APPLIED NONLINEAR DYNAMICS

Analytical, Computational, and Experimental Methods

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APPLIED NONLINEAR DYNAMICS

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To our wives

Samirah and Sundari

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PREFACE

Systems that can be modeled by nonlinear algebraic and/or nonlinear differential equations are called **nonlinear systems**. Examples of such systems occur in many disciplines of engineering and science. In this book, we deal with the dynamics of nonlinear systems. Poincaré (1899) studied nonlinear dynamics in the context of the n-body problem in celestial mechanics. Besides developing and illustrating the use of perturbation methods, Poincaré presented a geometrically inspired qualitative point of view.

In the nineteenth and twentieth centuries, many pioneering contributions were made to nonlinear dynamics. A partial list includes those due to Rayleigh, Duffing, van der Pol, Lyapunov, Birkhoff, Krylov, Bogoliubov, Mitropolski, Levinson, Kolomogorov, Andronov, Arnold, Pontryagin, Cartwright, Littlewood, Smale, Bowen, Piexoto, Ruelle, Takens, Hale, Moser, and Lorenz. While studying forced oscillations of the van der Pol oscillator, Cartwright and Littlewood (1945) observed a constrained random-like behavior, which is now called **chaos**. Subsequently, Lorenz (1963) studied a deterministic, third-order system in the context of weather dynamics and showed through numerical simulations that this deterministic system displayed random-like behavior too. Unaware of Lorenz's work, Smale (1967) introduced the horseshoe map as an abstract prototype to explain chaos-like behavior. No doubt Poincaré knew about chaos too, but it is only through numerical simulations on modern computers and experiments with physical systems that the presence of chaos has been discovered to be pervasive in many dynamical systems of physical interest. The observation of Poincaré that small differences in the initial conditions may produce great changes in the final phenomena is now known to be a characteristic of systems that exhibit chaotic behavior. The phenomenon of chaos, which has become very popular now, rejuvenated interest in nonlinear dynamics. The growing numbers of books and research papers published in the last two decades reflect a strong interest in nonlinear dynamics at the present time. The many important contributions that have been made through analytical, experimental, and numerical studies have been documented through many books, including those by Collet and Eckmann (1980), Mees (1981), Sparrow (1982), Guckenheimer and Holmes (1983), Lichtenberg and Lieberman (1983, 1992), Bergé, Pomeau, and Vidal (1984), Holden (1986), Kaneko (1986), Thompson and Stewart (1986), Moon (1987, 1992), Arnold (1988), Barnsley (1988), Schuster (1988), Seydel (1988), Wiggins (1988, 1990), Devaney (1989), Jackson (1989, 1990), Nicolis and Prigogine (1989), Parker and Chua (1989), Ruelle (1989a, 1989b), Tabor (1989), Arrowsmith and Place (1990), Baker and Gollub (1990), El Naschie (1990), Rasband (1990), Hale and Kocak (1991), Schroeder (1991), Troger and Steindl (1991), Drazin (1992), Kim and Stringer (1992), Medvéd (1992), Tufillaro, Abbott, and Reilly (1992), Ueda (1992), Mullin (1993), Ott (1993), Palis and Takens (1993), and Ott, Sauer, and Yorke (1994).

We are of the opinion that the books on nonlinear dynamics published thus far have a strong bias toward analytical methods, or experimental methods, or numerical methods. As these methods are complementary to each other, a person being taught nonlinear dynamics should be provided with a flavor of all the different methods. This is one of the intentions in writing this book. Another intention was to include some of the recent developments in the area of control of nonlinear dynamics of systems. In Chapter 1, we introduce dynamical systems. In Chapters 2–5, we address equilibrium solutions, periodic and quasiperiodic solutions, and chaos. We present some relevant theorems and their implications in Chapters 2 and 3. Proofs are not provided in this book, but references that provide them are included. Further, these chapters are not written within a mathematically rigorous framework. Continuation methods for equilibrium and periodic solutions are also presented in some detail in Chapter 6. We examine the different tools that can be used to characterize nonlinear motions in Chapter 7. In Chapter 8, we discuss methods for bifurcation control, chaos control, and synchronization to chaos.

