QUANTUM FIELD THEORY

From Operators to Path Integrals

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Cambridge, Massachusetts

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To *Rosemay*

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Contents

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Preface

Quantum field theory, the quantum mechanics of continuous systems, arose at the beginning of the quantum era, in the problem of blackbody radiation. It became fully developed in quantum electrodynamics, the most successful theory in physics. Since that time, it has been united with statistical mechanics through Feynman's path integral, and its domain has been expanded to cover particle physics, condensed-matter physics, astrophysics, and wherever path integrals are spoken.

This book is a textbook on the subject, aimed at readers conversant with what is usually called "advanced quantum mechanics," the equivalent of a first-year graduate course. Previous exposure to the Dirac equation and "second quantization" would be very helpful, but not absolutely necessary. The mathematical level is not higher than what is required in advanced quantum mechanics; but a degree of maturity is assumed.

In physics, a continuous system is one that appears to be so at long wavelengths or low frequencies. To model it as mathematically continuous, one runs into difficulties, in that the high-frequency modes often give rise to infinities. The usual procedure is to start with a discrete version, by discarding the high-frequency modes beyond some cutoff, and then try to approach the continuum limit, through a process called *renormalization*.

Renormalization is a relatively new concept, but its workings were already evident in classical physics. At the beginning of the atomic era, Boltzmann noted that classical equipartition of energy presents conceptual difficulties, when one seriously considers the atomic structure of matter. Since atoms are expected to contain smaller subunits, which in turn should composed of even smaller subunits, and so *ad infinitum*, and each degree of freedom contributes equally to the thermal energy of a substance, the specific heat of matter would be infinite.. The origin of this divergence lies in the extrapolation of known physical laws into the high-frequency domain, a characteristic shared by the infinities in quantum field theory.

Boltzmann's "paradox," however, matters not a whit when it comes to practical calculations, as evidenced by the great success of classical physics. The reason is that most equations of macroscopic physics, such as those in thermodynamics and hydrodynamics, make no explicit reference to atoms, but depend on coefficients like the specific heat, which can be obtained from experiments. From a modern perspective, we say that such theories are "renormalizable," in that the microstructure can be absorbed into measurable quantities.

One goal of this book is to explain what renormalization is, how it works, and what makes some systems appear "renormalizable" and others not. We follow the historical route, discovering it in quantum electrodynamics through necessity, and then realizing its physical meaning through Wilson's path-integral formulation.

This book, then, starts with a thorough introduction to the usual operator formalism, including Feynman graphs, from Chapters 1–10. This is followed by Chapters 11–14 on quantum electrodynamics, which illustrates how to do practical calculations, and includes a complete discussion of perturbative renormalization. The last part, Chapters 15–19, introduces the Feynman path integral, and discusses "modern" subjects, including the physical approach to renormalization, spontaneous symmetry breaking, and topological excitations. I have entirely omitted non-Abelian gauge fields and the standard model of particle physics, because these subjects are discussed in another book: K. Huang, *Quarks, Leptons, and Gauge Field,* 2nd ed. (World Scientific, Singapore, 1992).

I have chosen to introduce path integrals only after the canonical approach is fully developed and applied. Others might want them discussed earlier. To accommodate different tastes, I have tried to make each chapter self-contained in as much as possible, so that a knowledgeable reader can pick and skip.

There is definitely a change in flavor when quantum field theory is conveyed through the path integral. Apart from the union with statistical mechanics, which immeasurably enriches the subject, it liberates our imagination by making it possible to contemplate virtual but fantastic deformations, such as altering the structure of space–time. I am reminded of the classification of things as "gray" or "green" by Freeman Dyson, in his book *Disturbing the Universe* (Harper & Row, New York, 1979). He classified physics gray (and I suppose that included quantum field theory,) as opposed to things green, such as poems and horse manure. In a private letter dated August 3, 1983, Dyson wrote, "Everyone has to make his own choice of what to call gray and green. I took my choice from Goethe:

Grau, tenererfreund, ist alle Theorie,
Und grün des Lebens Goldner Baum.

Dear friend, all theory is grey,
And green is the golden tree of life.

I must admit that Hilbert space does seem a bit dreary at times; but, with Feynman's path integral, quantum field theory has surely turned green.

