Perturbation Methods

ALI HASAN NAYFEH

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PREFACE

Many of the problems faced today by physicists, engineers, and applied mathematicians involve difficulties, such as nonlinear governing equations, variable coefficients, and nonlinear boundary conditions at complex known or unknown boundaries, which preclude their solutions exactly. To solve these problems we are forced to resort to a form of approximation, a numerical solution, or a combination of both. Foremost among the approximation techniques is the systematic method of perturbations (asymptotic expansions) in terms of a small or a large parameter or coordinate. This book is concerned only with these perturbation techniques.

According to these perturbation techniques, the solution of the full problem is represented by the first few terms of a perturbation expansion, usually the first two terms. Although these perturbation expansions may be divergent, they can be more useful for a qualitative as well as a quantitative representation of the solution than expansions that are uniformly and absolutely convergent.

It is the rule rather than the exception that the straightforward (pedestrian) expansions in powers of a parameter have limited regions of validity and break down in certain regions called regions of nonuniformity. To render these expansions uniformly valid, investigators working in different branches of physics, engineering, and applied mathematics have developed a number of techniques. Some of these techniques are radically different, while others are different interpretations of the same basic idea.

The purpose of this book is to present in a unified way an account of some of these techniques, pointing out their similarities, differences, and advantages, as well as their limitations. The different techniques are described using examples which start with model simple ordinary equations that can be solved exactly and progress toward complex partial differential equations. The examples are drawn from different branches of physics and engineering. For each example a short description of the physical problem is first presented.

The different techniques are described as formal procedures without any attempt at justifying them rigorously. In fact, there are no rigorous mathematical justifications available yet for the expansions obtained for some of the complex examples treated in this book.

At the end of each chapter, a number of exercises have been included, which progress in complexity and provide further references.

The reader need not understand the physical bases of the examples used to describe the techniques, but it is assumed that he has a knowledge of basic calculus as well as the elementary properties of ordinary and partial differential equations.

Chapter 1 presents the notations, definitions, and manipulations of asymptotic expansions. The sources of nonuniformity in perturbation expansions are classified and discussed in Chapter 2. Chapter 3 deals with the method of strained coordinates where uniformity is achieved by expanding the dependent as well as the independent variables in terms of new independent parameters. Chapter 4 describes the methods of matched and composite asymptotic expansions; the first method expresses the solution in terms of several expansions valid in different regions but related by matching procedures, while the second method expresses the solution in terms of a single expansion valid everywhere. In Chapter 5 the idea of fast and slow variables is used in conjunction with the variation of parameters method to study the slow variations of the amplitudes and phases of weakly nonlinear waves and oscillations. The methods of Chapter 3, 4, and 5 are generalized in Chapter 6 into one of three versions of the method of multiple scales. Chapter 7 treats available methods for obtaining asymptotic solutions of linear ordinary and partial differential equations.

My first technical debt is to Dr. W. S. Saric and to my brothers Dr. Adnan Nayfeh and Mr. Munir Nayfeh for their comments and encouragement throughout the writing of this book. I am indebted to several colleagues for helpful comments and criticism, including in particular Drs. D. T. Mook, D. P. Telionis, A. A. Kamel, and B. H. Stephan and Messers O. R. Asfar and M. S. Tsai. This book would not have been written without the patience and encouragement of my wife, and the insistence of my parents Hasan and Khadrah, in spite of their illiteracy, that I acquire a higher education. Therefore I dedicate this book to my parents and wife.

ALI HASAN NAYFEH

Blacksburg, Virginia
May 1972

CONTENTS

CHAPTER 1

Introduction

Most of the physical problems facing engineers, physicists, and applied mathematicians today exhibit certain essential features which preclude exact analytical solutions. Some of these features are nonlinearities, variable coefficients, complex boundary shapes, and nonlinear boundary conditions at known or, in some cases, unknown boundaries. Even if the exact solution of a problem can be found explicitly, it may be useless for mathematical and physical interpretation or numerical evaluation. Examples of such problems are Bessel functions of large argument and large-order and doubly periodic functions. Thus, in order to obtain information about solutions of equations, we are forced to resort to approximations, numerical solutions, or combinations of both. Foremost among the approximation methods are perturbation (asymptotic) methods which are the subject of this book. According to these techniques, the solution is represented by the first few terms of an asymptotic expansion, usually not more than two terms. The expansions may be carried out in terms of a parameter (small or large) which appears naturally in the equations, or which may be artificially introduced for convenience. Such expansions are called *parameter perturbations*. Alternatively, the expansions may be carried out in terms of a coordinate (either small or large); these are called *coordinate perturbations*. Examples of parameter and coordinate expansions and their essential characteristics are presented in Sections 1.1 and 1.2. To formalize the concepts of limits and error estimates, definitions of order symbols and other notations are introduced in Section 1.3. Section 1.4 contains definitions of an asymptotic expansion, an asymptotic sequence, and a power series, while Section 1.5 presents a comparison of convergent and asymptotic series. Uniform and nonuniform asymptotic expansions are then defined in Section 1.6. A short summary of operations with asymptotic expansions is given in Section 1.7.

