must classify sets in the configuration space that are invariant with respect to subgroups of admitting groups. This procedure allows one to find the sub-space of the configuration space that contains a certain NNM. If the dimension of this subspace is equal to 1, the subspace coincides with a NNM. *Since the theory of discrete group-invariant solutions is applicable to both linear and nonlinear systems, one*

obtains an additional nonlinear generalization of normal modes. Moreover, considering a general nonlinear conservative system, one can formulate the following "inverse" problem: Is it possible to compute a special set of system parameters that leads to an extension of the admitting group? The answer to this problem allows one (at least in principle) to classify all nonlinear systems possessing specified symmetries in the configuration space and to compute their NNMs (Manevitch et al., 1989; Manevitch and Pinsky, 1972a; Pilipchuk, 1985).

The previous exposition shows that there exist several distinct ways for extending the concept of normal mode vibrations to nonlinear systems. In that context, NNMs can be regarded, (a) as mere synchronous periodic solutions of the equations of motion (formal approach), (b) as solutions that possess exponential temporal dependence, or (c) as solutions that preserve invariance of the equations of motion with respect to certain continuous or discrete symmetry groups (group-theoretic approach). By extending the notion of normal mode to nonlinear theory one is able to better classify and study the symmetries and the forced resonances of discrete and continuous oscillators. In addition, NNMs provide the necessary framework for studying nonlinear mode localization and motion confinement phenomena in weakly coupled oscillators and can be employed to establish a link between localized periodic responses of discrete or continuous oscillators and solitary waves or solitons in nonlinear wave theory. Additional applications of NNMs on the study of the global dynamics and chaotic responses of nonlinear oscillators are discussed in later chapters.

1.2 EXAMPLE: NNMs OF A TWO-DOF DYNAMICAL SYSTEM

The concept of nonlinear normal modes is now demonstrated by considering the dynamics of a simple nonlinear oscillator. To this end, the two DOF hamiltonian system depicted in Figure 1.2.1 will be studied, with governing equations of motion given by:

$$\begin{aligned}\ddot{x} + x + x^m + K(x - y)^m &= 0 \\ \ddot{y} + y + y^m - K(x - y)^m &= 0\end{aligned}\tag{1.2.1}$$

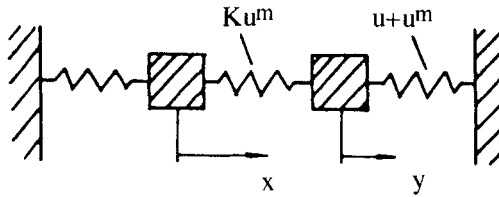


Figure 1.2.1 The two DOF nonlinear oscillator.

where the exponent m is assumed to be an odd number. This system possesses *similar* NNMs, corresponding to the following *linear relation* between the depended variables x and y :

$$y = c x \quad (1.2.2)$$

The similar modes (1.2.2) are represented by *straight modal lines* in the configuration plane of the system, and are the only types of normal modes encountered in linear theory. As shown in chapter 3, similar NNMs are not generic in nonlinear discrete oscillators, since they exist only in systems with special symmetries (such as the system depicted in Figure 1.2.1). More typical in nonlinear systems are *nonsimilar* NNMs, which correspond to *nonlinear relations* between depended variables of the form $y = f(x)$, and are represented in the configuration space by *modal curves*. Asymptotic methodologies for computing nonsimilar NNMs are also developed in chapter 3. As shown by Vakakis and Rand (1992), the similar NNMs (1.2.2) are the only type of normal modes that system (1.2.1) can possess.

Since the linear relation (1.2.2) is assumed to hold at all times, one can use it to eliminate the y variable from the equations of motion and to obtain the following equivalent set of equations:

$$\begin{aligned} \ddot{x} + x + [1 + K(1-c)^m] x^m &= 0 \\ \ddot{x} + x - (1/c) [K(1-c)^m + c^m] x^m &= 0, \quad c \neq 0 \end{aligned} \quad (1.2.3)$$

For motion on a NNM both equations (1.2.3) must provide the same response $x = x(t)$, a requirement that is satisfied by matching the respective coefficients of linear and nonlinear terms. Since both equations possess

identical linear parts, one obtains a single equation satisfied by the modal constant c :

$$K(1+c)(c-1)^m = c(1-c^{m-1}), \quad c \neq 0 \quad (1.2.4)$$

As pointed out by Vakakis (1990), the simultaneous matching of *all* linear and nonlinear coefficients in a discrete system generally leads to a set of overdetermined algebraic equations governing the modal constants, which can only be solved if the problem under consideration possesses certain symmetries. The algebraic equation (1.2.4) always possesses the solutions $c = \pm 1$, which correspond to in-phase and antiphase NNMs. These are the only normal modes that the corresponding linear system (with $m = 1$) can possess. Interestingly enough, the nonlinear system ($m = 3, 5, \dots$) can possess additional NNMs, with modal constants computed by solving the following algebraic equations:

$$\sum_{k=1}^{(m-1)/2} c^{2k-1} + K(1-c)^{m-1} = 0, \quad c \neq 0, \quad m = 3, 5, 7, \dots \quad (1.2.5)$$

It turns out that the additional normal modes (1.2.5) always occur in reciprocal pairs and bifurcate from the antiphase mode $c = -1$ at the critical value,

$$K = K_c = 2^{1-m} \sum_{k=1}^{(m-1)/2} (-1)^{2k-1}$$

in hamiltonian pitchfork bifurcations. The stability of the computed NNMs can be studied by performing, a local (linearized) analysis (Rosenberg and Hsu, 1961; Pecelli and Thomas, 1979; Zhupiev and Mikhlin, 1981, 1984; Caughey et al., 1990), an analysis based on Ince-algebraisation of the variational equations (Zhupiev and Mikhlin, 1981, 1984) or a global (nonlinear) analysis based on analytical or numerical Poincaré' maps (Month, 1979; Hyams and Month, 1984; Vakakis and Rand, 1992). In Figure 1.2.2 the NNMs of systems with $m = 1, 3, 5$ and 7 are depicted. These results are summarized in the following remarks.

