

Heinz Georg Schuster and Wolfram Just

Deterministic Chaos

An Introduction

Fourth, Revised and Enlarged Edition



WILEY-
VCH

WILEY-VCH Verlag GmbH & Co. KGaA

H. G. Schuster and W. Just
Deterministic Chaos

An Introduction

Heinz Georg Schuster and Wolfram Just

Deterministic Chaos

An Introduction

Fourth, Revised and Enlarged Edition



WILEY-
VCH

WILEY-VCH Verlag GmbH & Co. KGaA

Authors

Prof. Dr. H. G. Schuster
Christian Albrecht University Kiel, Germany
Department of Theoretical Physics

Lecturer Wolfram Just
Queen Mary / University of London, United Kingdom
School of Mathematical Sciences

Cover Picture

Detail of the "tail" in Plate IX (after Peitgen and Richter)

All books published by Wiley-VCH are carefully produced. Nevertheless, authors, editors, and publisher do not warrant the information contained in these books, including this book, to be free of errors. Readers are advised to keep in mind that statements, data, illustrations, procedural details or other items may inadvertently be inaccurate.

**Library of Congress Card No.: applied for
British Library Cataloging-in-Publication Data:**

A catalogue record for this book is available from the British Library

**Bibliographic information published by
Die Deutsche Bibliothek**

Die Deutsche Bibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data is available in the Internet at <<http://dnb.ddb.de>>.

© 2005 WILEY-VCH Verlag GmbH & Co. KGaA,
Weinheim

All rights reserved (including those of translation into other languages). No part of this book may be reproduced in any form – nor transmitted or translated into machine language without written permission from the publishers. Registered names, trademarks, etc. used in this book, even when not specifically marked as such, are not to be considered unprotected by law.

Printed in the Federal Republic of Germany
Printed on acid-free paper

Composition Michael Bär, Wiesloch
Printing Strauss GmbH, Mörlenbach
Bookbinding Litges & Dopf Buchbinderei GmbH,
Heppenheim

ISBN-13: 978-3-527-40415-5

ISBN-10: 3-527-40415-5

Contents

Table of Contents	v
Preface	ix
Color Plates	xiii
1 Introduction	1
2 Experiments and Simple Models	7
2.1 Experimental Detection of Deterministic Chaos	7
2.1.1 Driven Pendulum	7
2.1.2 Rayleigh–Bénard System in a Box	9
2.1.3 Stirred Chemical Reactions	11
2.1.4 Hénon–Heiles System	12
2.2 The Periodically Kicked Rotator	16
2.2.1 Logistic Map	17
2.2.2 Hénon Map	17
2.2.3 Chirikov Map	18
3 Piecewise Linear Maps and Deterministic Chaos	19
3.1 The Bernoulli Shift	19
3.2 Characterization of Chaotic Motion	21
3.2.1 Liapunov Exponent	21
3.2.2 Invariant Measure	25
3.2.3 Correlation Function	27
3.3 Deterministic Diffusion	29
4 Universal Behavior of Quadratic Maps	33
4.1 Parameter Dependence of the Iterates	35
4.2 Pitchfork Bifurcation and the Doubling Transformation	37
4.2.1 Pitchfork Bifurcations	37
4.2.2 Supercycles	40
4.2.3 Doubling Transformation and α	41
4.2.4 Linearized Doubling Transformation and δ	43

4.3	Self-Similarity, Universal Power Spectrum, and the Influence of External Noise	46
4.3.1	Self-Similarity in the Positions of the Cycle Elements	46
4.3.2	Hausdorff Dimension	48
4.3.3	Power Spectrum	50
4.3.4	Influence of External Noise	52
4.4	Behavior of the Logistic Map for $r_\infty \leq r$	55
4.4.1	Sensitive Dependence on Parameters	55
4.4.2	Structural Universality	57
4.4.3	Chaotic Bands and Scaling	59
4.5	Parallels between Period Doubling and Phase Transitions	61
4.6	Experimental Support for the Bifurcation Route	64
5	The Intermittency Route to Chaos	69
5.1	Mechanisms for Intermittency	69
5.1.1	Type-I Intermittency	70
5.1.2	Length of the Laminar Region	73
5.2	Renormalization-Group Treatment of Intermittency	75
5.3	Intermittency and $1/f$ -Noise	79
5.4	Experimental Observation of the Intermittency Route	84
5.4.1	Distribution of Laminar Lengths	84
5.4.2	Type-I Intermittency	86
5.4.3	Type-III Intermittency	86
6	Strange Attractors in Dissipative Dynamical Systems	89
6.1	Introduction and Definition of Strange Attractors	89
6.1.1	Baker's Transformation	92
6.1.2	Dissipative Hénon Map	94
6.2	The Kolmogorov Entropy	96
6.2.1	Definition of K	96
6.2.2	Connection of K to the Liapunov Exponents	97
6.2.3	Average Time over which the State of a Chaotic System can be Predicted	100
6.3	Characterization of the Attractor by a Measured Signal	102
6.3.1	Reconstruction of the Attractor from a Time Series	103
6.3.2	Generalized Dimensions and Distribution of Singularities in the Invariant Density	106
6.3.3	Generalized Entropies and Fluctuations around the K -Entropy	115
6.3.4	Kaplan–Yorke Conjecture	120
6.4	Pictures of Strange Attractors and Fractal Boundaries	122
7	The Transition from Quasiperiodicity to Chaos	127
7.1	Strange Attractors and the Onset of Turbulence	127
7.1.1	Hopf Bifurcation	127
7.1.2	Landau's Route to Turbulence	128
7.1.3	Ruelle–Takens–Newhouse Route to Chaos	129
7.1.4	Possibility of Three-Frequency Quasiperiodic Orbits	131
7.1.5	Break-up of a Two-Torus	133

7.2	Universal Properties of the Transition from Quasiperiodicity to Chaos	136
7.2.1	Mode Locking and the Farey Tree	139
7.2.2	Local Universality	141
7.2.3	Global Universality	146
7.3	Experiments and Circle Maps	150
7.3.1	Driven Pendulum	151
7.3.2	Electrical Conductivity in Barium Sodium Niobate	153
7.3.3	Dynamics of Cardiac Cells	154
7.3.4	Forced Rayleigh–Bénard Experiment	156
7.4	Routes to Chaos	157
7.4.1	Crises	158
8	Regular and Irregular Motion in Conservative Systems	161
8.1	Coexistence of Regular and Irregular Motion	163
8.1.1	Integrable Systems	163
8.1.2	Perturbation Theory and Vanishing Denominators	165
8.1.3	Stable Tori and KAM Theorem	166
8.1.4	Unstable Tori and Poincaré–Birkhoff Theorem	167
8.1.5	Homoclinic Points and Chaos	170
8.1.6	Arnold Diffusion	171
8.1.7	Examples of Classical Chaos	172
8.2	Strongly Irregular Motion and Ergodicity	174
8.2.1	Cat Map	174
8.2.2	Hierarchy of Classical Chaos	176
8.2.3	Three Classical K -Systems	180
9	Chaos in Quantum Systems?	183
9.1	The Quantum Cat Map	184
9.2	A Quantum Particle in a Stadium	186
9.3	The Kicked Quantum Rotator	187
10	Controlling Chaos	193
10.1	Stabilization of Unstable Orbits	194
10.2	The OGY Method	197
10.3	Time-Delayed Feedback Control	199
10.3.1	Rhythmic Control	200
10.3.2	Extended Time-Delayed Feedback Control	201
10.3.3	Experimental Realization of Time-Delayed Feedback Control	202
10.4	Parametric Resonance from Unstable Periodic Orbits	203
11	Synchronization of Chaotic Systems	207
11.1	Identical Systems with Symmetric Coupling	207
11.1.1	On–Off Intermittency	208
11.1.2	Strong vs. Weak Synchronization	209
11.2	Master–Slave Configurations	210