1.1. Parameter Perturbations

Many physical problems involving the function $u(x, \epsilon)$ can be represented mathematically by the differential equation $L(u, x, \epsilon) = 0$ and the boundary condition $B(u, \epsilon) = 0$, where x is a scalar or vector independent variable and ϵ is a parameter. In general, this problem cannot be solved exactly. However, if there exists an $\epsilon = \epsilon_0$ (ϵ can be scaled so that $\epsilon_0 = 0$) for which the above problem can be solved exactly or more readily, one seeks to find the solution for small ϵ in, say, powers of ϵ; that is

$$u(x; \epsilon) = u_0(x) + \epsilon u_1(x) + \epsilon^2 u_2(x) + \cdots \tag{1.1.1}$$

where u_n is independent of ϵ and $u_0(x)$ is the solution of the problem for $\epsilon = 0$. One then substitutes this expansion into $L(u, x, \epsilon) = 0$ and $B(u, \epsilon) = 0$, expands for small ϵ, and collects coefficients of each power of ϵ. Since these equations must hold for all values of ϵ, each coefficient of ϵ must vanish independently because sequences of ϵ are linearly independent. These usually are simpler equations governing u_n, which can be solved successively. This is demonstrated in the next two examples.

1.1.1. AN ALGEBRAIC EQUATION

Let us consider first the solution of the algebraic equation

$$u = 1 + \epsilon u^3 \tag{1.1.2}$$

for small ϵ. If $\epsilon = 0$, $u = 1$. For ϵ small, but different from zero, we let

$$u = 1 + \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + \cdots \tag{1.1.3}$$

and (1.1.2) becomes

$$\epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + \cdots = \epsilon(1 + \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + \cdots)^3 \tag{1.1.4}$$

Expanding for small ϵ, we rewrite (1.1.4) as

$$\epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + \cdots = \epsilon[1 + 3\epsilon u_1 + 3\epsilon^2(u_2 + u_1^2) + \cdots] \tag{1.1.5}$$

Collecting coefficients of like powers of ϵ, we have

$$\epsilon(u_1 - 1) + \epsilon^2(u_2 - 3u_1) + \epsilon^3(u_3 - 3u_2 - 3u_1^2) + \cdots = 0 \tag{1.1.6}$$

Since this equation is an identity in ϵ, each coefficient of ϵ vanishes independently. Thus

$$u_1 - 1 = 0 \tag{1.1.7}$$

$$u_2 - 3u_1 = 0 \tag{1.1.8}$$

$$u_3 - 3u_2 - 3u_1^2 = 0 \tag{1.1.9}$$

The solution of (1.1.7) is

$$u_1 = 1 \tag{1.1.10}$$

Then the solution of (1.1.8) is

$$u_2 = 3u_1 = 3 \tag{1.1.11}$$

and the solution of (1.1.9) is

$$u_3 = 3u_2 + 3u_1^2 = 12 \tag{1.1.12}$$

Therefore (1.1.3) becomes

$$u = 1 + \epsilon + 3\epsilon^2 + 12\epsilon^3 + \cdots \tag{1.1.13}$$

where the ellipsis dots stand for all terms with powers of ϵ^n for which $n \geq 4$. Thus (1.1.13) is an approximation to the solution of (1.1.2), which is equal to 1 when $\epsilon \equiv 0$.

1.1.2. THE VAN DER POL OSCILLATOR

As a second example, we consider van der Pol's (1922) equation

$$\frac{d^2u}{dt^2} + u = \epsilon(1 - u^2)\frac{du}{dt} \tag{1.1.14}$$

for small ϵ. If $\epsilon = 0$ this equation reduces to

$$\frac{d^2u}{dt^2} + u = 0 \tag{1.1.15}$$

with the general solution

$$u = a\cos(t + \varphi) \tag{1.1.16}$$

where a and φ are constants. To determine an improved approximation to the solution of (1.1.14), we seek a perturbation expansion of the form

$$u(t;\epsilon) = u_0(t) + \epsilon u_1(t) + \epsilon^2 u_2(t) + \cdots \tag{1.1.17}$$

where the ellipsis dots stand for terms proportional to powers of ϵ greater than 2. Substituting this expansion into (1.1.14), we have

$$\begin{aligned} &\frac{d^2u_0}{dt^2} + u_0 + \epsilon\left(\frac{d^2u_1}{dt^2} + u_1\right) + \epsilon^2\left(\frac{d^2u_2}{dt^2} + u_2\right) + \cdots \\ &\quad = \epsilon[1 - (u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots)^2]\left[\frac{du_0}{dt} + \epsilon\frac{du_1}{dt} + \epsilon^2\frac{du_2}{dt} + \cdots\right] \end{aligned} \tag{1.1.18}$$

Expanding for small ϵ, we obtain

$$\begin{aligned} &\frac{d^2u_0}{dt^2} + u_0 + \epsilon\left(\frac{d^2u_1}{dt^2} + u_1\right) + \epsilon^2\left(\frac{d^2u_2}{dt^2} + u_2\right) + \cdots \\ &\quad = \epsilon(1 - u_0^2)\frac{du_0}{dt} + \epsilon^2\left[(1 - u_0^2)\frac{du_1}{dt} - 2u_0u_1\frac{du_0}{dt}\right] + \cdots \end{aligned} \tag{1.1.19}$$

Since u_n is independent of ϵ and (1.1.19) is valid for all small values of ϵ, the coefficients of like powers of ϵ must be the same on both sides of this equation. Equating the coefficients of like powers of ϵ on both sides of (1.1.19), we have

