METHODS OF MATHEMATICAL PHYSICS

By R. COURANT and D. HILBERT

VOLUME II

PARTIAL DIFFERENTIAL EQUATIONS

By R. Courant

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METHODS OF MATHEMATICAL PHYSICS

VOLUME II

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TO

KURT OTTO FRIEDRICHS

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PREFACE

The present volume is concerned with the theory of partial differential equations, in particular with parts of this wide field that are related to concepts of physics and mechanics. Even with this restriction, completeness seems unattainable; to a certain extent the material selected corresponds to my personal experience and taste. The intention is to make an important branch of mathematical analysis more accessible by emphasizing concepts and methods rather than presenting a collection of theorems and facts, and by leading from an elementary level to key points on the frontiers of our knowledge

Almost forty years ago I discussed with David Hilbert the plan of a. work on mathematical physics. Although Hilbert could not participate in carrying out the plan, I hope the work, and in particular the present volume, reflects his scientific ethos, which was always firmly directed towards the relevant nucleus of a mathematical problem and averse to merely formal generality. We shall introduce our topics by first concentrating on typical specific cases which are suggestive by their concrete freshness and yet exhibit the core of the underlying abstract situation. Individual phenomena are not relegated to the role of special examples; rather, general theories emerge by steps as we reach higher vantage points from which the details on a "lower level" can be better viewed, unified, and mastered. Thus, corresponding to the organic process of learning and teaching, an inductive approach is favored, sometimes at the expense of the conciseness which can be gained by a deductive, authoritarian mode of presentation.

This book is essentially self-contained; it corresponds to Volume II of the German edition of the "Methoden der Mathematischen Physik" which appeared in 1937. The original work was subsequently suppressed by the Ministry of Culture in Nazi Germany; later my loyal friend Ferdinand Springer was forced out as the head of his famous publishing house. The reprinting by Interscience Publishers under license of the United States government (1943) secured the survival of the book. Ever since, a completely new version in the English language has been in preparation. During this long period, knowledge in the field has advanced considerably, and I too have been struggling towards more comprehensive understanding. Naturally the book reflects these developments to the extent to which I have shared in them as an active and as a learning participant.

The table of contents indicates the scope of the present book. It differs in almost every important detail from the German original. For example, the theory of characteristics and their role for the theory of wave propagation is now treated much more adequately than was possible twenty-five years ago. Also the concept of weak solutions of differential equations, clarified by Sobolev and Friedrichs and already contained in the German edition, now appears in the context of the theory of ideal functions which, introduced and called "distributions" by Laurent Schwartz, have become an indispensable tool of advanced calculus. An appendix to Chapter VI contains an elementary presentation of this theory. On the other hand, the material of the last chapter of the German edition, in particular the discussion of existence of solutions of elliptic differential equations, did not find room in this volume. A short third volume on the construction of solutions will treat these topics, including an account of recent mathematics.

The book as now submitted to the public is certainly uneven in style, completeness and level of difficulty. Still, I hope that it will be useful to my fellow students, whether they are beginners, scholars, mathematicians, other scientists or engineers. Possibly the presence of various levels in the book might make the terrain all the more accessible by way of the lower regions.

I am apologetically conscious of the fact that some of the progress achieved outside of my own sphere may have been inadequately reported or even overlooked in this book. Some of these shortcomings will be remedied by other publications in the foreseeable future such as a forthcoming book by Gårding and Leray about their fascinating work.

The present publication would have been impossible without the sustained unselfish cooperation given to me by friends. Throughout all my career I have had the rare fortune to work with younger people who were successively my students, scientific companions and instructors. Many of them have long since attained high prominence and yet have continued their helpful attitude. Kurt O. Friedrichs and Fritz John, whose scientific association with me began more

than thirty years ago, are still actively interested in this work on mathematical physics.—That this volume is dedicated to K. O. Friedrichs is a natural acknowledgment of a lasting scientific and personal friendship.

To the cooperation of Peter D. Lax and Louis Nirenberg I owe much more than can be expressed by quoting specific details. Peter Ungar has greatly helped me with productive suggestions and criticisms. Also, Lipman Bers has rendered most valuable help and, moreover, has contributed an important appendix to Chapter IV.

Among younger assistants I must particularly mention Donald Ludwig whose active and spontaneous participation has led to a number of significant contributions.

Critical revision of parts of the manuscript in different stages was undertaken by Konrad Jörgens, Herbert Kranzer, Anneli Lax, Hanan Rubin. Proofs were read by Natascha Brunswick, Susan Hahn, Reuben Hersh, Alan Jeffrey, Peter Rejto, Brigitte Rellich, Leonard Sarason, Alan Solomon and others. Jane Richtmyer assisted in preparing the list of references and in many other aspects of the production. A great deal of the editing was done by Lori Berkowitz.

Most of the technical preparation was in the hands of Ruth Murray, who typed and retyped thousands of pages of manuscript, drew the figures and altogether was most instrumental in the exasperating process of transforming hardly legible drafts into the present book.

To all these helpers and to others, whose names may have been omitted, I wish to extend my profound thanks.