11.3	Generalized Synchronization	212
11.3.1	Strange Nonchaotic Attractors	212
11.4	Phase Synchronization of Chaotic Systems	213
12	Spatiotemporal Chaos	217
12.1	Models for Space–Time Chaos	217
12.1.1	Coupled Map Lattices	217
12.1.2	Coupled Oscillator Models	218
12.1.3	Complex Ginzburg–Landau Equation	220
12.1.4	Kuramoto–Sivashinsky Equation	220
12.2	Characterization of Space–Time Chaos	221
12.2.1	Liapunov Spectrum	222
12.2.2	Co-moving Liapunov Exponent	223
12.2.3	Chronotopic Liapunov Analysis	224
12.3	Nonlinear Nonequilibrium Space–Time Dynamics	225
12.3.1	Fully Developed Turbulence	225
12.3.2	Spatiotemporal Intermittency	227
12.3.3	Molecular Dynamics	227
	Outlook	231
	Appendix	233
A	Derivation of the Lorenz Model	233
B	Stability Analysis and the Onset of Convection and Turbulence in the Lorenz Model	235
C	The Schwarzian Derivative	236
D	Renormalization of the One-Dimensional Ising Model	238
E	Decimation and Path Integrals for External Noise	240
F	Shannon’s Measure of Information	243
F.1	Information Capacity of a Store	243
F.2	Information Gain	244
G	Period Doubling for the Conservative Hénon Map	245
H	Unstable Periodic Orbits	249
	Remarks and References	257
	Index	283

Preface

Since 1994 when the last edition of the present monograph was published, the field of Nonlinear Science has developed tremendously. It is nowadays no longer possible to give a comprehensible introduction into, and a balanced overview of the different branches within this field. Following the general practice of the previous editions it is the scope of this fourth augmented edition to introduce aspects of Nonlinear Dynamics at a level which is accessible to a wide audience. We have intensified and added three new topics:

- Control of chaos is one of the most popular branches of Nonlinear Science. As a particular new aspect we have included a comprehensive discussion of time-delayed feedback control which is widely used in applications.
- Topics in synchronization became recently quite popular from a fundamental as well as an applied point of view. We introduce basic concepts as well as novel notions like phase synchronization or strange nonchaotic, attractors, at an elementary level.
- Spatiotemporal chaos covers a wide range of topics, from classical fields in physics such as hydrodynamics to current research topics in theoretical biophysics, which are commonly related with the nonlinear dynamics of a large number of degrees of freedom. We introduce here basic features of relevant model systems as well as selected concepts for quantitative analysis. But our exposition is far from complete.

The fourth edition benefits from data and figures that have been provided by several colleagues, in particular by R. Klages, J. Kurths, A. Pikovski, H. Posch, and M. Rosenblum. It is a pleasure to thank E. Schöll for his kind hospitality during a stay at Berlin University of Technology, where parts of the new edition were written. We are indebted to the publisher, in particular to Dr. M. Bär and R. Schulz, for their continual help in preparing the manuscript. Despite the remarkable support from various people the present edition could still contain mistakes. We apologize in advance for such inconsistencies and we invite the reader to report to us any deficiencies.

Kiel/London, October 2004
H. G. Schuster, W. Just

Preface to the Third Edition

Since the last edition of this book in 1989 the field of deterministic chaos has continued to grow. Within the wealth of new results there are three major trends.

- Unstable periodic orbits have been rediscovered as building blocks of chaotic dynamics, especially through the work of Cvitanovich *et al.* (1990). They developed an expansion of physical averages in terms of primitive cycles (see also Appendix H).
- Exploiting the concept of unstable periodic orbits, Ott, Grebogi and Yorke demonstrated in 1990 that deterministic chaos can be controlled. They found that small time-dependent changes in the control parameter of the system can stabilize previously unstable periodic cycles in such a way that the system becomes nonchaotic (see Chapter 10).
- There are new theoretical and experimental results in the field of quantum chaos which are described excellently in the new books by Gutzwiller (1990), Haake (1991) and Reichl (1992).

During the preparation of the new edition, J. C. Gruel helped with the pictures of the new chapter, Mrs. H. Heimann typed the new text, M. Poulson and R. Wengenmayr from VCH Publishers took care of the editorial work. H. J. Stockmann and H. J. Stein contributed the fascinating pictures of simulations of quantum chaos in microwave resonators. I would like to thank all these people for their cooperation and patience.

Kiel, August 1994
H. G. Schuster

Preface to the Second Edition

This is a revised and updated version of the first edition, to which new sections on sensitive parameter dependence, fat fractals, characterization of attractors by scaling indices, the Farey tree, and the notion of global universality have been added. I thank P. C. T. de Boer, J. L. Grant, P. Grassberger, W. Greulich, F. Kaspar, K. Pawelzik, K. Schmidt, and S. Smid for helpful hints and remarks, and Mrs. Adlfinger and Mrs. Boffo for their patient help with the manuscript.

Kiel, August 1987
H. G. Schuster

Preface to the First Edition

Daily experience shows that, for many physical systems, small changes in the initial conditions lead to small changes in the outcome. If we drive a car and turn the steering wheel only a little, our course will differ only slightly from that which the car would have taken without this change. But there are cases for which the opposite of this rule is true: For a coin which is placed on its rim, a slight touch is sufficient to determine the side on which it will fall. Thus