Coefficient of ϵ^0

$$\frac{d^2u_0}{dt^2} + u_0 = 0 \tag{1.1.20}$$

Coefficient of ϵ

$$\frac{d^2u_1}{dt^2} + u_1 = (1 - u_0^2)\frac{du_0}{dt} \tag{1.1.21}$$

Coefficient of ϵ^2

$$\frac{d^2u_2}{dt^2} + u_2 = (1 - u_0^2)\frac{du_1}{dt} - 2u_0u_1\frac{du_0}{dt} \tag{1.1.22}$$

Note that (1.1.20) is the same as (1.1.15) and its general solution is given by (1.1.16); that is

$$u_0 = a\cos(t + \varphi) \tag{1.1.23}$$

Substituting for u_0 into (1.1.21) gives

$$\frac{d^2u_1}{dt^2} + u_1 = -[1 - a^2\cos^2(t + \varphi)]a\sin(t + \varphi)$$

Using the trigonometric identity

$$\cos^2(t + \varphi)\sin(t + \varphi) = \frac{\sin(t + \varphi) + \sin 3(t + \varphi)}{4}$$

we can rewrite this equation as

$$\frac{d^2u_1}{dt^2} + u_1 = \frac{a^3 - 4a}{4}\sin(t + \varphi) + \tfrac{1}{4}a^3\sin 3(t + \varphi) \tag{1.1.24}$$

Its particular solution is

$$u_1 = -\frac{a^3 - 4a}{8}t\cos(t + \varphi) - \tfrac{1}{32}a^3\sin 3(t + \varphi) \tag{1.1.25}$$

With u_0 and u_1 known the right-hand side of (1.1.22) is known, and one can solve it for u_2 in a similar fashion. The usefulness of such an expansion is the subject of this book.

1.2. Coordinate Perturbations

If the physical problem is represented mathematically by a differential equation $L(u, x) = 0$ subject to the boundary conditions $B(u) = 0$, where x

is a scalar, and if $u(x)$ takes a known form u_0 as $x \to x_0$ (x_0 is scaled to 0 or ∞), one attempts to determine the deviation of u from u_0 for x near x_0 in terms of powers of x if $x_0 = 0$, or x^{-1} if $x_0 = \infty$. This technique is demonstrated by the next two examples.

1.2.1. THE BESSEL EQUATION OF ZEROTH ORDER

We consider the solutions of

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0 \tag{1.2.1}$$

This equation has a regular singular point at $x = 0$, which suggests that a power series solution for y can be obtained using the method of Frobenius (e.g., Ince, 1926, Section 16.1). Thus we let

$$y = \sum_{m=0}^{\infty} a_m x^{\mu+m} \tag{1.2.2}$$

where the number μ and the coefficients a_m must be determined so that (1.2.2) is a solution of (1.2.1).

Substituting (1.2.2) into (1.2.1) gives

$$\sum_{m=0}^{\infty}(\mu + m)(\mu + m - 1)a_m x^{\mu+m-1} + \sum_{m=0}^{\infty}(\mu + m)a_m x^{\mu+m-1} + \sum_{m=0}^{\infty} a_m x^{\mu+m+1} = 0$$

or

$$\sum_{m=0}^{\infty}(\mu + m)^2 a_m x^{\mu+m-1} + \sum_{m=0}^{\infty} a_m x^{\mu+m+1} = 0 \tag{1.2.3}$$

which can be written as

$$\mu^2 a_0 x^{\mu-1} + (\mu + 1)^2 a_1 x^{\mu} + \sum_{m=2}^{\infty}(\mu + m)^2 a_m x^{\mu+m-1} + \sum_{m=0}^{\infty} a_m x^{\mu+m+1} = 0$$

Replacing m by $m + 2$ in the first summation of this equation, we can rewrite it as

$$\mu^2 a_0 x^{\mu-1} + (\mu + 1)^2 a_1 x^{\mu} + \sum_{m=0}^{\infty}[(\mu + m + 2)^2 a_{m+2} + a_m]x^{\mu+m+1} = 0 \tag{1.2.4}$$

Since (1.2.4) is an identity in x, the coefficient of each power of x must vanish independently; that is

$$\mu^2 a_0 = 0 \tag{1.2.5}$$

$$(\mu + 1)^2 a_1 = 0 \tag{1.2.6}$$

$$(\mu + m + 2)^2 a_{m+2} + a_m = 0, \qquad m = 0, 1, 2, \ldots \tag{1.2.7}$$

The first equation demands that $\mu = 0$ if $a_0 \neq 0$; then (1.2.6) gives $a_1 = 0$ and (1.2.7) gives

$$a_{m+2} = -\frac{a_m}{(\mu + m + 2)^2}, \qquad m = 0, 1, 2, \ldots \tag{1.2.8}$$

Therefore

$$a_{2m+1} = 0, \qquad m = 1, 2, 3, \ldots,$$

$$a_2 = -\frac{a_0}{2^2}, \qquad a_4 = \frac{a_0}{2^2 \cdot 4^2}, \qquad a_6 = -\frac{a_0}{2^2 \cdot 4^2 \cdot 6^2}$$

$$a_{2n} = (-1)^n \frac{a_0}{2^2 \cdot 4^2 \cdot 6^2 \cdots (2n)^2} \tag{1.2.9}$$

The solution thus obtained if $a_0 = 1$ is a Bessel function of zeroth order, and it is often denoted by J_0. Thus

$$J_0 = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots + (-1)^n \frac{x^{2n}}{2^2 \cdot 4^2 \cdot 6^2 \cdots (2n)^2} + \cdots \tag{1.2.10}$$

Since the ratio of the nth term to the $(n - 1)$th term is $-x^2/(2n)^2$ and tends to zero as $n \to \infty$ irrespective of the value and sign of x, the series (1.2.10) for J_0 converges uniformly and absolutely for all values of x.