The authors are deeply indebted to several colleagues for helpful comments and criticisms, including, in particular, Professor Sherif Noah and his students, Dr. Marwan Bikdash, Mr. Haider Arafat, Mr. Samir A. Nayfeh, Mr. Ghaleb Abdallah, Professors Jose Baltezar, Anil Bajaj, Eyad Abed, Dean Mook, and Muhammad Hajj. One of us (BB) would like to thank Professors Davinder Anand and Patrick Cunniff of the University of Maryland for their encouragement and support during the final stages of preparation of this book. We wish to thank Dr. Char–Ming Chin for generating many of the figures dealing with crises, intermittency, and Shilnikov chaos. Thanks are due also to fifteen year old Nader Nayfeh for scanning, editing, and preparing the eps files for all the illustrations in this book. Last but not least, we wish to thank Mrs. Sally G. Shrader for her patient typing of the drafts of the manuscript and fine preparation of the final camera–ready copy of this book.

Ali H. Nayfeh
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October 1994

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Chapter 1

INTRODUCTION

A **dynamical system** is one whose state evolves (changes) with time t. The evolution is governed by a set of rules (not necessarily equations) that specifies the state of the system for either discrete or continuous values of t. A **discrete–time evolution** is usually described by a system of algebraic equations (map), while a **continuous–time evolution** is usually described by a system of differential equations.

The asymptotic behavior of a dynamical system as $t \to \infty$ is called the **steady state** of the system. Often, this steady state may correspond to a bounded set, which may be either a static solution or a dynamic solution. The behavior of the dynamical system prior to reaching the steady state is called the **transient state**, and the corresponding solution of the dynamical system is called the **transient solution**.

A solution of a dynamical system can be either constant or time varying. **Fixed points**, **equilibrium solutions**, and **stationary solutions** are other names for constant solutions, while **dynamic solutions** is another name for time–varying solutions. We explore equilibrium solutions in Chapter 2 and dynamic solutions in Chapters 3–5. In Sections 1.1 and 1.2, we explain the notion of a dynamical system. In Section 1.3, we discuss attracting sets, and in Sections 1.4 and 1.5, we examine the concepts of stability and attractors.

1.1 DISCRETE–TIME SYSTEMS

A discrete–time evolution is governed by

$$\mathbf{x}_{k+1} = \mathbf{F}(\mathbf{x}_k) \tag{1.1.1}$$

where $\mathbf{x}$ is a finite–dimensional vector. At the discrete times t_k and t_{k+1}, $\mathbf{x}_k$ and $\mathbf{x}_{k+1}$ represent the states of the system, respectively. Let the dimension of the finite–dimensional state vector be n. Then, we need n real numbers to specify the state of the system. Formally, the state vector $\mathbf{x} \in \mathcal{R}^n$ and the time $t \in \mathcal{R}$, where the symbol $\in$ means **belongs to** and the symbol $\mathcal{R}^n$ refers to an n–dimensional **Euclidean space**; that is, a real–number space equipped with the **Euclidean norm**

$$\| \mathbf{x} \| = \sqrt{(x_1^2 + x_2^2 + \cdots + x_n^2)} \tag{1.1.2}$$

where the x_i are the scalar components of $\mathbf{x}$. If the discrete values of time correspond to integers rather than real numbers, we say that $t \in \mathcal{Z}$, where $\mathcal{Z}$ is the set of all integers. We note that the evolution of a dynamical system may also be studied in other spaces, such as cylindrical, toroidal, and spherical spaces. In these cases, one or more state variables are angular coordinates. However, according to topological concepts, local regions of these spaces have the structure of a Euclidean space.