the sequence of heads and tails which we obtain when tossing a coin exhibits an irregular or chaotic behavior in time, because extremely small changes in the initial conditions can lead to completely different outcomes. It has become clear in recent years, partly triggered by the studies of nonlinear systems using high-speed computers, that a sensitive dependence on the initial conditions, which results in a chaotic time behavior, is by no means exceptional but is a typical property of many systems. Such behavior has, for example, been found in periodically stimulated cardiac cells, in electronic circuits, at the onset of turbulence in fluids and gases, in chemical reactions, in lasers, etc. Mathematically, all nonlinear dynamical systems with more than two degrees of freedom, i. e., especially many biological, meteorological or economic models, can display chaos and, therefore, become unpredictable over longer time scales. "Deterministic chaos" is now a very active field of research with many exciting results. Methods have been developed to classify different types of chaos, and it has been discovered that many systems show, as a function of an external control parameter, similar transitions from order to chaos. This universal behavior is reminiscent of ordinary second-order phase transitions, and the introduction of renormalization and scaling methods from statistical mechanics has brought new perspectives into the study of deterministic chaos. It is the aim of this book to provide a self-contained introduction to this field from a physicist's point of view. The book grew out of a series of lectures, which I gave during the summer terms of 1982 and 1983 at the University of Frankfurt, and it requires no knowledge which a graduate student in physics would not have. A glance at the table of contents shows that new concepts such as the Kolmogorov entropy, strange attractors, etc., or new techniques such as the functional renormalization group, are introduced at an elementary level. On the other hand, I hope that there is enough material for research workers who want to know, for example, how deterministic chaos can be distinguished experimentally from white noise, or who want to learn how to apply their knowledge about equilibrium phase transitions to the study of (nonequilibrium) transitions from order to chaos. During the preparation of this book the manuscripts, preprints and discussion, the remarks of G. Eilenberger, K. Kehr, H. Leschke, W. Selke, and M. Schmutz were of great help. P. Berge, M. Dubois, W. Lauterborn, W. Martienssen, G. Pfister and their coworkers supplied several, partly unpublished, pictures of their experiments. H. O. Peitgen, P. H. Richter and their group gave permission to include some of their most fascinating computer pictures in this book (see cover and Section 6.4). All contributions are gratefully appreciated. Furthermore, I want to thank W. Greulich, D. Hackenbracht, M. Heise, L. L. Hirst, R. Liebmann, I. Neil, and especially I. Procaccia for carefully reading parts of the manuscript and for useful criticism and comments. I also acknowledge illuminating discussions with V. Emery, P. Grassberger, D. Gempel, S. Grossmann, S. Fishman, and H. Horner. It is a pleasure to thank R. Hornreich for the kind hospitality extended to me during a stay at the Weizmann Institute, where several chapters of this book were written, with the support of the Minerva foundation. Last but not least, I thank Mrs. Boffo and Mrs. Knolle for their excellent assistance in preparing the illustrations and the text.

Frankfurt, October 1984
H. G. Schuster

Legends to Plates I–XX

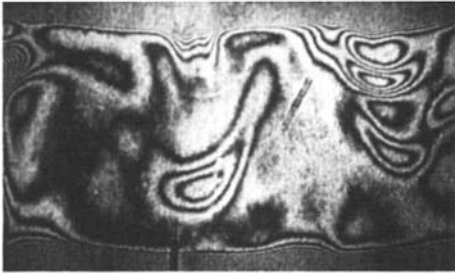
Many of these plates are part of Chapter 6. Accordingly, references mentioned in the legends are to be found on pages 89–125.

- I. *Biperiodic flow in a Bénard experiment.* Figs. 1–8 show interferometric pictures of a Bénard cell in the biperiodic regime; that is, there are two incommensurate frequencies in the power spectrum (see also pages 2.1.2–2.1.2). The time between successive pictures is 10 s. The first period lasts 40 s after which the “mouth” in the middle of the pictures repeats itself (see Figs. 1 and 5). But the details, e. g., in the upper right corners of Figs. 1 and 5 are not the same; that is, the motion is not simply periodic. (From a film taken by P. Berge and M. Dubois, CEN Saclay, Gif-sur-Yvette, France.) [page xv]
- II. *Nonlinear electronic oscillator* (see also Fig. 46 on page 66): The current-versus-voltage phase portraits (at the nonlinear diode) are shown on the oscilloscope screen. For increasing driving voltage one observes the period-doubling route. The nonlinearity of the diode that has been used in this experiment differs from eq. 4.119. (Picture taken by W. Meyer-Ilse, after Klinker *et al.*, 1984.) [page xvi]
- III. *Taylor instability.* a) Formation of rolls, b) the rolls start oscillating, c) a more complicated oscillatory motion, d) chaos. (After Pfister, 1984; see also pages 133–135.) [page xvi]
- IV. *Disturbed heartbeats.* The voltage difference (black) across the cell membrane of one cell of an aggregate of heart cells from embryonic chicken shows a) phase locking with the stimulating pulse and b) irregular dynamics, displaying escape or interpolation beats if the time between successive periodic stimuli (red) is changed from 240 ms in a) to 560 ms in b). (After Glass *et al.*, 1983; see also page 7.3.3.) [page xvii]
- V. *Chaotic electrical conduction in BSN crystals.* The birefringence pattern of a ferroelectric BSN crystal shows domain walls which mirror the charge transport near the onset of chaos (see also Fig. 116 on page 153). For clarity, the dark lines in the original pattern have been redrawn in red. (After Martin *et al.*, 1984.) [page xviii]
- VI. *Power spectra of cavitation noise:* The noise amplitude is depicted in colors, and the input pressure is increasing linearly with time. One observes (with increasing pressure) a subharmonic route $f_0 \rightarrow f_0/2 \rightarrow f_0/4 \rightarrow \text{chaos}$. This picture is the colored version of Fig. 47c on page 67. (Picture taken by E. Suchla, after Lauterborn and Cramer, 1981.) [page xviii]
- VII. *The Cassini division.* The ring of Saturn (a) shows a major gap (b), the so-called Cassini division, because the motion on this orbit is unstable (see also Fig. 142 on page 174).

(NASA pictures no. P-23068 and P-23207 with permission from Bildarchiv, Baader Planetarium.) [page xviii]

Plates VIII–XV show fractal boundaries in the complex plane:

- VIII. *Newton's algorithm* for $f(z) = z^3 - 1 = 0$. The basins of attraction for the three roots of $z^3 = 1$ are shown in red, green and blue (after Peitgen and Richter, 1984; see also pages 122–124). [page xix]
- IX. *Mandelbrot's set* (black) in the complex plane. (After Peitgen and Richter, 1984; see also page 125). [page xix]
- X–XII. Enlargements of regions A, D, E in Plate IX (after Peitgen and Richter, 1984). [pages xx–xxi]
- XIII. Enlargement of the “tail of the seahorse” in Plate IX (after Peitgen and Richter, 1984). [page xxi]
- XIV. “*Eye of the seahorse*” in Plate IX (after Peitgen and Richter, 1984). [page xxii]
- XV. Detail of the “tail” in Plate IX (after Peitgen and Richter, 1984). [page xxii]
- XVI. *Liapunov exponent* λ (depicted in colors) of the circle map $(\theta_{n+1} = \theta_n + \Omega - (K/2\pi) \times \sin(2\pi\theta_n))$ as a function of the parameters $K = 0 \dots 10$ (y-axis) and $\Omega = 0 \dots 1$ (x-axis). The arrow indicates the critical line at $K = l$. (After K. Schmidt, priv. comm.; see also Fig. 107 on page 138.) [page xxiii]
- XVII. *Liapunov exponent* λ (depicted in colors) of the driven pendulum with additional external torque $(\ddot{\theta} + 1.58\dot{\theta} + K \sin \theta = \Omega \cos(1.76t) + \Omega)$ as a function of the parameters $K = 0 \dots 20$ (y-axis) and $\Omega = 0 \dots 20$ (x-axis). (After K. Schmidt, priv. comm.; see also pages 151–153.) [page xxiii]
- XVIII. *Experiments with microwaves* in irregularly shaped resonators make it possible to simulate quantum mechanical properties of classically chaotic systems in an easy fashion. The picture shows the color-coded distribution of amplitudes of standing electromagnetic waves which follow the wave equation 9.10 in a stadium-shaped resonator for different excitation frequencies. (After H. J. Stockmann (1993), the pictures were taken by J. Stein, Marburg.) [page xxiv]
- XIX. *Fractal diffusion coefficient* of a chaotic dynamical system (“chain of boxes”) generalizing a simple random walk. The dotted red line shows a trajectory performing deterministic diffusion. The diffusion coefficient as a function of the slope of the map yields a “wave-like” structure. Self-similarity is illustrated by the inset showing a blow up of a small region of the curve (after Klages, 2004). [page xxv]
- XX. *Lifetime of the fixed point* of the Hénon map in the plane of control parameters. Within the triangle the fixed point becomes stable and the lifetime becomes infinite (cf. Chapter 10 and Fig. 164 for further details). [page xxv]