An expansion valid for large values of x is obtained in Section 7.1.2 and compared with the above expansion in Section 1.5.

1.2.2. A SIMPLE EXAMPLE

As a second example, we consider the solution of

$$\frac{dy}{dx} + y = \frac{1}{x} \tag{1.2.11}$$

for large x. For large x we seek a solution in the form

$$y = \sum_{m=1}^{\infty} a_m x^{-m} \tag{1.2.12}$$

Substituting this expansion into (1.2.11) yields

$$\sum_{m=1}^{\infty} -m a_m x^{-m-1} + \sum_{m=2}^{\infty} a_m x^{-m} + (a_1 - 1)x^{-1} = 0 \tag{1.2.13}$$

Replacing m by $m + 1$ in the second summation series, we can rewrite this equation as

$$(a_1 - 1)x^{-1} + \sum_{m=1}^{\infty}(a_{m+1} - ma_m)x^{-m-1} = 0 \tag{1.2.14}$$

Since this equation is an identity in x, the coefficient of each x^{-m} must vanish independently; that is

$$a_1 = 1, \qquad a_{m+1} = ma_m \quad \text{for} \quad m \geq 1 \tag{1.2.15}$$

Hence

$$a_2 = 1, \qquad a_3 = 2!, \qquad a_4 = 3!, \qquad a_n = (n-1)!$$

and (1.2.12) becomes

$$y = \frac{1}{x} + \frac{1!}{x^2} + \frac{2!}{x^3} + \frac{3!}{x^4} + \cdots + \frac{(n-1)!}{x^n} + \cdots \tag{1.2.16}$$

Since the ratio of the nth to the $(n-1)$th term is $(n-1)x^{-1}$ and it tends to infinity as $n \to \infty$ irrespective of the value of x, the series (1.2.16) diverges for all values of x. In spite of its divergence, this series is shown in Section 1.4 to be useful for numerical calculations, and it is called an *asymptotic series.*

1.3. Order Symbols and Gauge Functions

Suppose we are interested in a function of the single real parameter ϵ, denoted by $f(\epsilon)$. In carrying out our approximations, we are interested in the limit of $f(\epsilon)$ as ϵ tends to zero, denoted by $\epsilon \to 0$. This limit might depend on whether ϵ tends to zero from below, denoted by $\epsilon \uparrow 0$, or from above, denoted by $\epsilon \downarrow 0$. If the limit of $f(\epsilon)$ exists (i.e., it does not have an essential singularity at $\epsilon = 0$ such as $\sin \epsilon^{-1}$), then there are three possibilities

$$\left.\begin{aligned} f(\epsilon) &\to 0 \\ f(\epsilon) &\to A \\ f(\epsilon) &\to \infty \end{aligned}\right\} \quad \text{as} \quad \epsilon \to 0, 0 < A < \infty \tag{1.3.1}$$

In the first and last cases, the rate at which $f(\epsilon) \to 0$ and $f(\epsilon) \to \infty$ is expressed by comparing $f(\epsilon)$ with known functions called *gauge functions.* The simplest and most useful of these are

$$\ldots, \epsilon^{-n}, \ldots, \epsilon^{-2}, \epsilon^{-1}, 1, \epsilon, \epsilon^2, \ldots, \epsilon^n, \ldots$$

In some cases these must be supplemented by

$$\log \epsilon^{-1}, \log(\log \epsilon^{-1}), e^{\epsilon^{-1}}, e^{-\epsilon^{-1}}, \text{and so on}$$

Other gauge functions are

$$\sin \epsilon, \cos \epsilon, \tan \epsilon, \sinh \epsilon, \cosh \epsilon, \tanh \epsilon, \text{and so on}$$

The behavior of a function $f(\epsilon)$ is compared with a gauge function $g(\epsilon)$ as $\epsilon \to 0$, employing either of the Landau symbols: O or o.

The Symbol O

We write

$$f(\epsilon) = O[g(\epsilon)] \quad \text{as} \quad \epsilon \to 0 \tag{1.3.2}$$

if there exists a positive number A independent of ϵ and an $\epsilon_0 > 0$ such that

$$|f(\epsilon)| \le A\,|g(\epsilon)| \quad \text{for all} \quad |\epsilon| \le \epsilon_0 \tag{1.3.3}$$

This condition can be replaced by

$$\lim_{\epsilon \to 0} \left| \frac{f(\epsilon)}{g(\epsilon)} \right| < \infty \tag{1.3.4}$$

For example, as $\epsilon \to 0$

$$\begin{aligned}
&\sin \epsilon = O(\epsilon), && \sin \epsilon^2 = O(\epsilon^2)\\
&\sin 7\epsilon = O(\epsilon), && \sin 2\epsilon - 2\epsilon = O(\epsilon^3)\\
&\cos \epsilon = O(1), && 1 - \cos \epsilon = O(\epsilon^2)\\
&J_0(\epsilon) = O(1), && J_0(\epsilon) - 1 = O(\epsilon^2)\\
&\sinh \epsilon = O(\epsilon), && \cosh \epsilon = O(1)\\
&\tanh \epsilon = O(\epsilon), && \tan \epsilon = O(\epsilon)\\
&\coth \epsilon = O(\epsilon^{-1}), && \cot \epsilon = O(\epsilon^{-1})
\end{aligned}$$