KERSON HUANG

December, 1997
Marblehead, Massachusetts

Acknowledgments

I wish to thank my colleagues and students for helping me to learn the stuff in this book over the years, in particular Herman Feshbach, from whom I took the first course on the subject. I thank Jeffrey Goldstone, Roman Jackiw, and Ken Johnson, who can usually be relied on to provide correct and simple answers to difficult questions; and Patrick Lee, who taught a course with me on the subject, thereby broadening my horizon. Last but not least, I thank my editor Greg Franklin for his understanding and support.

K. H.

QUANTUM FIELD THEORY

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CHAPTER ONE

Introducing Quantum Fields

1.1 THE CLASSICAL STRING

We obtain a quantum field by quantizing a classical field, of which the simplest example is the classical string. To be on firm mathematical grounds, we define the latter as the long-wavelength limit of a discrete chain. Consider $N + 2$ masses described by the classical Lagrangian

$$L(q, \dot{q}) = \sum_{j=0}^{N+1} \left[\frac{m}{2} \dot{q}_j^2 - \frac{\kappa}{2} (q_j - q_{j+1})^2 \right] \tag{1.1}$$

where m is the mass and κ a force constant. The coordinate $q_j(t)$ represents the lateral displacement of the jth mass along a one-dimensional chain. We impose fixed-endpoint boundary conditions, by setting

$$q_0(t) = q_{N+1}(t) = 0 \tag{1.2}$$

The equations of motion for the N remaining movable masses are then

$$m\ddot{q}_j - \kappa(q_{j+1} - 2q_j + q_{j-1}) = 0 \qquad (j = 1, \ldots, N) \tag{1.3}$$

The normal modes have the form

$$q_j(t) = \cos(\omega t) \sin(jp) \tag{1.4}$$

To satisfy the boundary conditions, choose p to have one of the N possible values

$$p_n = \frac{\pi n}{N + 1} \qquad (n = 1, \ldots, N) \tag{1.5}$$

Substituting this into the equations of motion, we obtain N independent normal frequencies ω_n:

$$\omega_n^2 = \omega_0^2 \sin^2\left(\frac{\pi}{2}\frac{n}{N+1}\right) \qquad (n = 1 \cdots N) \tag{1.6}$$

where

$$\omega_0 = 2\sqrt{\frac{\kappa}{m}} \tag{1.7}$$

This is a cutoff frequency, for the modes with $n > N$ merely repeat the lower ones. For $N = 4$, for example, the independent modes correspond to $n = 1,2,3,4$. The case $n = 5$ is trivial, since $p = \pi$, and hence $q_j(t) = 0$ by (1.4). The case $n = 6$ is the same as that for $n = 4$, since $\omega_6 = \omega_4$, and $\sin(jp_6) = -\sin(jp_4)$.

When N is large, and we are not interested in the behavior near the endpoints, it is convenient to use periodic boundary conditions:

$$q_{j+N}(t) = q_j(t) \tag{1.8}$$

In this case the normal modes are

$$q_j(t) = e^{i(jp-\omega t)} \tag{1.9}$$

For N even, the boundary conditions can be satisfied by putting

$$p_n = \frac{2\pi n}{N} \qquad \left(n = 0, \pm 1, \ldots, \pm \frac{N}{2}\right) \tag{1.10}$$

The corresponding normal frequencies are

$$\omega_n^2 = \omega_0^2 \sin^2\left(\frac{\pi n}{N}\right) \tag{1.11}$$

Compared to the fixed-end case, the spacing between normal frequencies is now doubled; but each frequency is twofold degenerate, and the number of normal modes remains the same. A comparison of the two cases for $N = 8$ is shown in Fig. 1.1.

The equilibrium distance a between masses does not explicitly appear in the Lagrangian; it merely supplies a length scale for physical distances. For example, it appears in the definition of the distance of a mass from an end of the chain:

$$x \equiv ja \qquad (j = 1, \ldots, N) \tag{1.12}$$

The total length of the chain is then defined as

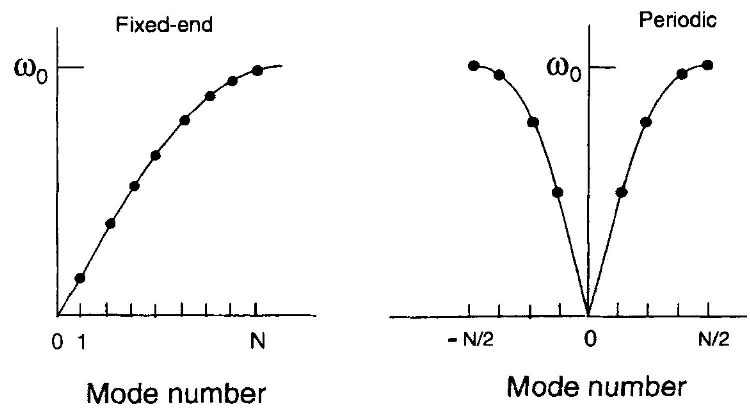

Figure 1.1 Normal modes of the classical chain for fixed-end and periodic boundary conditions.