Thanks are also due to my patient friend Eric S. Proskauer of Interscience.

Finally I wish to thank the Office of Naval Research and the National Science Foundation, in particular F. Joachim Weyl and Arthur Grad, for the effective and understanding support given in the preparation of this book.

New Rochelle, New York R. COURANT
November 1961

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CONTENTS

The present volume, essentially independent of the first, treats the theory of partial differential equations from the point of view of mathematical physics. A shorter third volume will be concerned with existence proofs and with the construction of solutions by finite difference methods and other procedures.

CHAPTER I

Introductory Remarks

We begin with an introductory chapter describing basic concepts, problems, and lines of approach to their solution.

A partial differential equation is given as a relation of the form

$$(1) \qquad F(x, y, \cdots, u, u_x, u_y, \cdots, u_{xx}, u_{xy}, \cdots) = 0,$$

where F is a function of the variables $x, y, \cdots, u, u_x, u_y, \cdots, u_{xx}, u_{xy}, \cdots$; a function $u(x, y, \cdots)$ of the independent variables $x, y, \cdots$ is sought such that equation (1) is identically satisfied in these independent variables if $u(x, y, \cdots)$ and its partial derivatives

$$u_x = \frac{\partial u}{\partial x}, \qquad u_y = \frac{\partial u}{\partial y}, \qquad \cdots,$$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2}, \qquad u_{xy} = \frac{\partial^2 u}{\partial x \partial y}, \qquad \cdots,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

are substituted in F.

Such a function $u(x, y, \cdots)$ is called a *solution of the partial differential equation* (1). We shall not only look for a *single "particular" solution* but investigate the *totality of solutions* and, in particular, characterize *individual solutions* by further conditions which may be imposed in addition to (1).

The partial differential equation (1) becomes an ordinary differential equation if the number of independent variables is one.

The order of the highest derivative occurring in a differential equation is called the *order* of the differential equation.

Frequently we shall restrict the independent variables $x, y, \cdots$ to a specific region of the $x, y, \cdots$-space; similarly, we shall consider F

only in a restricted part of the $x, y, \cdots, u, u_x, u_y, \cdots$-space. This restriction means that we admit only those functions $u(x, y, \cdots)$ of the basic region in the $x, y, \cdots$-space which satisfy the conditions imposed on the corresponding arguments of F. *Once and for all we stipulate that all our considerations refer to regions chosen sufficiently small. Similarly, we shall assume that, unless the contrary is specifically stated, all occurring functions $F, u, \cdots$ are continuous and have continuous derivatives of all occurring orders.*[1]

The differential equation is called *linear* if F is linear in the variables $u, u_x, u_y, \cdots, u_{xx}, u_{xy}, \cdots$ with coefficients depending only on the independent variables $x, y, \cdots$. If F is linear in the highest order derivatives (say the n-th), with coefficients depending upon $x, y, \cdots$ and possibly upon u and its derivatives up to order $n - 1$, then the differential equation is called *quasi-linear.*

We shall deal mainly with either linear or quasi-linear differential equations; more general differential equations will usually be reduced to equations of this type.

In the case of merely two independent variables x, y, the solution $u(x, y)$ of the differential equation (1) is visualized geometrically as a surface, an "*integral surface*" in the x, y, u-space.

§1. *General Information about the Variety of Solutions*

1. Examples. For an ordinary differential equation of n-th order, the totality of solutions (except possible "singular" solutions) is a function of the independent variable x which also depends on n arbitrary integration constants $c_1, c_2, \cdots, c_n$. Conversely, for every n-parameter family of functions

$$u = \phi(x; c_1, c_2, \cdots, c_n),$$

there is an n-th order differential equation with the solution $u = \phi$ obtained by eliminating the parameters $c_1, c_2, \cdots, c_n$ from the equation $u = \phi(x; c_1, c_2, \cdots, c_n)$ and from the n equations

$$\begin{aligned} u' &= \phi'(x; c_1, c_2, \cdots, c_n), \\ &\cdots\cdots\cdots\cdots\cdots \\ u^{(n)} &= \phi^{(n)}(x; c_1, c_2, \cdots, c_n). \end{aligned}$$

[1] Also, when systems of equations are inverted, we will always consider a neighborhood of a point in which the corresponding Jacobian does not vanish.

For partial differential equations the situation is more complicated. Here too, one may seek the totality of solutions or the "*general solution*"; i.e., one may seek a solution which, after certain "arbitrary" elements are fixed, represents every individual solution (again with the possible exception of certain "singular" solutions). In the case of partial differential equations such arbitrary elements can no longer occur in the form of constants of integration, but must involve arbitrary functions; in general, the number of these arbitrary functions is equal to the order of the differential equation. These arbitrary functions depend on one independent variable less than the solution u. A more precise statement of the situation is implied in the existence theorem of §7. In the present section however, we merely collect information by studying a few examples.

1) The differential equation

$$u_y = 0$$

for a function $u(x, y)$ states that u does not depend on y; hence,

$$u = w(x),$$

where $w(x)$ is an arbitrary function of x.