If f is a function of another variable x in addition to ϵ, and $g(x, \epsilon)$ is a gauge function, we also write

$$f(x, \epsilon) = O[g(x, \epsilon)] \quad \text{as} \quad \epsilon \to 0 \tag{1.3.5}$$

if there exists a positive number A independent of ϵ and an $\epsilon_0 > 0$ such that

$$|f(x, \epsilon)| \le A\,|g(x, \epsilon)| \quad \text{for all} \quad |\epsilon| \le \epsilon_0 \tag{1.3.6}$$

If A and ϵ_0 are independent of x, the relationship (1.3.5) is said to hold *uniformly*. For example

$$\sin(x + \epsilon) = O(1) = O[\sin(x)] \quad \text{uniformly as} \quad \epsilon \to 0$$

while

$$e^{-\epsilon t} - 1 = O(\epsilon) \quad \text{nonuniformly as} \quad \epsilon \to 0$$

$$\sqrt{x + \epsilon} - \sqrt{x} = O(\epsilon) \quad \text{nonuniformly as} \quad \epsilon \to 0$$

The Symbol o

We write

$$f(\epsilon) = o[g(\epsilon)] \quad \text{as} \quad \epsilon \to 0 \tag{1.3.7}$$

if for every positive number δ, independent of ϵ, there exists an ϵ_0 such that

$$|f(\epsilon)| \le \delta\, |g(\epsilon)| \quad \text{for} \quad |\epsilon| \le \epsilon_0 \tag{1.3.8}$$

This condition can be replaced by

$$\lim_{\epsilon\to 0} \left| \frac{f(\epsilon)}{g(\epsilon)} \right| = 0 \tag{1.3.9}$$

Thus as $\epsilon \to 0$

$$\begin{array}{lll}
\sin \epsilon = o(1), & \sin \epsilon^2 = o(\epsilon) & \\
\cos \epsilon = o(\epsilon^{-1/2}), & J_0(\epsilon) = o(\epsilon^{-1}) & \\
\coth \epsilon = o(\epsilon^{-3/2}), & \cot \epsilon = o[\epsilon^{-(n+1)/n}] & \text{for positive } n \\
1 - \cos 3\epsilon = o(\epsilon), & \exp(-\epsilon^{-1}) = o(\epsilon^n) & \text{for all } n
\end{array}$$

If $f = f(x, \epsilon)$ and $g = g(x, \epsilon)$, then (1.3.7) is said to hold uniformly if δ and ϵ_0 are independent of x. For example

$$\sin(x + \epsilon) = o(\epsilon^{-1/3}) \quad \text{uniformly as} \quad \epsilon \to 0$$

while

$$e^{-\epsilon t} - 1 = o(\epsilon^{1/2}) \quad \text{nonuniformly as} \quad \epsilon \to 0$$

$$\sqrt{x + \epsilon} - \sqrt{x} = o(\epsilon^{3/4}) \quad \text{nonuniformly as} \quad \epsilon \to 0$$

1.4. Asymptotic Expansions and Sequences

1.4.1. ASYMPTOTIC SERIES

We found in Section 1.2.2 that a particular solution of

$$\frac{dy}{dx} + y = \frac{1}{x} \tag{1.4.1}$$

is

$$y = \frac{1}{x} + \frac{1!}{x^2} + \frac{2!}{x^3} + \frac{3!}{x^4} + \cdots + \frac{(n-1)!}{x^n} + \cdots \tag{1.4.2}$$

which diverges for all values of x. To investigate whether this series is of any value for computing a particular solution of our equation, we determine the remainder if we truncate the series after n terms. To do this we note that a particular integral of our differential equation is given by

$$y = e^{-x}\int_{-\infty}^{x} x^{-1}e^{x}\,dx \tag{1.4.3}$$

which converges for negative x. Integrating (1.4.3) by parts, we find that

$$\begin{aligned} y &= \frac{1}{x} + e^{-x}\int_{-\infty}^{x} x^{-2}e^{x}\,dx = \frac{1}{x} + \frac{1}{x^2} + 2e^{-x}\int_{-\infty}^{x} x^{-3}e^{x}\,dx \\ &= \frac{1}{x} + \frac{1}{x^2} + \frac{2}{x^3} + 3!\,e^{-x}\int_{-\infty}^{x} x^{-4}e^{x}\,dx \\ &= \frac{1}{x} + \frac{1!}{x^2} + \frac{2!}{x^3} + \frac{3!}{x^4} + \cdots + \frac{(n-1)!}{x^n} \\ &\quad + n!\,e^{-x}\int_{-\infty}^{x} x^{-n-1}e^{x}\,dx \end{aligned} \tag{1.4.4}$$

Therefore if we truncate the series after n terms, the remainder is

$$R_n = n!\,e^{-x}\int_{-\infty}^{x} x^{-n-1}e^{x}\,dx \tag{1.4.5}$$

which is a function of n and x. For the series to converge, $\lim_{n\to\infty} R_n$ must be zero. This is not true in our example; in fact, $R_n \to \infty$ as $n \to \infty$ so that the series diverges for all x in agreement with what we found in Section 1.2.2 using the ratio test. Therefore, if the series (1.4.2) is to be useful, n must be fixed. For negative x

$$|R_n| \le n!\,|x^{-n-1}|\,e^{-x}\int_{-\infty}^{x} e^{x}\,dx = \frac{n!}{|x^{n+1}|} \tag{1.4.6}$$