$$R = Na \tag{1.13}$$

In the continuum limit

$$a \to 0 \qquad N \to \infty \qquad (R = Na \text{ fixed}) \tag{1.14}$$

the discrete chain approaches a continuous string, and the coordinate approaches a classical field defined by

$$q(x, t) \equiv q_j(t) \tag{1.15}$$

The Lagrangian in the continuum limit can be obtained by making the replacements

$$(q_{n+1} - q_j)^2 \to a^2 \left[\frac{\partial q(x, t)}{\partial x} \right]^2$$

$$\sum_j \to \frac{1}{a} \int_0^R dx \tag{1.16}$$

Assuming that the mass density ρ and string tension σ approach finite limits

$$\rho = \frac{m}{a} \tag{1.17}$$

$$\sigma = \kappa a \tag{1.18}$$

we obtain the limit Lagrangian

$$L_{\text{cont}} = \frac{1}{2}\int_0^R dx\left[\rho\left(\frac{\partial q(x,t)}{\partial t}\right)^2 - \sigma\left(\frac{\partial q(x,t)}{\partial x}\right)^2\right] \tag{1.19}$$

This leads to the equation of motion

$$\frac{\partial^2 q(x,t)}{\partial t^2} - \frac{1}{c^2}\frac{\partial^2 q(x,t)}{\partial x^2} = 0 \tag{1.20}$$

which is a wave equation, with propagation velocity

$$c = \sqrt{\frac{\sigma}{\rho}} \tag{1.21}$$

The general solutions are the real and imaginary parts of

$$q(x,t) = e^{i(kx-\omega t)} \tag{1.22}$$

with a linear dispersion law

$$\omega = ck \tag{1.23}$$

For fixed-end boundary conditions

$$q(0,t) = q(R,t) = 0 \tag{1.24}$$

the normal modes of the continuous string are

$$q_n(x,t) = \cos(\omega_n t)\sin(k_n x) \tag{1.25}$$

with $\omega_n = ck_n$, and

$$k_n = \frac{\pi n}{R} \qquad (n = 0,1,2,\ldots) \tag{1.26}$$

The normal frequencies ω_n are the same as those for the discrete chain for $n/N \ll 1$, as given in (1.6). However, the number of modes of the continuum string is infinite, and only the first N modes have correspondence with those of the discrete string. This is illustrated in Fig. 1.2 for $N = 4$. Thus, there is a cutoff frequency

$$\omega_c \equiv \omega_N = \frac{\pi c}{a} \tag{1.27}$$

This is of the same order, but not same as the maximum frequency defined earlier, $\omega_0 = 2c/a$, for ω_c is based on a linear dispersion law. The continuum model is an accurate representation of the discrete chain only for $\omega \ll \omega_c$.

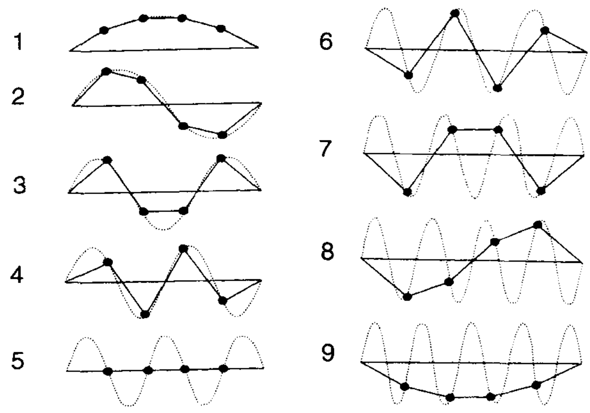

Figure 1.2 Normal modes of a discrete chain of four masses, compared with those of a continuous string. The former repeat themselves after the first four modes. (After J. C. Slater and N. H. Frank, Mechanics, McGraw-Hill, New York, 1947.)