1



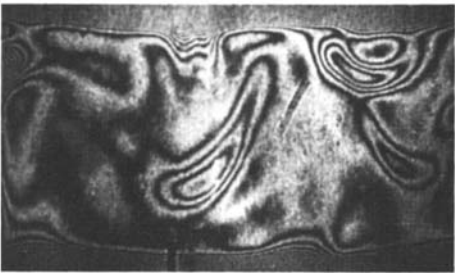
2



3



4



5



6



7



8

Plate I.

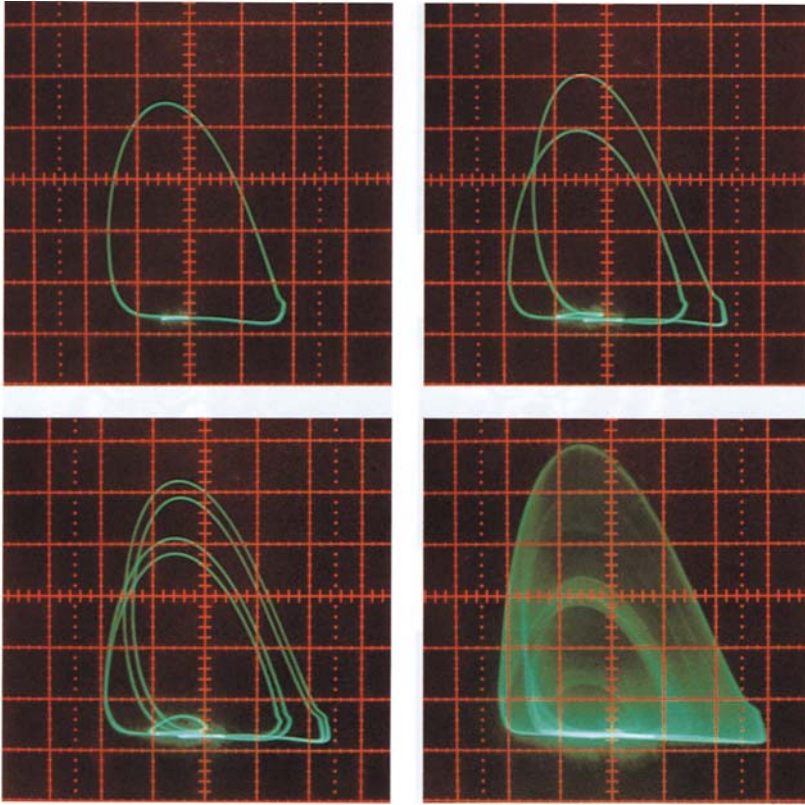


Plate II.



Plate III.

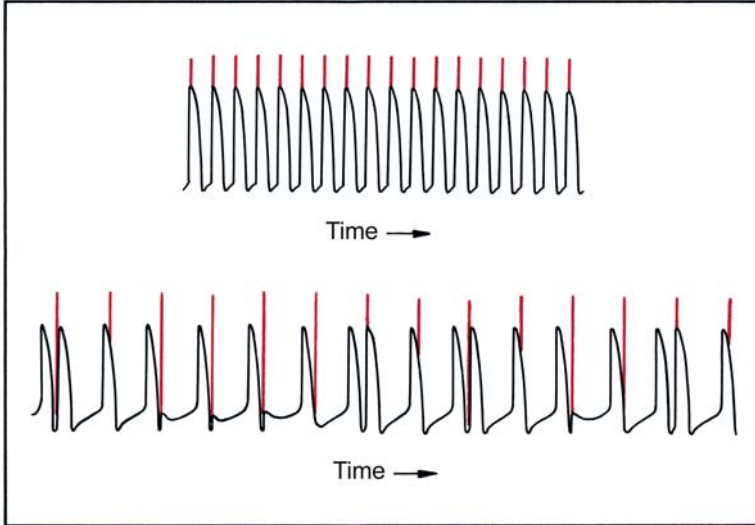


Plate IV.

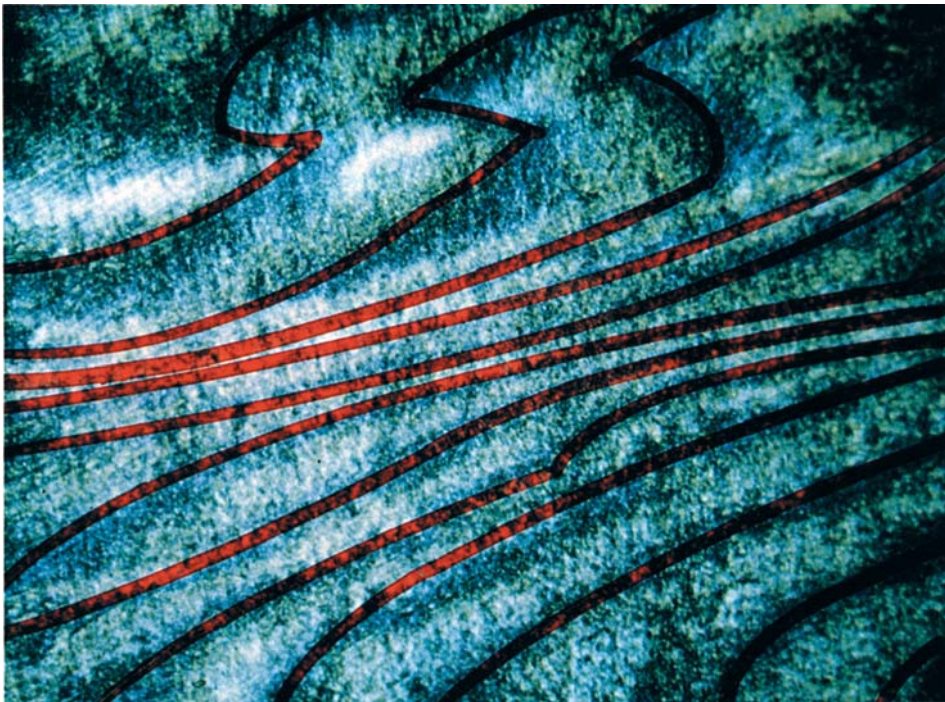


Plate V.