Equation (1.1.1) is a **transformation** or a **map** that transforms the current state of the system to the subsequent state. In the literature, the words **map**, **mapping**, and **function** are often used interchangeably. To a certain extent, the words **set** and **space** are also used interchangeably. Formally, a map $\mathbf{F}$ from points in a region M to points in a region N is represented by $\mathbf{F} : M \to N$. We note that M and N are contained in $\mathcal{R}^n$. Formally, $M \subset \mathcal{R}^n$ and $N \subset \mathcal{R}^n$, where the symbol $\subset$ is called the **subset operator** and means **inclusion**. The map $\mathbf{F}$ is said to map M **onto** N if for every point $\mathbf{y} \in N$ there exists at least one point $\mathbf{x} \in M$ that is mapped to $\mathbf{y}$ by $\mathbf{F}$. Furthermore, $\mathbf{F}$ is said to be **one–to–one** if no two points in M map to the same point in N. A map that is one–to–one and onto is **invertible** (e.g., Dugundji, 1966, Chapter I); that is, given $\mathbf{x}_{k+1}$, we can solve (1.1.1) to determine $\mathbf{x}_k$ uniquely. Denoting the inverse of $\mathbf{F}$ in (1.1.1) by $\mathbf{F}^{-1}$, we have

$$\mathbf{x}_k = \mathbf{F}^{-1}(\mathbf{x}_{k+1})$$

The map $\mathbf{F}^{-1}$ is also onto and one–to–one. A map $\mathbf{F}$ that is not invertible is called a **noninvertible map**.

When each of the scalar components of $\mathbf{F}$ is r times continuously differentiable with respect to the scalar components of $\mathbf{x}$, $\mathbf{F}$ is said to be a C^r function. When each of the scalar components of $\mathbf{F}$ is continuous with respect to the scalar components of $\mathbf{x}, \mathbf{F}$ is said to be a C^0 function. For $r \geq 1$, the map $\mathbf{F}$ is called a **differentiable map**. The map $\mathbf{F}$ is called a **homeomorphism** if it is invertible and both $\mathbf{F}$ and $\mathbf{F}^{-1}$ are continuous; that is, $\mathbf{F}$ is C^0. If both $\mathbf{F}$ and $\mathbf{F}^{-1}$ are C^r functions where $r \geq 1$, then we call the map a C^r **diffeomorphism**. In subsequent chapters, we discuss what are called **Poincaré maps**. These maps, which are discretized versions of associated systems of ordinary–differential equations, are diffeomorphisms. In one discretized version, a Poincaré map describes the evolution of a system for discrete values of time. The other cases are discussed in detail in Chapters 3, 4, 5, and 7.

An **orbit of an invertible map** initiated at $\mathbf{x} = \mathbf{x}_0$ is made up of the discrete points

$$\Big\{\cdots,\ \mathbf{F}^{-m}(\mathbf{x}_0),\ \cdots,\ \mathbf{F}^{-2}(\mathbf{x}_0),\ \mathbf{F}^{-1}(\mathbf{x}_0),\ \mathbf{x}_0,\ \mathbf{F}(\mathbf{x}_0),\ \mathbf{F}^2(\mathbf{x}_0),\ \cdots,\ \mathbf{F}^m(\mathbf{x}_0),\ \cdots\Big\}$$

where $m \in \mathcal{Z}^+$ and $\mathcal{Z}^+$ is the set of all positive integers. When $k > 0$, $\mathbf{F}^k$ means the kth successive application of the map $\mathbf{F}$. Similarly, when $k < 0$, $\mathbf{F}^k$ means the kth successive application of the map $\mathbf{F}^{-1}$. An **orbit of a noninvertible map** initiated at $\mathbf{x} = \mathbf{x}_0$ is made up of the discrete points

$$\Big\{\mathbf{x}_0,\ \mathbf{F}(\mathbf{x}_0),\ \mathbf{F}^2(\mathbf{x}_0),\ \cdots,\ \mathbf{F}^m(\mathbf{x}_0),\ \cdots\Big\}$$

Successive applications of $\mathbf{F}$ are also referred to as the **forward iterates** of the corresponding map.

With reference to (1.1.1), we note that $\mathbf{F}$ is also called an **evolution operator**. Sometimes, we wish to study the evolution as we change or control a certain set of parameters $\mathbf{M}$. To make this explicit, we write the map as

$$\mathbf{x}_{k+1} = \mathbf{F}(\mathbf{x}_k; \mathbf{M}) \tag{1.1.3}$$

where $\mathbf{M}$ is the vector of control parameters.

