NUMERICAL SEMIGROUPS

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Para Loly, Patricia y Carlos

Para María y Alba

Contents

Introduction

Let $\mathbb{N}$ be the set of nonnegative integers. A numerical semigroup is a nonempty subset S of $\mathbb{N}$ that is closed under addition, contains the zero element, and whose complement in $\mathbb{N}$ is finite.

If $n_1,\ldots,n_e$ are positive integers with $\gcd\{n_1,\ldots,n_e\}=1$, then the set $\langle n_1,\ldots,n_e\rangle = \{\lambda_1 n_1+\cdots+\lambda_e n_e \mid \lambda_1,\ldots,\lambda_e\in\mathbb{N}\}$ is a numerical semigroup. Every numerical semigroup is of this form.

The simplicity of this concept makes it possible to state problems that are easy to understand but whose resolution is far from being trivial. This fact attracted several mathematicians like Frobenius and Sylvester at the end of the 19th century. This is how for instance the Frobenius problem arose, concerned with finding a formula depending on $n_1,\ldots,n_e$ for the largest integer not belonging to $\langle n_1,\ldots,n_e\rangle$ (see [52] for a nice state of the art on this problem).

During the second half of the past century, numerical semigroups came back to the scene mainly due to their applications in algebraic geometry. Valuations of analytically unramified one-dimensional local Noetherian domains are numerical semigroups under certain conditions, and many properties of these rings can be characterized in terms of their associated numerical semigroups. For a field K, the valuation of the ring $K[[t^{n_1},\ldots,t^{n_e}]]$ is precisely $\langle n_1,\ldots,n_e\rangle$. This link can be used to construct one-dimensional Noetherian local domains with the desired properties, and it is basically responsible for how some invariants in a numerical semigroup have been termed. Such invariants include the multiplicity, embedding dimension, degree of singularity, type and conductor. Some families of numerical semigroups also were considered partly because of this connection: symmetric numerical semigroups, pseudo-symmetric numerical semigroups, numerical semigroups with maximal embedding dimension and with the Arf property, saturated numerical semigroups, and complete intersections, each having their counterpart in ring theory. A good translator for these concepts between both ring and semigroup theory is [5]. It is worth mentioning that these semigroups are important not only for their applications in algebraic geometry, but also because their definitions appear in a very natural way in the scope of numerical semigroups. One of the aims of this volume is to show this.

J.C. Rosales, P.A. García-Sánchez, *Numerical Semigroups,*
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Recently, the study of factorizations on integral domains has moved to the setting of commutative cancellative monoids (this is mainly due to the fact that addition is not needed to study factorizations into irreducibles). Numerical semigroups are cancellative monoids. Problems of factorizations in a monoid are closely related to presentations of the monoid. By taking advantage of the results obtained in the past decades for the computation of minimal presentations of a numerical semigroup, numerical semigroups have become a nice source of examples in factorization theory. This is not the only connection with number theory. Recently, the study of certain Diophantine modular inequalities gave rise to the concept of proportionally modular numerical semigroups, which are related with the Stern-Brocot tree, and whose finite intersections can be realized as the positive cone of certain amenable C^*-algebras.

Finding the set of factorizations of an element in a numerical semigroup can be done with linear integer programming. We will also show another relation with linear integer programming, by proving that the set of numerical semigroups with given multiplicity is in one-to-one connection with the set of integer points in a rational cone.

From a classic point of view, people working in semigroup theory have been mainly concerned with characterizing families of semigroups via the properties they fulfill. In the last chapter of this monograph, we present several characterizations of numerical semigroups as finitely generated commutative monoids with some extra properties.

At the end of each chapter, the reader will find a series of exercises. Some cover concepts not included in the theory presented in the book, but whose relevance has been highly motivated in the literature, and can be solved by using the tools presented in this monograph. Others are simply thought of as a tool to practice and to deepen the definitions given in the chapter. There is also a series of exercises that covers some recent results, and a reference to where they can be found is given. Sometimes these problems are split in smaller parts so that the readers can produce their own proofs.

Some of the proofs presented in this volume can be performed by using commutative algebra tools. Our goal has been to write a self-contained monograph on numerical semigroups that needs no auxiliary background other than basic integer arithmetic. This is mainly why we have not taken advantage of commutative algebra or algebraic geometry.

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Chapter 1
Notable elements

Introduction

The study of numerical semigroups is equivalent to that of nonnegative integer solutions to a linear nonhomogeneous equation with positive integer coefficients. Thus it is a classic problem that has been widely treated in the literature (see [12, 13, 22, 28, 42, 101, 102]). Following this classic line, two invariants play a role of special relevance in a numerical semigroup. These are the Frobenius number and the genus. Besides, in the literature one finds many manuscripts devoted to the study of analytically unramified one-dimensional local domains via their value semigroups, which turn out to be numerical semigroups (just to mention some of them, see [5, 6, 19, 27, 32, 44, 105, 107]). Playing along this direction other invariants of a numerical semigroup arise: the multiplicity, embedding dimension, degree of singularity, conductor, Apéry sets, pseudo-Frobenius numbers and type. These invariants have their interpretation in this context, and this is the reason why their names may seem bizarre in the scope of monoids. In this sense the monograph [5] serves as an extraordinary dictionary between these apparently two different parts of Mathematics.