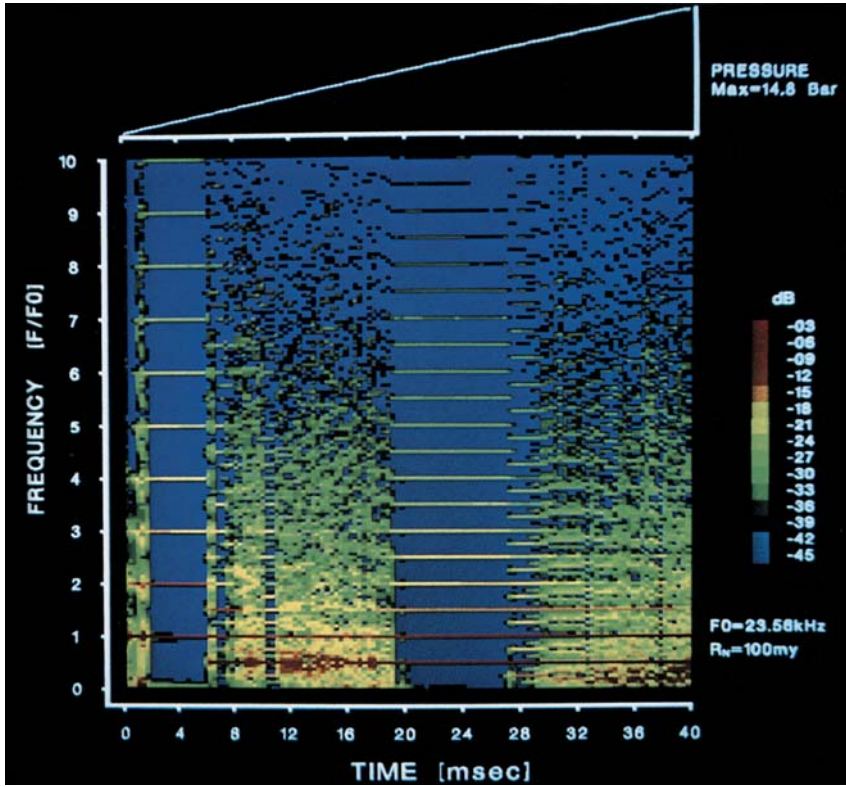


Plate VI.

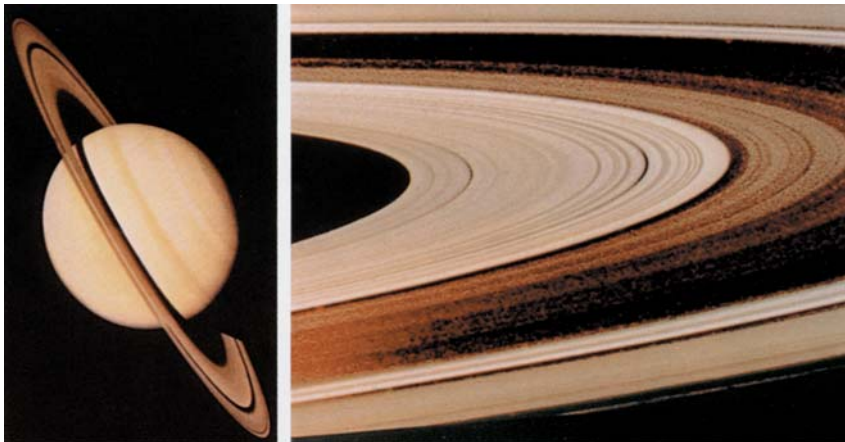


Plate VII.

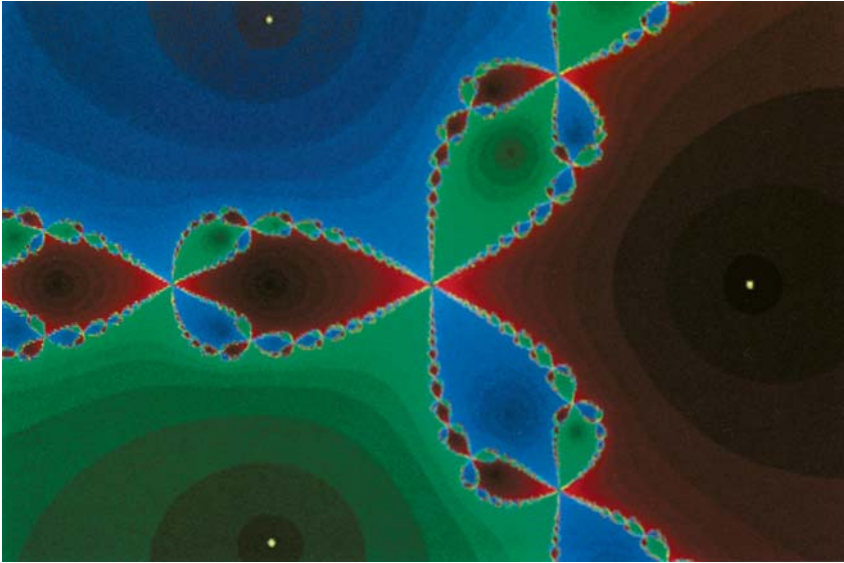


Plate VIII.

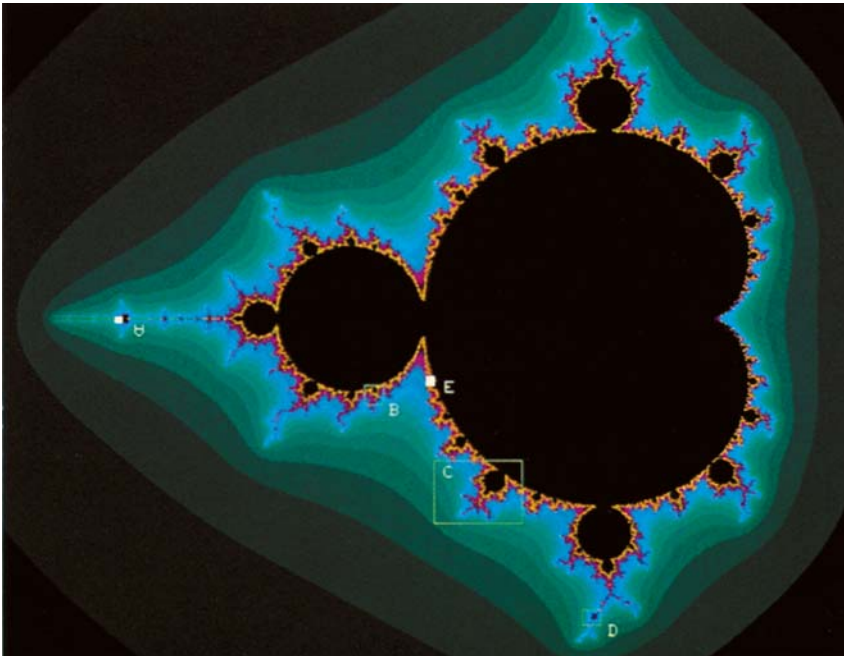


Plate IX.