Example 1.1. For illustration, we consider the one-dimensional map

$$x_{k+1} = 4\alpha x_k(1 - x_k) \tag{1.1.4}$$

where $0 \leq x_k \leq 1$ and $0 < \alpha \leq 1$. For $\alpha = 0.50$, the orbit of the map initiated at $x_0 = 0.25$ is

$$\{0.25,\ 0.375,\ 0.46875,\ \cdots\}$$

Equation (1.1.4) is the famous **logistic map**, which has been the subject of many studies (e.g., May, 1976). This map is a noninvertible map because it is not a one-to-one map. In fact, this map is a **two-to-one map** because it maps the two points x and $(1-x)$ to the same point $4\alpha x(1-x)$. Further, (1.1.4) is an example of a differentiable map.

Example 1.2. We consider the **Hénon map** (Hénon, 1976)

$$x_{k+1} = 1 + y_k - \alpha x_k^2 \tag{1.1.5}$$

$$y_{k+1} = \beta x_k \tag{1.1.6}$$

where α and β are scalar parameters. When $\beta = 0$, (1.1.5) and (1.1.6) reduce to the one-dimensional map

$$x_{k+1} = 1 - \alpha x_k^2$$

which is noninvertible. It is called the **quadratic map**. However, when $\beta \neq 0$, the map (1.1.5) and (1.1.6) is invertible. The inverse is

$$x_k = \frac{1}{\beta} y_{k+1}$$

$$y_k = x_{k+1} - 1 + \frac{\alpha}{\beta^2} y_{k+1}^2$$

We note that $\{x_k \;\; y_k\}^T$ uniquely determines $\{x_{k+1} \;\; y_{k+1}\}^T$ and vice versa. Further, because both $\mathbf{F}$ and $\mathbf{F}^{-1}$ are differentiable, the Hénon map is a diffeomorphism when $\beta \neq 0$. For $\alpha = 0.2$ and $\beta = 0.3$, the orbit of the map initiated at

$$\left\{ \begin{array}{c} x_0 \\ y_0 \end{array} \right\} = \left\{ \begin{array}{c} 1.0 \\ 0.0 \end{array} \right\}$$

is

$$\left\{ \ldots, \left\{ \begin{array}{c} -3.33 \\ 1.22 \end{array} \right\}, \left\{ \begin{array}{c} 0.0 \\ -1.0 \end{array} \right\}, \left\{ \begin{array}{c} 0.0 \\ 0.0 \end{array} \right\}, \right.$$
$$\left. \left\{ \begin{array}{c} 1.0 \\ 0.0 \end{array} \right\}, \left\{ \begin{array}{c} 0.8 \\ 0.3 \end{array} \right\}, \left\{ \begin{array}{c} 1.17 \\ 0.24 \end{array} \right\}, \left\{ \begin{array}{c} 0.97 \\ 0.35 \end{array} \right\}, \ldots \right\}$$

In Figure 1.1.1, we show some of the discrete points that make up the orbit of (x_0, y_0).

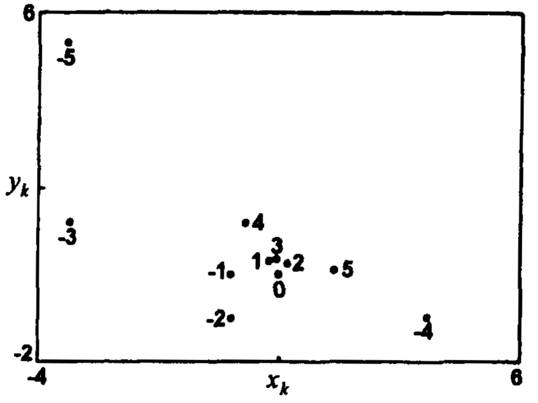

Figure 1.1.1: Some of the discrete points that make up the orbit of (1, 0) of the Hénon map for $\alpha = 0.2$ and $\beta = 0.3$. The index k associated with each point is also shown.

We note that the dynamics of many Poincaré maps show qualitative similarities to the dynamics of the logistic and Hénon maps.

1.2 CONTINUOUS–TIME SYSTEMS

For continuous values of time, the evolution of a system is governed by either an autonomous or a nonautonomous system of differential equations.