For periodic boundary conditions

$$q(0, t) = q(R, t) \tag{1.28}$$

the allowed wave numbers are

$$k_n = \frac{2\pi n}{R} \qquad (n = 0, \pm 1, \pm 2, \ldots) \tag{1.29}$$

We obtain the cutoff frequency ω_c by setting $n = N/2$.

The high-frequency cutoff is a theoretical necessity. Without it, the specific heat of the string will diverge, since each normal mode contributes an amount kT. The value of the cutoff cannot be determined from the long-wavelength effective theory, because only the combination $c = a\omega_c/\pi$ occurs. Absorbing the cutoff into measurable parameters, as done in (1.17), is called *renormalization*. A theory for which this can be done is said to be *renormalizable*.

Nonrenormalizable systems exhibit behavior that is sensitive to details on an atomic scale. Such behavior would appear to be random on a macroscopic scale, as in the propagation of cracks in materials, and the nucleation of raindrops.

1.2 THE QUANTUM STRING

We now quantize the classical chain, to obtain a quantum field in the continuum limit. The Hamiltonian of the classical discrete chain is given by

$$H(p, q) = \sum_{j=1}^{N} \left[\frac{p_j^2}{2m} + \frac{\kappa}{2}(q_j - q_{j+1})^2 \right] \tag{1.30}$$

where $p_j = m\dot{q}_j$. The system can be quantized by replacing p_j and q_j by Hermitian operators satisfying the commutation relations

$$[p_j, q_k] = -i\delta_{jk} \tag{1.31}$$

We impose periodic boundary conditions, and expand these operators in Fourier series:

$$q_j = \frac{1}{\sqrt{N}} \sum_{n=-N/2}^{N/2} Q_n e^{i2\pi nj/N}$$

$$p_j = \frac{1}{\sqrt{N}} \sum_{n=-N/2}^{N/2} P_n e^{i2\pi nj/N} \tag{1.32}$$

where P_n and Q_n are operators satisfying

$$\begin{aligned} [P_n^\dagger, Q_m] &= -i\delta_{nm} \\ P_n^\dagger &= P_{-n} \\ Q_n^\dagger &= Q_{-n} \end{aligned} \tag{1.33}$$

The system is reduced to a sum of independent harmonic oscillators:

$$\begin{aligned} H &= \sum_{n=-N/2}^{N/2} \left[\frac{1}{2m} P_n^\dagger P_n + \frac{1}{2} m\omega_n^2 Q_n^\dagger Q_n \right] \\ \omega_n^2 &= \frac{4\kappa}{m} \sin^2\left(\frac{\pi n}{N} \right) \end{aligned} \tag{1.34}$$

The eigenvalues are labeled by a set of occupation numbers $\{\alpha_n\}$:

$$E_\alpha = \sum_{n=-N/2}^{N/2} \omega_n(\alpha_n + \tfrac{1}{2}) \tag{1.35}$$

where $\alpha_n = 0,1,2,\ldots$. The frequency ω_n is taken to be the positive root of ω_n^2, since H is positive-definite.

In the continuum limit (1.14) the Hamiltonian becomes

$$H_{\text{cont}} = \int_0^R dx \left[\frac{1}{2\rho} p^2(x, t) + \frac{\sigma}{2} \left(\frac{\partial q(x, t)}{\partial x} \right)^2 \right] \tag{1.36}$$

where, with $x = ja$,

$$p(x, t) = \frac{p_j(t)}{a} = \rho \frac{\partial q(x, t)}{\partial t} \tag{1.37}$$

The quantum field $q(x, t)$ and its canonical conjugate $p(x, t)$ satisfy the equal-time commutation relation

$$[p(x, t), q(x', t)] = -i\delta(x - x') \tag{1.38}$$

Just as in the classical case, we have to introduce a cutoff frequency ω_c. General properties of the quantum field will be discussed more fully in Chapter 2.

1.3 SECOND QUANTIZATION

Another way to obtain a quantum field is to consider a collection of identical particles in quantum mechanics. In this case, the quantum field is an equivalent description of the system. Identical particles are defined by a Hamiltonian that is (1) invariant under a permutation of the particle coordinates and (2) has the same form for any number of particles. The quantized-field description is called "second quantization" for historical reasons, but quantization was actually done only once.