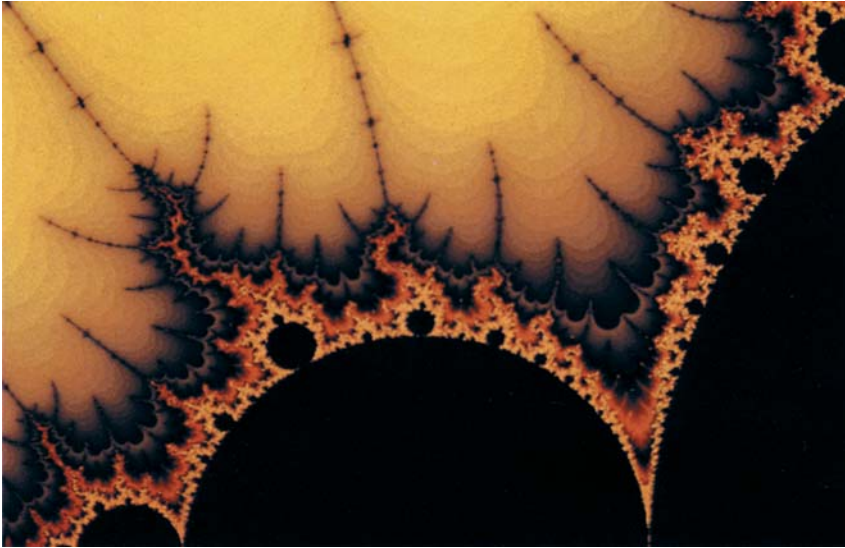


Plate X.

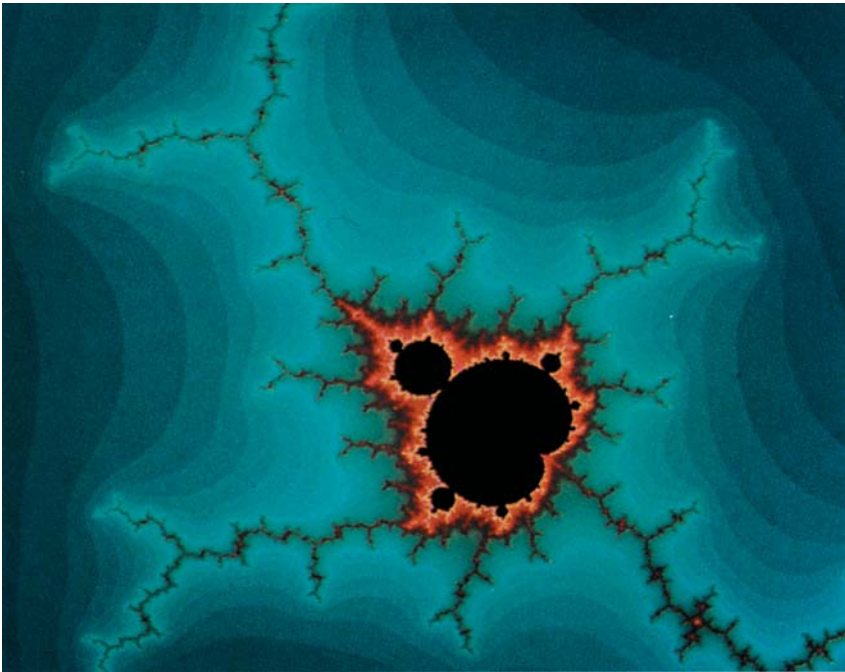


Plate XI.



Plate XII.

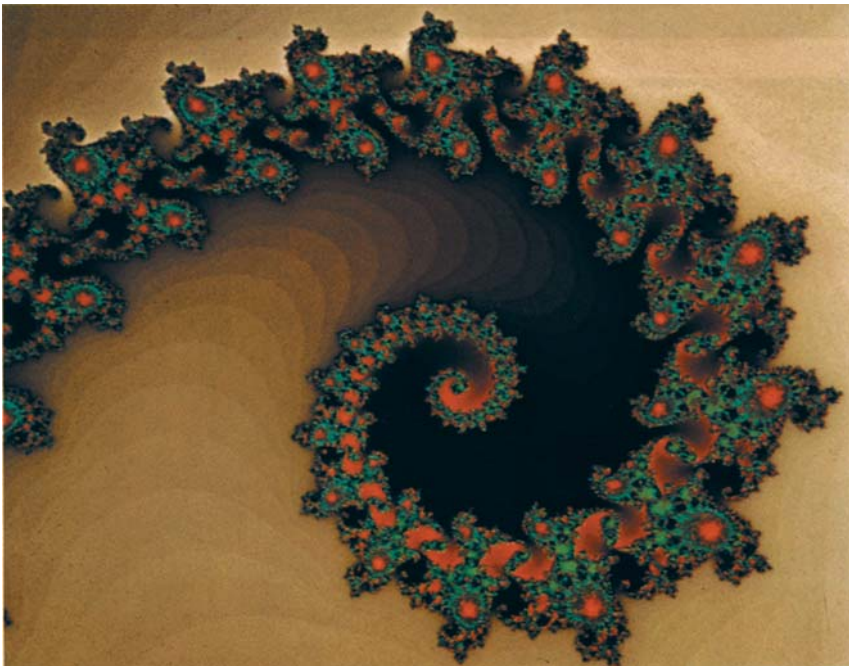


Plate XIII.

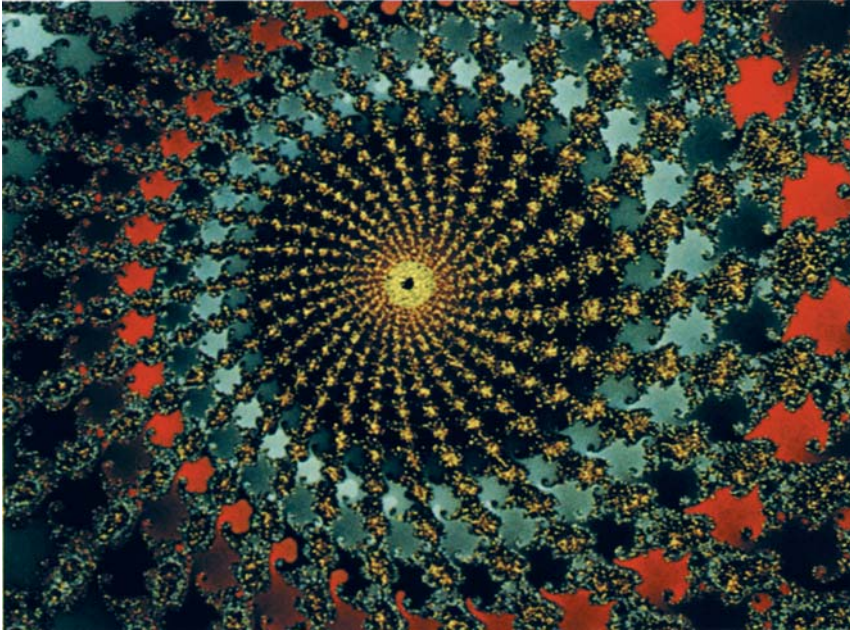


Plate XIV.



Plate XV.