1.2.1 Nonautonomous Systems

In the nonautonomous case, the equations are of the form

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, t) \tag{1.2.1}$$

where $\mathbf{x}$ is finite dimensional, $\mathbf{x} \in \mathcal{R}^n$, $t \in \mathcal{R}$, and $\mathbf{F}$ explicitly depends on t. The vector $\mathbf{F}$ is often referred to as **vector field**, the vector $\mathbf{x}$ is called a **state vector** because it describes the state of the system, and the space $\mathcal{R}^n$ in which $\mathbf{x}$ evolves is called a **state space**. A state space is called a **phase space** when one–half of the states are displacements and the other one–half are velocities. The $(n+1)$–dimensional space $\mathcal{R}^n \times \mathcal{R}^1$, where the additional dimension corresponds to t, is often referred to as an **extended state space**. In (1.2.1), if $\mathbf{F}$ is a linear function of $\mathbf{x}$ it is called a **linear vector field**, and if $\mathbf{F}$ is a nonlinear function of $\mathbf{x}$ it is called a **nonlinear vector field.**

Let the initial state of the system at time t_0 be $\mathbf{x}_0$, and let I represent a time interval that includes t_0. Then one can think of a solution of (1.2.1) as a map from different points in I into different points in the n–dimensional state space $\mathcal{R}^n$. A graph of a solution of (1.2.1) in the extended state space is known as an **integral curve**. On an integral curve, the vector function $\mathbf{F}$ specifies the tangent vector (velocity vector) at every point $(\mathbf{x}, t)$. A geometric interpretation of a **vector field** is that it is a collection of tangent vectors on different integral curves.

In general, a projection of a solution $\mathbf{x}(t, t_0, \mathbf{x}_0)$ of (1.2.1) onto the n–dimensional state space is referred to as a **trajectory** or an **orbit** of the system through the point $\mathbf{x} = \mathbf{x}_0$. In other words, the solution could be thought of as a point that moves along a trajectory, occupying different positions at different times similar to the way a planet moves through space. We use the symbol $\gamma(\mathbf{x}_0)$ or Γ to denote an orbit. The orbit obtained for times $t \geq 0$ passing through the point $\mathbf{x}_0$ at $t = 0$ is called a **positive orbit** and is denoted by $\gamma^+(\mathbf{x}_0)$; the orbit obtained for times $t \leq 0$ is called a **negative orbit** and is denoted by $\gamma^-(\mathbf{x}_0)$. Also, $\Gamma = \gamma(\mathbf{x}_0) = \gamma^+(\mathbf{x}_0) \cup \gamma^-(\mathbf{x}_0)$, where the symbol $\cup$ stands for the union operator.

Example 1.3. For illustration, we consider the following periodically forced linear oscillator:

$$\ddot{x} + 2\mu\dot{x} + \omega^2 x = F\cos(\Omega t)$$

Letting $x = x_1$ and $\dot{x} = x_2$, we express this second-order equation as a system of two first-order equations in terms of the state variables x_1 and x_2. The result is

$$\dot{x}_1 = x_2 \tag{1.2.2}$$

$$\dot{x}_2 = -\omega^2 x_1 - 2\mu x_2 + F\cos(\Omega t) \tag{1.2.3}$$

For $\omega^2 = 8$, $\mu = 2$, $F = 10$, and $\Omega = 2$, the solution of (1.2.2) and (1.2.3) is

$$x_1 = e^{-2t}\left[a\cos(2t) + b\sin(2t)\right] + 0.5\cos(2t) + \sin(2t)$$

$$x_2 = -2e^{-2t}\left[(a-b)\cos(2t) + (a+b)\sin(2t)\right] - \sin(2t) + 2\cos(2t)$$

where the constants a and b are determined by the initial condition (x_{10}, x_{20}). We note that as $t \to \infty$, the exponential term decays to zero. Therefore, the steady state does not depend on the initial condition. In Figure 1.2.1a, we show an integral curve initiated at $(x_{10}, x_{20}, t_0) = (1, 0, 0)$ in the (x_1, x_2, t) space for $0 \leq t \leq 10$. The arrows on the curve indicate the direction of evolution for positive times. The tangent vector is also shown at two different locations on the integral curve. It should be noted that the apparent intersections in Figure 1.2.1a are a consequence of the chosen viewing angle. In