2) For the equation

$$u_{xy} = 0,$$

one immediately obtains the general solution

$$u = w(x) + v(y).$$

3) Similarly, the solution of the nonhomogeneous differential equation

$$u_{xy} = f(x, y)$$

is

$$u(x, y) = \int_{x_0}^{x} \int_{y_0}^{y} f(\xi, \eta)\, d\xi\, d\eta + w(x) + v(y)$$

with arbitrary functions w and v and fixed values x_0, y_0.

More generally, one may replace the integral by an area integral if one takes, as the region of integration $\triangleleft$, a "triangle" such as that in Figure 1, whose curved boundary consists of a curve C: $y = g(x)$ or $x = h(y)$ which is not intersected more than once by any of the

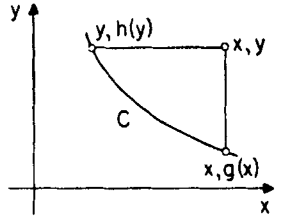

FIGURE 1

curves $x = \text{const.}$ or $y = \text{const.}$ Then,

$$(2) \qquad \begin{gathered} u(x, y) = \iint_{\square} f(\xi, \eta)\, d\xi\, d\eta + w(x) + v(y), \\ u_x = \int_{g(x)}^{y} f(x, \eta)\, d\eta + w'(x), \qquad u_y = \int_{h(y)}^{x} f(\xi, y)\, d\xi + v'(y). \end{gathered}$$

The special solution of the differential equation for $w(x) = v(y) = 0$ satisfies the condition $u = u_x = u_y = 0$ for all points (x, y) on the curve C.

4) The partial differential equation

$$u_x = u_y$$

is transformed into the equation

$$2\omega_\eta = 0$$

by the transformation of variables

$$x + y = \xi, \qquad x - y = \eta, \qquad u(x, y) = \omega(\xi, \eta).$$

The "general solution" of the transformed equation is $\omega = w(\xi)$; therefore

$$u = w(x + y).$$

Similarly, if α and β are constants, the general solution of the differential equation

$$\alpha u_x + \beta u_y = 0$$

is

$$u = w(\beta x - \alpha y).$$

5) According to elementary theorems of the differential calculus, the partial differential equation

$$u_x g_y - u_y g_x = 0,$$

where $g(x, y)$ is any given function of x, y states that the Jacobian $\partial(u, g)/\partial(x, y)$ of u, g with respect to x, y vanishes. This means that u depends on g, i.e., that

$$u = w[g(x, y)], \tag{3}$$

where w is an arbitrary function of the quantity g. Since, conversely, every function u of the form (3) satisfies the differential equation $u_x g_y - u_y g_x = 0$, we obtain the totality of solutions by means of the arbitrary function w.

It is noteworthy that the same result holds for the more general—quasi-linear—differential equation

$$u_x g_y(x, y, u) - u_y g_x(x, y, u) = 0,$$

where g now depends explicitly not only on x, y but on the unknown function $u(x, y)$ as well. For, as one sees, the Jacobian of any solution $u(x, y)$ and $\gamma(x, y) = g[x, y, u(x, y)]$ vanishes since

$$u_x \gamma_y - u_y \gamma_x = u_x g_y - u_y g_x + u_x g_u u_y - u_y g_u u_x = 0.$$

Thus, even in this case, the solution is given by the relation

$$u(x, y) = W[g(x, y, u)], \tag{4}$$

which is an implicit definition of u by means of the arbitrary function W.

For instance, the solution $u(x, y)$ of the differential equation

$$\alpha(u)u_x - \beta(u)u_y = 0$$

is implicitly defined by

$$u = W[\alpha(u)y + \beta(u)x], \tag{5}$$

(or by $\alpha(u)y + \beta(u)x = w(u)$), so that u depends on the arbitrary function W in a rather involved way. (An application will be given in §7, 1.)

A special case of the differential equation $\alpha(u)u_x - \beta(u)u_y = 0$ is

$$u_y + uu_x = 0;$$

the solution is given implicitly by

$$u = W(-x + uy),$$

where W is arbitrary. If $u = u(x(y), y)$ is interpreted as the velocity of a particle at a point $x = x(y)$ moving with the time y, then the differential equation states that the acceleration of all the particles is zero.

6) The partial differential equation of second order

$$u_{xx} - u_{yy} = 0$$

is transformed into

$$4\omega_{\xi\eta} = 0$$

by the transformation

$$x + y = \xi, \qquad x - y = \eta, \qquad u(x, y) = \omega(\xi, \eta).$$

Hence, according to example *2)*, its solutions are

$$u(x, y) = w(x + y) + v(x - y).$$

7) In a similar way the general solution of the differential equation

$$u_{xx} - \frac{1}{t^2} u_{yy} = 0$$

for any value of the parameter t is

$$u = w(x + ty) + v(x - ty).$$

In particular, the functions

$$u = (x + ty)^n$$

and

$$u = (x - ty)^n$$

are solutions; i.e.,

$$t^2 u_{xx} - u_{yy}$$

vanishes for all x, y and for all real t.

8) According to elementary algebra, if a polynomial in t vanishes for all real values of t, then it vanishes for all complex values of t as