Thus the error committed in truncating the series after n terms is numerically less than the first neglected term, namely, the $(n + 1)$th term. Moreover, as $|x| \to \infty$ with n fixed, $R_n \to 0$. Therefore, although the series (1.4.2) diverges, *for a fixed n the first n terms in the series can represent y with an error which can be made arbitrarily small by taking $|x|$ sufficiently large.* Such a series is called an *asymptotic series of the Poincaré type* (Poincaré, 1892) and is denoted by

$$y \sim \sum_{n=1}^{\infty}\frac{(n-1)!}{x^n} \quad \text{as} \quad |x| \to \infty \tag{1.4.7}$$

In general, given a series $\sum_{m=0}^{\infty} (a_m/x^m)$, where a_m is independent of x, we say that the series is an *asymptotic series* and write

$$y \sim \sum_{m=0}^{\infty} \frac{a_m}{x^m} \quad \text{as} \quad |x| \to \infty \tag{1.4.8}$$

if and only if

$$y = \sum_{m=0}^{n} \frac{a_m}{x^m} + o(|x|^{-n}) \quad \text{as} \quad |x| \to \infty \tag{1.4.9}$$

The condition (1.4.9) can be rewritten as

$$y = \sum_{m=0}^{n-1} \frac{a_m}{x^m} + O(|x|^{-n}) \quad \text{as} \quad |x| \to \infty \tag{1.4.10}$$

As another example of an asymptotic series, we consider, after Euler (1754), the evaluation of the integral

$$f(\omega) = \omega \int_0^\infty \frac{e^{-x}}{\omega + x}\, dx \tag{1.4.11}$$

for large positive ω. Since

$$\frac{\omega}{\omega + x} = \sum_{m=0}^{\infty} \frac{(-1)^m x^m}{\omega^m} \quad \text{if} \quad x < \omega \tag{1.4.12}$$

and

$$\int_0^\infty x^m e^{-x}\, dx = m! \tag{1.4.13}$$

$$f(\omega) = \sum_{m=0}^{\infty} \frac{(-1)^m m!}{\omega^m} \tag{1.4.14}$$

Since the ratio of the mth to the $(m - 1)$th term, $-m\omega^{-1}$, tends to infinity as $m \to \infty$, the series (1.4.14) diverges for all values of ω.

To investigate whether (1.4.14) is an asymptotic series, we estimate the remainder if the series is truncated after the nth term. To do this we note that

$$\frac{\omega}{\omega + x} = \sum_{m=0}^{n-1} \frac{(-1)^m x^m}{\omega^m} + \frac{(-1)^n x^n}{\omega^{n-1}(\omega + x)} \tag{1.4.15}$$

Hence

$$f(\omega) = \sum_{m=0}^{n-1} \frac{(-1)^m m!}{\omega^m} + R_n \tag{1.4.16}$$

where

$$|R_n| = \left| \frac{(-1)^n}{\omega^{n-1}} \int_0^\infty \frac{x^n e^{-x}}{\omega + x}\, dx \right| < \frac{1}{\omega^n} \int_0^\infty x^n e^{-x}\, dx = \frac{n!}{\omega^n} \tag{1.4.17}$$

Hence the error committed by truncating the series after the first n terms is numerically less than the first neglected term, and

$$f(\omega) = \sum_{m=0}^{n-1} \frac{(-1)^m m!}{\omega^m} + O(\omega^{-n}) \tag{1.4.18}$$

Therefore the series (1.4.14) is an asymptotic series, and we write

$$f(\omega) \sim \sum_{m=0}^{\infty} \frac{(-1)^m m!}{\omega^m} \tag{1.4.19}$$

1.4.2. ASYMPTOTIC EXPANSIONS

One does not need to use a power series to represent a function. Instead, one can use a general sequence of functions $\delta_n(\epsilon)$ as long as

$$\delta_n(\epsilon) = o[\delta_{n-1}(\epsilon)] \quad \text{as} \quad \epsilon \to 0 \tag{1.4.20}$$

Such a sequence is called an *asymptotic sequence*. Examples of such asymptotic sequences are

$$\epsilon^n,\ \epsilon^{n/3},\ (\log \epsilon)^{-n},\ (\sin \epsilon)^n,\ (\cot \epsilon)^{-n} \tag{1.4.21}$$

In terms of asymptotic sequences, we can define asymptotic expansions. Thus, given $\sum_{m=0}^{\infty} a_m \delta_m(\epsilon)$ where a_m is independent of ϵ and $\delta_m(\epsilon)$ is an asymptotic sequence, we say that this expansion is an *asymptotic expansion* and write

$$y \sim \sum_{m=0}^{\infty} a_m \delta_m(\epsilon) \quad \text{as} \quad \epsilon \to 0 \tag{1.4.22}$$

if and only if

$$y = \sum_{m=0}^{n-1} a_m \delta_m(\epsilon) + O[\delta_n(\epsilon)] \quad \text{as} \quad \epsilon \to 0 \tag{1.4.23}$$

Clearly, an asymptotic series is a special case of an asymptotic expansion.