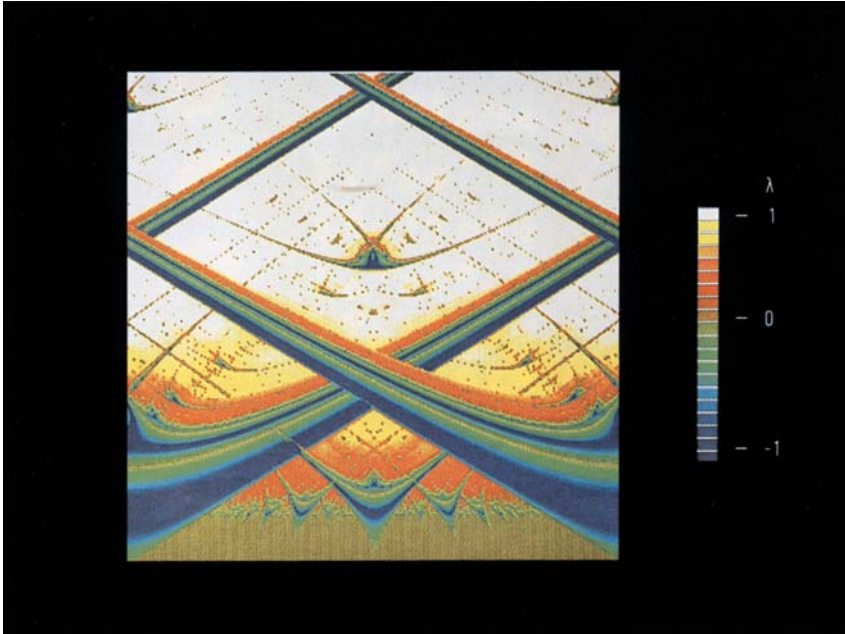


Plate XVI.

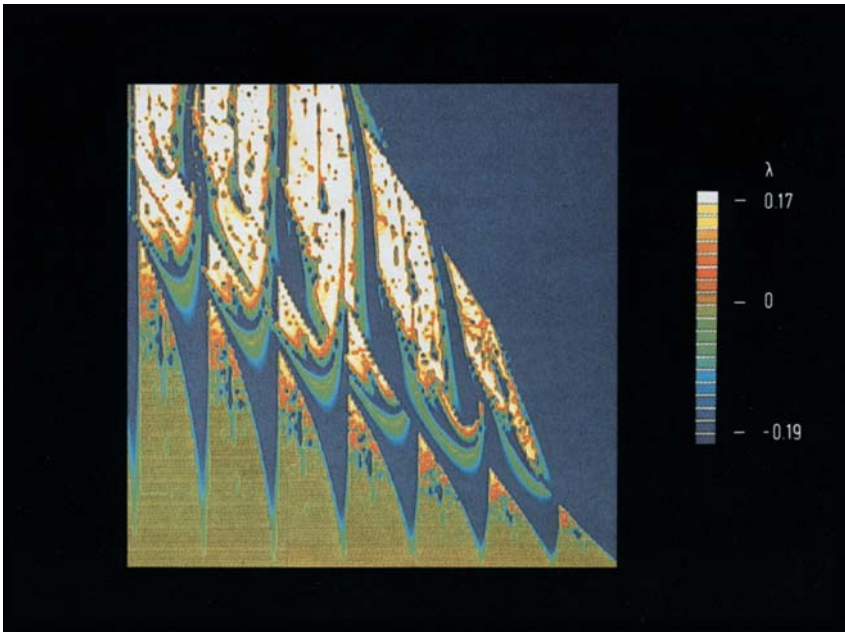


Plate XVII.

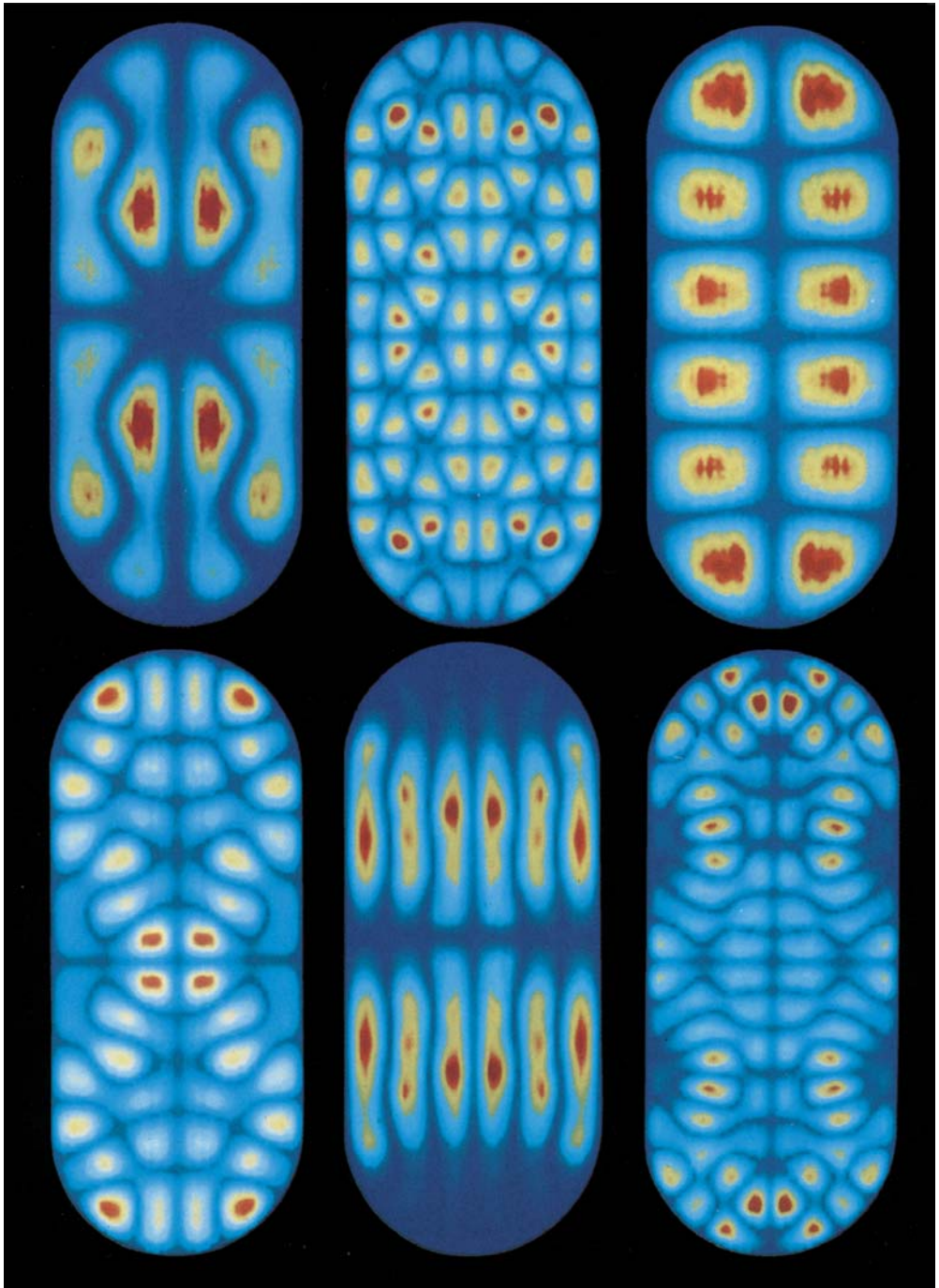


Plate XVIII.