Let $\mathcal{H}_N$ be the Hilbert space of a system of N identical nonrelativistic particles. The union of all $\mathcal{H}_N$ is called the Fock space:

$$\mathcal{F} = \bigcup_{N=0}^{\infty} \mathcal{H}_N \tag{1.39}$$

The subspace with $N = 0$ contains the vacuum state as its only member. We assume that N is the eigenvalues of a "number operator" N_{op}, which commutes with the Hamiltonian. It is natural to introduce operators on Fock space that connect subspaces of different N. An elementary operator of this kind creates or annihilates one particle at a point in space. Such an operator is a quantum field operator, since it is a spatial function. This is why a quantum-mechanical many-particle system automatically gives rise to a quantum field.

For definiteness, consider N nonrelativistic particles in three spatial dimensions, with coordinates $\{\mathbf{r}_1, \ldots, \mathbf{r}_N\}$. The Hamiltonian is

$$H = -\frac{1}{2m} \sum_{i=1}^{N} \nabla_i^2 + V(\mathbf{r}_1, \ldots, \mathbf{r}_N) \tag{1.40}$$

where ∇_i^2 is the Laplacian with respect to $\mathbf{r}_i$, and where V is a symmetric function of its arguments. The eigenfunctions Ψ_n are defined by

$$H\Psi_n(\mathbf{r}_1, \ldots, \mathbf{r}_N) = E_n \Psi_n(\mathbf{r}_1, \ldots, \mathbf{r}_N) \tag{1.41}$$

For Bose or Fermi statistics, Ψ_n is respectively symmetric or antisymmetric under an interchange of any two coordinates $\mathbf{r}_i$ and $\mathbf{r}_j$. The particles are called *bosons* or *fermions*, respectively.

We now describe the equivalent quantum field theory, and justify it later. Let $\psi(\mathbf{r})$ be the Schrödinger-picture operator that annihilates one particle at $\mathbf{r}$. Its Hermitian conjugate $\psi^\dagger(\mathbf{r})$ will create one particle at $\mathbf{r}$. They are defined through the commutation relations

$$[\psi(\mathbf{r}), \psi^\dagger(\mathbf{r}')]_\pm = \delta^3(\mathbf{r} - \mathbf{r}')$$
$$[\psi(\mathbf{r}), \psi(\mathbf{r}')]_\pm = 0 \tag{1.42}$$

where $[A,B]_\pm = AB \pm BA$, with the plus sign corresponding to bosons and the minus sign to fermions. The Fock-space Hamiltonian is defined in such a manner that it reduces to (1.40) in the N-particle subspace.

A general N-particle Hamiltonian has the structure

$$H = \sum_i f(\mathbf{r}_i) + \sum_{i<j} g(\mathbf{r}_i, \mathbf{r}_j) + \sum_{i<j<k} h(\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}_k) + \cdots \tag{1.43}$$

where the functions g, h, and so on are symmetric functions of their arguments. The first term is a "one-particle operator," a sum of operators of the form $f(\mathbf{r})$, which act on one particle only. The second term is a "two-particle operator," a sum of operators of the form $g(\mathbf{r}_1, \mathbf{r}_2)$, over all distinct pairs. Generally, an "n-particle operator" is a sum of operators that depend only on a set of n coordinates. To construct the Hamiltonian on Fock space, we associate an n-particle operator with an operator on Fock space, with the following correspondences:

$$\sum_i f(\mathbf{r}_i) \to \int d^3r\, \psi^\dagger(\mathbf{r}) f(\mathbf{r}) \psi(\mathbf{r})$$

$$\sum_{i<j} g(\mathbf{r}_i, \mathbf{r}_j) \to \frac{1}{2}\int d^3r_1\, d^3r_2\, \psi_1^\dagger \psi_2^\dagger g_{12} \psi_2 \psi_1$$

$$\sum_{i<j<k} h(\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}_k) \to \frac{1}{3!}\int d^3r_1\, d^3r_2\, d^3r_3\, \psi_1^\dagger \psi_2^\dagger \psi_3^\dagger h_{123} \psi_3 \psi_2 \psi_1$$

$$\vdots \tag{1.44}$$

where for brevity we have written $\psi_1 = \psi(\mathbf{r}_1)$, $g_{12} = g(\mathbf{r}_1, \mathbf{r}_2)$, and so on.

As an example, suppose the potential in (1.40) is a sum of two-body potentials:

$$V(\mathbf{r}_1, \ldots, \mathbf{r}_N) = \sum_{i<j} v(\mathbf{r}_i, \mathbf{r}_j) \tag{1.45}$$

Then the corresponding Fock-space Hamiltonian, also denoted H, takes the form