As an example of an asymptotic expansion that is not an asymptotic power series, we return to the integral (1.4.11). Following van der Corput (1962), we represent $f(\omega)$ in terms of the factorial asymptotic sequence $[(\omega + 1)(\omega + 2) \cdots (\omega + n)]^{-1}$ as $\omega \to \infty$. To do this we note that

$$\begin{aligned}
\frac{1}{\omega + x} &= \frac{1}{\omega} - \frac{x}{\omega(\omega + x)} \\
&= \frac{1}{\omega} - \frac{x}{\omega(\omega + 1)} + \frac{x(x-1)}{\omega(\omega + 1)(\omega + x)} \\
&= \frac{1}{\omega} - \frac{x}{\omega(\omega + 1)} + \frac{x(x-1)}{\omega(\omega + 1)(\omega + 2)} - \frac{x(x-1)(x-2)}{\omega(\omega + 1)(\omega + 2)(\omega + x)}
\end{aligned} \tag{1.4.24}$$

In general

$$\frac{\omega}{\omega + x} = \sum_{m=0}^{n} \frac{(-1)^m x(x-1)\cdots(x+1-m)}{(\omega+1)(\omega+2)\cdots(\omega+m)} + \frac{(-1)^{n+1}x(x-1)\cdots(x-n)}{(\omega+1)(\omega+2)\cdots(\omega+n)(\omega+x)} \tag{1.4.25}$$

This equation can be proved by induction as follows. If (1.4.25) is valid for n, we show that it is valid for $(n + 1)$. To do this we note that

$$\begin{aligned}\frac{\omega}{\omega + x} = \sum_{m=0}^{n} \frac{(-1)^m x(x-1)\cdots(x+1-m)}{(\omega+1)(\omega+2)\cdots(\omega+m)} &+ \frac{(-1)^{n+1}x(x-1)\cdots(x-n)}{(\omega+1)(\omega+2)\cdots(\omega+n+1)} \\ &- \frac{(-1)^{n+1}x(x-1)\cdots(x-n)}{(\omega+1)(\omega+2)\cdots(\omega+n+1)} \\ &+ \frac{(-1)^{n+1}x(x-1)\cdots(x-n)}{(\omega+1)(\omega+2)\cdots(\omega+n)(\omega+x)}\end{aligned}$$

By combining the last two terms and extending the summation to $n + 1$, we can rewrite this expression as

$$\frac{\omega}{\omega + x} = \sum_{m=0}^{n+1} \frac{(-1)^m x(x-1)\cdots(x+1-m)}{(\omega+1)(\omega+2)\cdots(\omega+m)} + (-1)^{n+2}\frac{x(x-1)\cdots(x-n-1)}{(\omega+1)(\omega+2)\cdots(\omega+n+1)(\omega+x)} \tag{1.4.26}$$

Thus if (1.4.25) is true for n, (1.4.26) shows that it is true for $n + 1$. Since (1.4.25) is true for $n = 0$, 1, and 2 according to (1.4.24), it is true for $n = 3$, 4, 5, ... Therefore it is true for all n.

Multiplying (1.4.25) by $\exp(-x)$ and integrating from $x = 0$ to $x = \infty$, we have

$$f(\omega) = \sum_{m=0}^{n} a_m \delta_m(\omega) + R_n(\omega) \tag{1.4.27}$$

where

$$a_m = \int_0^\infty x(x-1)\cdots(x-m+1)e^{-x}\,dx \tag{1.4.28}$$

$$\delta_m(\omega) = (-1)^m[(\omega+1)(\omega+2)\cdots(\omega+m)]^{-1} \tag{1.4.29}$$

$$R_n = -\delta_n(\omega)\int_0^\infty \frac{x(x-1)\cdots(x-n)}{\omega+n+x-n}\,e^{-x}\,dx \tag{1.4.30}$$

Since ω is a positive large number

$$|R_n| < |\delta_n(\omega)| \left| \int_0^\infty x(x-1)\cdots(x-n+1)e^{-x}\,dx \right|$$
$$= |a_n|\,|\delta_n(\omega)| \tag{1.4.31}$$

Thus the error committed by keeping the first n terms is numerically less than the nth term, hence

$$f(\omega) = \sum_{m=0}^{n-1} a_m \delta_m(\omega) + O[\delta_n(\omega)] \tag{1.4.32}$$

Since $\delta_m(\omega)$ is an asymptotic sequence as $\omega \to \infty$

$$f(\omega) \sim \sum_{m=0}^{\infty} a_m \delta_m(\omega) \quad \text{as} \quad \omega \to \infty \tag{1.4.33}$$

1.4.3. UNIQUENESS OF ASYMPTOTIC EXPANSIONS

We have shown in the previous two sections that

$$f(\omega) \sim \sum_{m=0}^{\infty} \frac{(-1)^m m!}{\omega^m} \quad \text{as} \quad \omega \to \infty \tag{1.4.34}$$

and

$$f(\omega) \sim \sum_{m=0}^{\infty} \frac{(-1)^m \int_0^\infty x(x-1)\cdots(x+1-m)e^{-x}\,dx}{(\omega+1)(\omega+2)\cdots(\omega+m)} \quad \text{as} \quad \omega \to \infty \tag{1.4.35}$$