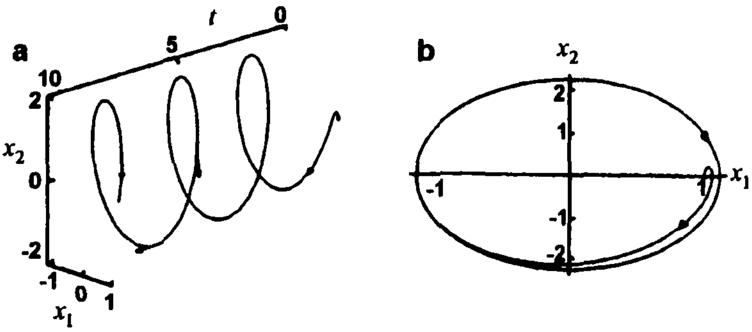

Figure 1.2.1: Solution of (1.2.2) and (1.2.3) initiated from (1,0) at $t = 0$ for $\omega^2 = 8$, $\mu = 2$, $F = 10$, and $\Omega = 2$: (a) integral curve and (b) positive orbit.

Figure 1.2.1b, we show a projection of the integral curve onto the two-dimensional (x_1, x_2) space. This projection is a positive orbit of $(x_{10}, x_{20}) = (1, 0)$.

Again, we remind the reader that besides Euclidean state spaces there are other state spaces, such as cylindrical, toroidal, and spherical spaces. In Figure 1.2.2a, we show a **cylindrical space**. A motion evolving in this space is described by two Cartesian coordinates and an angular coordinate θ. One of the Cartesian coordinates is defined along the cylinder's axis, while the other one is defined along the radius of its cross-section. This cylindrical space is represented by $\mathcal{R}^2 \times S^1$. The variable θ belongs to the space S and is such that $0 \leq \theta < 2\pi$; formally, $\theta \in [0, 2\pi)$. A **toroidal space** is shown in Figure 1.2.2b. Specifically, we call this object a **two–torus**, and a dynamical system evolving in this space is described by two angular coordinates θ_1 and θ_2. We represent this space by $S^1 \times S^1$. One would require n angular coordinates to describe the motion evolving on an n–**torus**. A **spherical space** is shown in Figure 1.2.2c. We need two angular coordinates to describe a motion evolving on the spherical surface.

A local region of the cylindrical, toroidal, or spherical surface of Figure 1.2.2 has the appearance of a flat surface and can be treated as a two-dimensional Euclidean space. Smooth and continuous surfaces, such as those shown in Figure 1.2.2, are called **manifolds**. Manifolds can be thought of as generalized surfaces. (The reader is referred to

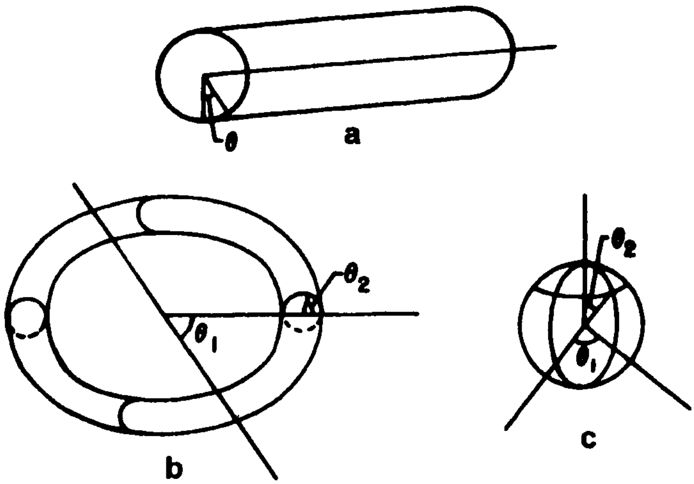

Figure 1.2.2: Different spaces: (a) cylindrical space, (b) toroidal space, and (c) spherical space.

Guillemin and Pollack (1974) for a precise description of a manifold.) In a two-dimensional space, a smooth object, such as a circle, is an example of a manifold, but an object with sharp corners, such as a rectangle, is not an example of a manifold. Locally, the circle may be approximated by a tangent line. Similarly, local regions of toroidal and spherical surfaces can be approximated by tangent planes. We note that an open flat surface is also a manifold.