- (1) The additional bifurcating NNMs of the nonlinear systems with $m = 3$ and 7 exist only at small values of the coupling parameter. The

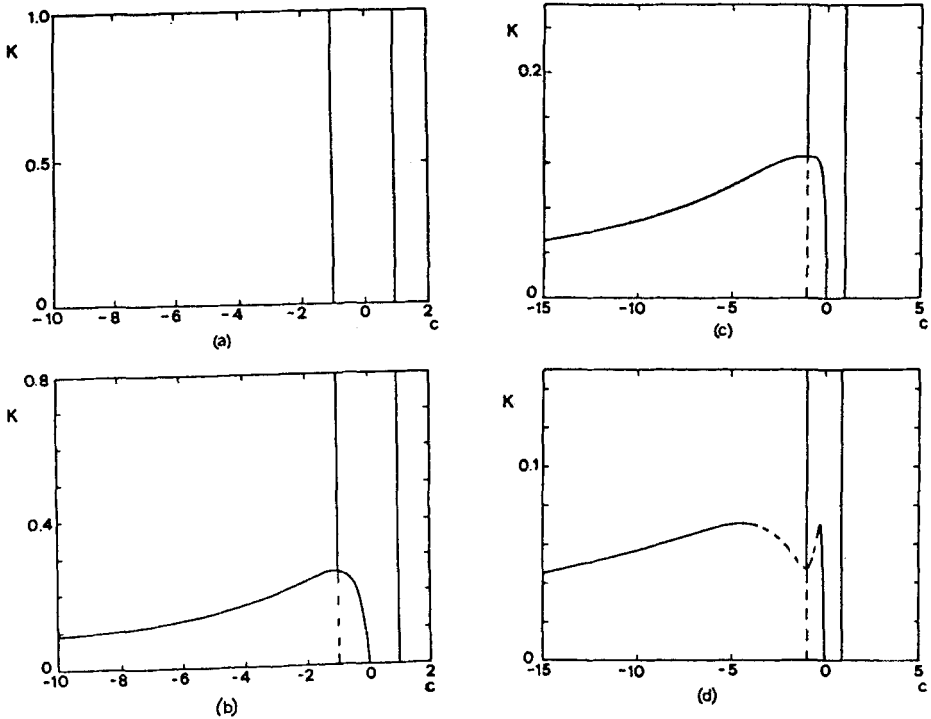


Figure 1.2.2 Bifurcations of NNMs for systems with (a) $m = 1$ (linear case), (b) $m = 3$, (c) $m = 5$, and (d) $m = 7$.

— Stable NNMs, - - - - Unstable NNMs.

bifurcating NNMs are essentially nonlinear and cannot be regarded as analytic continuations of any linear modes. This is in contrast to the modes $c = \pm 1$ which can be regarded as nonlinear continuations of the linear normal modes of the system with $m = 1$.

- (2) As $K \rightarrow 0$, a pair of bifurcating NNMs becomes strongly localized, with modal constants approaching the limits, $c \rightarrow 0$ and ∞ , respectively. It can be shown that these NNMs are orbitally stable and, thus, physically realizable.
- (3) The bifurcations of NNMs have important implications on the low- and high-energy global dynamics and on the forced nonlinear resonances of system (1.2.1).

To demonstrate the effects of the mode bifurcations on the global dynamics, the nonlinear system with $m = 3$ is considered in more detail. This system is hamiltonian with a four-dimensional phase space

$$(x, \dot{x}, y, \dot{y})$$

and its global dynamics can be studied by constructing numerical or analytical Poincare' maps (Month and Rand, 1977, 1980; Month, 1979). Here only a brief description of the construction of these maps will be given, and for a more detailed discussion, the reader is referred to the aforementioned references. By fixing the total energy of the dynamical system to a constant level, one restricts the flow in the phase space to a three-dimensional isoenergetic manifold. This is performed by imposing the following condition:

$$H(x, \dot{x}, y, \dot{y}) = h \quad (1.2.6)$$

where $H(\bullet)$ is the hamiltonian of the system, and h is the fixed energy level. The hamiltonian H is a first integral of the motion, and for autonomous oscillators represents conservation of energy during free oscillations. If an additional independent first integral of motion exists, the two-DOF system is said to be *integrable* and the isoenergetic manifold $H = h$ is fibered by invariant two-dimensional tori (Guckenheimer and Holmes, 1984). This integrability property is not generic in hamiltonian systems, and, in general, one does not expect the existence of an independent second integral of motion. However, for low energies, even nonintegrable oscillators appear to have an approximate second integral of motion. This is because for low energies the isoenergetic manifolds of these systems appear to be fibered by approximate invariant tori which, as the energy increases, "break," giving rise to randomlike chaotic motions (Lichtenberg and Lieberman, 1983).

Now suppose that one intersects the three-dimensional isoenergetic manifold defined by (1.2.6) with a two-dimensional cut-plane. If the intersection of the two manifolds is *transverse* (Guckenheimer and Holmes, 1984; Wiggins, 1990), the resulting cross-section, Σ , is two-dimensional, and the flow of the dynamical system intersecting the cut-plane defines a 'Poincare' map. Choosing the cut-plane as $T:\{x=0\}$, the Poincare' section Σ is defined as

$$\Sigma = \{x = 0, \dot{x} > 0\} \cap \{H = h\}$$