Thus the asymptotic representation of $f(\omega)$ as $\omega \to \infty$ is not unique. In fact, $f(\omega)$ can be represented by an infinite number of asymptotic expansions because there exists an infinite number of asymptotic sequences that can be used in the representation. However, given an asymptotic sequence $\delta_m(\omega)$, the representation of $f(\omega)$ in terms of this sequence is unique. In this case

$$f(\omega) \sim \sum_{m=0}^{\infty} a_m \delta_m(\omega) \quad \text{as} \quad \omega \to \infty \tag{1.4.36}$$

where the a_m are uniquely given by

$$a_0 = \lim_{\omega\to\infty} \frac{f(\omega)}{\delta_0(\omega)}, \qquad a_1 = \lim_{\omega\to\infty} \frac{f(\omega) - a_0\delta_0(\omega)}{\delta_1(\omega)}$$
$$a_n = \lim_{\omega\to\infty} \frac{f(\omega) - \sum_{m=0}^{n-1} a_m\delta_m(\omega)}{\delta_n(\omega)} \tag{1.4.37}$$

1.5. Convergent versus Asymptotic Series

We found in Section 1.2.1 that one of the solutions of Bessel's equation

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0 \tag{1.5.1}$$

is given by the series

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots + (-1)^n \frac{x^{2n}}{2^2 \cdot 4^2 \cdots (2n)^2} + \cdots \tag{1.5.2}$$

which is uniformly and absolutely convergent for all values of x.

Another representation of J_0 can be obtained if we note that the change of variable

$$y = x^{-1/2} y_1 \tag{1.5.3}$$

transforms (1.5.1) into

$$\frac{d^2y_1}{dx^2} + \left(1 + \frac{1}{4x^2}\right) y_1 = 0 \tag{1.5.4}$$

As $x \to \infty$, this equation tends to

$$\frac{d^2y_1}{dx^2} + y_1 = 0 \tag{1.5.5}$$

with the solutions

$$y_1 = e^{\pm ix} \tag{1.5.6}$$

This suggests the transformation

$$y_1 = e^{ix} y_2 \tag{1.5.7}$$

which gives

$$\frac{d^2y_2}{dx^2} + 2i\frac{dy_2}{dx} + \frac{1}{4x^2} y_2 = 0 \tag{1.5.8}$$

This equation can be satisfied formally by

$$y_2 = 1 - \frac{1}{8x} i - \frac{1 \cdot 3^2}{8^2 \cdot 2! \cdot x^2} + \frac{1 \cdot 3^2 \cdot 5^2}{8^3 \cdot 3! \cdot x^3} i + \frac{1 \cdot 3^2 \cdot 5^2 \cdot 7^2}{8^4 \cdot 4! \cdot x^4} + \cdots \tag{1.5.9}$$

By combining this series with that obtained by changing i into $-i$, we obtain the following two independent solutions

$$\begin{aligned} y^{(1)} &\sim x^{-1/2}(u \cos x + v \sin x) \\ y^{(2)} &\sim x^{-1/2}(u \sin x - v \cos x) \end{aligned} \tag{1.5.10}$$

where

$$u = 1 - \frac{1 \cdot 3^2}{8^2 \cdot 2! \cdot x^2} + \frac{1 \cdot 3^2 \cdot 5^2 \cdot 7^2}{8^4 \cdot 4! \cdot x^4} + \cdots$$
$$v = \frac{1}{8x} - \frac{1 \cdot 3^2 \cdot 5^2}{8^3 \cdot 3! \cdot x^3} + \cdots \tag{1.5.11}$$

To determine the connection between $J_0(x)$ and these two independent solutions, we use the integral representation

$$\pi J_0(x) = \int_0^\pi \cos(x \cos\theta)\, d\theta \tag{1.5.12}$$

and obtain (see Section 7.1.2)

$$J_0(x) \sim \sqrt{\frac{2}{\pi x}}\,[u \cos(x - \tfrac{1}{4}\pi) + v \sin(x - \tfrac{1}{4}\pi)] \tag{1.5.13}$$

The ratio test shows that y_2, u, and v, and hence the right-hand side of (1.5.13), are divergent for all values of x. However, for large x the leading terms in u and v decrease rapidly with increasing rank so that (1.5.13) is an asymptotic expansion for large x.

For small x the first few terms of (1.5.2) give fairly accurate results. In fact, the first 9 terms give a value for $J_0(2)$ correct to 11 significant figures. However, as x increases, the number of terms needed to yield the same accuracy increases rapidly. At $x = 4$, eight terms are needed to give an accuracy of three significant figures, whereas the first term of the asymptotic expansion (1.5.13) yields the same accuracy. As x increases further, an accurate result is obtained with far less labor by using the asymptotic divergent series (1.5.13).

1.6. Nonuniform Expansions

In parameter perturbations the quantities to be expanded can be functions of one or more variables besides the perturbation parameter. If we develop the asymptotic expansion of a function $f(x; \epsilon)$, where x is a scalar or vector variable independent of ϵ, in terms of the asymptotic sequence $\delta_m(\epsilon)$, we have

$$f(x; \epsilon) \sim \sum_{m=0}^{\infty} a_m(x)\delta_m(\epsilon) \quad \text{as} \quad \epsilon \to 0 \tag{1.6.1}$$

where the coefficients a_m are functions of x only. This expansion is said to be *uniformly valid* if

$$f(x; \epsilon) = \sum_{m=0}^{N-1} a_m(x)\delta_m(\epsilon) + R_N(x; \epsilon) \tag{1.6.2a}$$

$$R_N(x; \epsilon) = O[\delta_N(\epsilon)] \qquad \text{uniformly for all } x \text{ of interest} \tag{1.6.2b}$$