Returning to (1.2.1), we note that this equation is also referred to as an **evolution equation.** Let the evolution of the system described by this equation be controlled by a set of parameters **M**. To make this parameter dependence explicit, we describe the evolution by

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, t; \mathbf{M}) \tag{1.2.4}$$

where **M** is a vector of control parameters. Formally, $\mathbf{M} \in \mathcal{R}^m$, and the vector function **F** can be represented as $\mathbf{F} : \mathcal{R}^n \times \mathcal{R}^1 \times \mathcal{R}^m \rightarrow \mathcal{R}^n$.

Next, we state some facts from the theory of ordinary-differential equations. If the scalar components of **F** are $\mathcal{C}^0$ (i.e., continuous) in a domain D of $(\mathbf{x}, t)$ space, then a solution $\mathbf{x}(t, \mathbf{x}_0, t_0)$ satisfying the condition $\mathbf{x} = \mathbf{x}_0$ at $t = t_0$ exists in a small time interval around t_0 in D. Moreover, if the scalar components of $\mathbf{F}$ are C^1 in D, then the solution $\mathbf{x}(t, \mathbf{x}_0, t_0)$ is unique in a small time interval around t_0. The uniqueness of solutions is also assured in certain cases where $\mathbf{F}$ is C^0 (Coddington and Levinson, 1955, Chapter 1; Arnold, 1973, 1992, Chapters 2 and 4). If the existence and uniqueness of solutions of a system of the form (1.2.4) are ensured, then this system is **deterministic**. This means that two integral curves starting from two different initial conditions cannot intersect each other in the extended state space. However, the corresponding orbits may intersect each other in the corresponding state space.

If the scalar components of $\mathbf{F}$ are C^r functions of t and the scalar components of $\mathbf{x}$ and $\mathbf{M}$, then a solution of (1.2.4) satisfying the initial condition $\mathbf{x} = \mathbf{x}_0$ at $t = t_0$ is also a C^r function of t, t_0, $\mathbf{x}_0$, and $\mathbf{M}$ in a small interval around t_0. Moreover, if a solution of (1.2.4) originating at a certain initial condition exists for all times, then this solution can be extended indefinitely. If a solution exists and is defined only over a finite interval of time, then this solution starting from a location in this interval can be extended up to the boundaries of this interval (Arnold, 1973, 1992, Chapters 2 and 4).

Example 1.4. This system is an example of a deterministic dynamical system. The parameter values used to generate Figure 1.2.3 are the same as those used to generate Figure 1.2.1. In Figure 1.2.3, we graphically show the solutions of (1.2.2) and (1.2.3) initiated at $t = 0$ from (1.0, 0.0) and (1.5, 0.0). From Figure 1.2.3a, we note that the corresponding integral curves do not intersect each other anywhere in the (x_1, x_2, t) space. As in Figure 1.2.1, the apparent intersections in Figure 1.2.3a are a consequence of the chosen viewing angle. From the previous discussion of Example 1.3, it is clear that as $t \to \infty$, both integral curves converge to the steady state

$$x_1 = 0.5\cos(2t) + \sin(2t)$$
$$x_2 = -\sin(2t) + 2\cos(2t)$$

Although the two integral curves coincide only at $t = \infty$, on the scales of Figure 1.2.3a they are not distinguishable after about $t = 2.5$ units. In

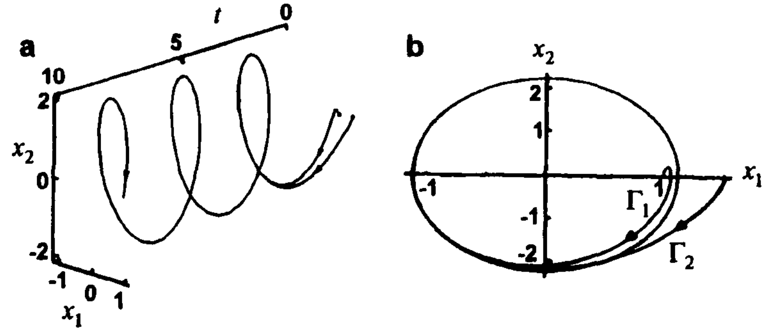

Figure 1.2.3: Solutions of (1.2.2) and (1.2.3) initiated from (1.0,0.0) and (1.5,0.0) at $t = 0$ for $\omega^2 = 8$, $\mu = 2$, $F = 10$, and $\Omega = 2$: (a) integral curves and (b) positive orbits. Γ_1 and Γ_2 are the positive orbits of (1.0,0.0) and (1.5,0.0), respectively.