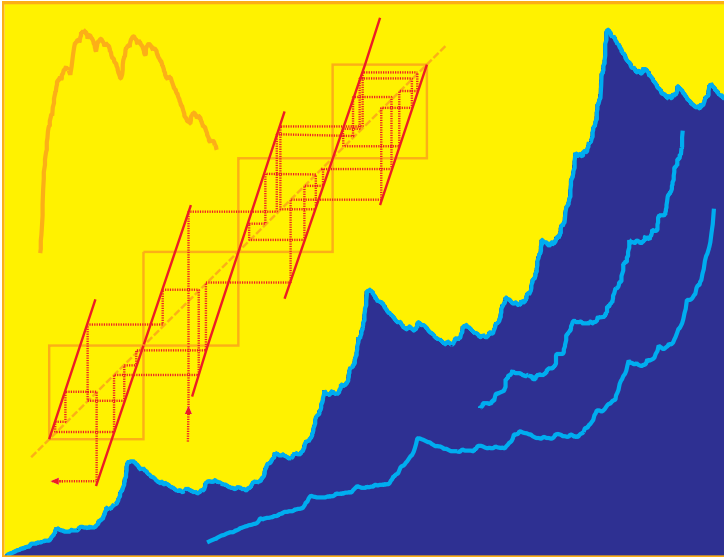


Plate XIX.

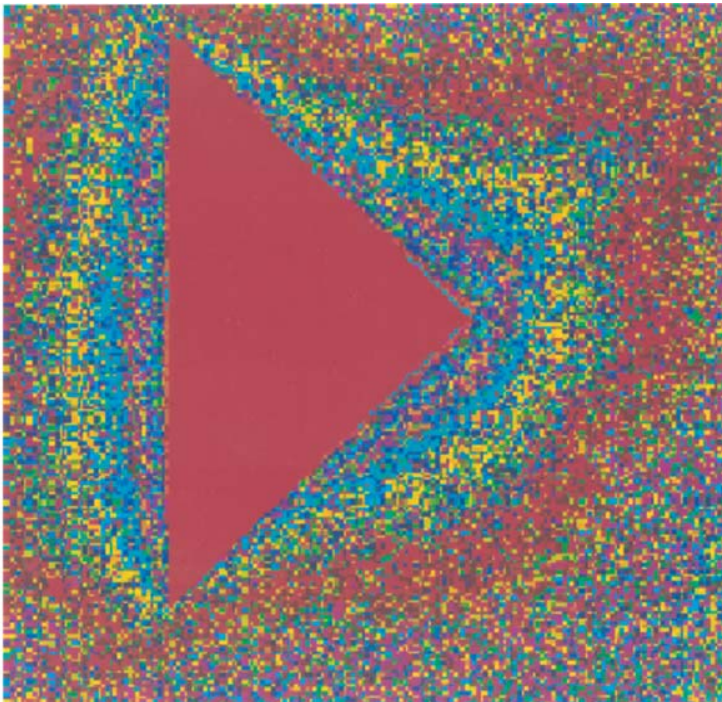


Plate XX.

1 Introduction

*Ante mare et terras et, quod tegit omnia,
caelum Unus erat toto naturae vultus in
orbe, Quem dixere Chaos, rudis
indigestaque moles Nec quicquam nisi
pondus iners congestaque eodem Non
bene iunctarum discordia semina rerum.*
Ovid

It seems appropriate to begin a book which is entitled “Deterministic Chaos” with an explanation of both terms. According to the Encyclopaedia Britannica the word “chaos” is derived from the Greek “ $\chi\alpha\omicron\varsigma$ ” and originally meant the infinite empty space which existed before all things. The later Roman conception interpreted chaos as the original crude shapeless mass into which the Architect of the world introduces order and harmony. In modern usage which we will adopt here, chaos denotes a state of disorder and irregularity.

In the following, we shall consider physical systems whose time dependence is deterministic, i. e., there exists a prescription, either in terms of differential or difference equations, for calculating their future behavior from given initial conditions. One could assume naively that deterministic motion (which is, for example, generated by continuous differential equations) is rather regular and far from being chaotic because successive states evolve continuously from each other. But it was already discovered at the turn of the century by the mathematician H. Poincaré (1892) that certain mechanical systems, whose time evolution is governed by Hamilton’s equations, could display chaotic motion. Unfortunately, this was considered by many physicists as a mere curiosity, and it took another 70 years until, in 1963, the meteorologist E. N. Lorenz found that even a simple set of three coupled, first-order, nonlinear differential equations can lead to completely chaotic trajectories. Lorenz’s paper, the general importance of which is recognized today, was also not widely appreciated until many years after its publication. He discovered one of the first examples of deterministic chaos in dissipative systems.

In the following, deterministic chaos denotes the irregular or chaotic motion that is generated by nonlinear systems whose dynamical laws uniquely determine the time evolution of a state of the system from a knowledge of its previous history. In recent years – due to new theoretical results, the availability of high speed computers, and refined experimental techniques – it has become clear that this phenomenon is abundant in nature and has far-reaching consequences in many branches of science (see the long list in Table 1, which is far from complete).

Table 1: Some nonlinear systems which display deterministic chaos. (For numerals, see “References” on page 259.)

Forced pendulum [1]
Fluids near the onset of turbulence [2]
Lasers [3]
Nonlinear optical devices [4]
Josephson junctions [5]
Chemical reactions [6]
Classical many-body systems (three-body problem) [7]
Particle accelerators [8]
Plasmas with interacting nonlinear waves [9]
Biological models for population dynamics [10]
Stimulated heart cells (see Plate IV at the beginning of the book) [11]

We note that nonlinearity is a necessary, but not a sufficient condition for the generation of chaotic motion. (Linear differential or difference equations can be solved by Fourier transformation and do not lead to chaos.) The observed chaotic behavior in time is neither due to external sources of noise (there are none in the Lorenz equations) nor to an infinite number of degrees of freedom (in Lorenz’s system there are only three degrees of freedom) nor to the uncertainty associated with quantum mechanics (the systems considered are purely classical). The actual source of irregularity is the property of the nonlinear system of separating initially close trajectories exponentially fast in a bounded region of phase space (which is, e. g., three-dimensional for Lorenz’s system).

It becomes therefore practically impossible to predict the long-time behavior of these systems, because in practice one can only fix their initial conditions with finite accuracy, and errors increase exponentially fast. If one tries to solve such a nonlinear system on a computer, the result depends for longer and longer times on more and more digits in the (irrational) numbers which represent the initial conditions. Since the digits in irrational numbers (the rational numbers are of measure zero along the real axis) are irregularly distributed, the trajectory becomes chaotic.

Lorenz called this sensitive dependence on the initial conditions the butterfly effect, because the outcome of his equations (which describe also, in a crude sense, the flow of air in the earth’s atmosphere, i. e., the problem of weather forecasting) could be changed by a butterfly flapping wings. This also seems to be confirmed sometimes by daily experience.

The results described above immediately raise a number of fundamental questions:

- Can one predict (e. g., from the form of the corresponding differential equations) whether or not a given system will display deterministic chaos?
- Can one specify the notion of chaotic motion more mathematically and develop quantitative measures for it?
- What is the impact of these findings on different branches of physics?
- Does the existence of deterministic chaos imply the end of long-time predictability in physics for some nonlinear systems, or can one still learn something from a chaotic signal?