Figure 1.2.3b, the positive orbits initiated from (1.0, 0.0) and (1.5, 0.0) are shown. We note the presence of a transverse intersection close to (0.7, −2.0) in Figure 1.2.3b.

1.2.2 Autonomous Systems

In the case of an autonomous system, the equations are of the form

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}; \mathbf{M}) \tag{1.2.5}$$

where $\mathbf{x}$, $\mathbf{F}$, and $\mathbf{M}$ are as defined before. Here, $\mathbf{F}$ does not explicitly depend on the independent variable t and can be represented by the map $\mathbf{F} : \mathcal{R}^n \times \mathcal{R}^m \to \mathcal{R}^n$. Hence, the system (1.2.5) is **time invariant, time independent**, or **stationary**. This means that if $\mathbf{X}(t)$ is a solution of (1.2.5), then $\mathbf{X}(t + \tau)$ is also a solution of (1.2.5) for any arbitrary τ. If the scalar components of $\mathbf{F}$ have continuous and bounded first partial derivatives with respect to the scalar components of $\mathbf{x}$, then the system (1.2.5) has a unique solution for a given initial condition $\mathbf{x}_0$. As a consequence, no two trajectories or orbits of an autonomous system can intersect each other in the n-dimensional state space of the system. Moreover, if the vector field $\mathbf{F}$ is a C^r function of $\mathbf{x}$ and $\mathbf{M}$,

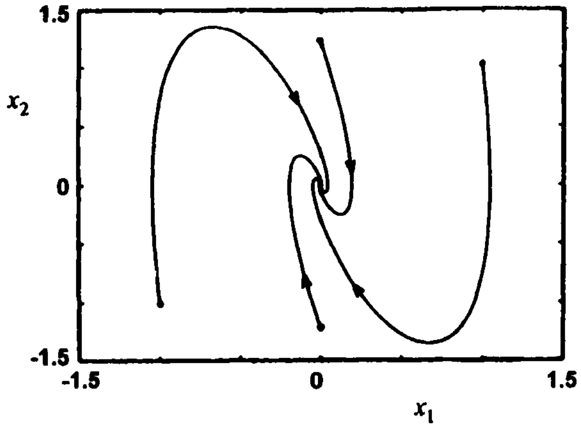

Figure 1.2.4: Positive orbits of (1.2.6) and (1.2.7) initiated at $t = 0$ from $(1.0, 1.0)$, $(0.0, -1.2)$, $(-1.0, -1.0)$, and $(0.0, 1.2)$ for $\omega^2 = 8$ and $\mu = 2$. All four orbits approach the origin as $t \to \infty$.

then the associated solution of (1.2.5) is also a C^r function of t, $\mathbf{x}$, and $\mathbf{M}$ (Arnold, 1973, 1992, Chapters 2 and 4).

Example 1.5. We consider the following autonomous system:

$$\dot{x}_1 = x_2 \tag{1.2.6}$$

$$\dot{x}_2 = -\omega^2 x_1 - 2\mu x_2 \tag{1.2.7}$$

In Figure 1.2.4, we show positive orbits of (1.2.6) and (1.2.7) initiated from four different initial conditions when $\omega^2 = 8$ and $\mu = 2$. These orbits do not intersect each other anywhere in the plane as they approach the origin, where they all meet. The direction of the orbits in the (x_1, x_2) space is given by

$$\frac{dx_2}{dx_1} = \frac{-(\omega^2 x_1 + 2\mu x_2)}{x_2}$$

which is well defined everywhere except at the origin. Hence, we call $(0, 0)$ a **singular point** of (1.2.6) and (1.2.7). Such solutions are discussed at length in Chapter 2.