1 Monoids and monoid homomorphisms

Numerical semigroups live in the world of monoids. Thus we spend some time here recalling some basic definitions and facts concerning them.

A *semigroup* is a pair $(S,+)$ with S a set and $+$ a binary operation on S that is associative. All semigroups considered in this book are commutative ($a+b=b+a$ for all $a,b \in S$). For this reason we will not keep repeating the adjective commutative in what follows. Usually we will also omit the binary operation $+$ while referring to a commutative semigroup and will write S instead of $(S,+)$. A subsemigroup T of a semigroup S is a subset that is closed under the binary operation considered on

J.C. Rosales, P.A. García-Sánchez, *Numerical Semigroups,*
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S. Clearly, the intersection of subsemigroups of a semigroup S is again a subsemigroup of S. Thus given A a nonempty subset of S, the smallest subsemigroup of S containing A is the intersection of all subsemigroups of S containing A. We denote this semigroup by $\langle A \rangle$, and call it the *subsemigroup generated* by A. It follows easily that

$$\langle A \rangle = \{\, \lambda_1 a_1 + \cdots + \lambda_n a_n \mid n \in \mathbb{N} \setminus \{0\}, \lambda_1, \ldots, \lambda_n \in \mathbb{N} \setminus \{0\}, a_1, \ldots, a_n \in A \,\}$$

(where $\mathbb{N}$ denotes the set of nonnegative integers). We say that S is generated by $A \subseteq S$ if $S = \langle A \rangle$. In this case, A is a *system of generators* of S. If A has finitely many elements, then we say that S is finitely generated.

A semigroup M is a *monoid* if it has an identity element, that is, there is an element in M, denoted by 0, such that $0 + a = a + 0 = a$ for all $a \in M$ (recall that we are assuming that the semigroups in this book are commutative, whence this also extends to monoids).

A subset N of M is a *submonoid* of M if it is a subsemigroup of M and $0 \in N$. Observe if M is a monoid, then $\{0\}$ is a submonoid of M. This is called the *trivial* submonoid of M. As for semigroups, the intersection of submonoids of a monoid is again one of its submonoids. Given a monoid M and a subset A of M, the smallest submonoid of M containing A is

$$\langle A \rangle = \{\, \lambda_1 a_1 + \cdots + \lambda_n a_n \mid n \in \mathbb{N},\ \lambda_1, \ldots, \lambda_n \in \mathbb{N} \text{ and } a_1, \ldots, a_n \in A \,\},$$

which we will call the submonoid of M generated by A. As in the semigroup case, the set A is a *system of generators* of M if $\langle A \rangle = M$, and we will also say that M is generated by A. Accordingly, a monoid M is *finitely generated* if there exists a system of generators of M with finitely many elements. Note that $\langle \emptyset \rangle = \{0\} = \langle 0 \rangle$.

Given two semigroups X and Y, a map $f : X \to Y$ is a *semigroup homomorphism* if $f(a+b) = f(a) + f(b)$ for all $a, b \in X$. We say that f is a *monomorphism*, an *epimorphism*, or an *isomorphism* if f is injective, surjective or bijective, respectively. Clearly, if f is an isomorphism so is its inverse f^{-1}. Two semigroups X and Y are said to be isomorphic if there exists an isomorphism between them. We will denote this fact by $X \cong Y$.

A map $f : X \to Y$ with X and Y monoids is a *monoid homomorphism* if it is a semigroup homomorphism and $f(0) = 0$. The concepts of monomorphism, epimorphism, and isomorphism of monoids are defined as for semigroups.

2 Multiplicity and embedding dimension

The set $\mathbb{N}$ with the operation of addition is a monoid. In this book we are mainly interested in the submonoids of $\mathbb{N}$. We see next that they can be classified up to isomorphism by those having finite complement in $\mathbb{N}$. A submonoid of $\mathbb{N}$ with finite complement in $\mathbb{N}$ is a *numerical semigroup*. In this section we show that every numerical semigroup (and thus every submonoid of $\mathbb{N}$) is finitely generated, admits a unique minimal system of generators and its cardinality is upper bounded by the least positive element in the monoid.

For A a nonempty subset of $\mathbb{N}$, $\langle A\rangle$, the submonoid of $\mathbb{N}$ generated by A, is a numerical semigroup if and only if the greatest common divisor of the elements of A is one.

Lemma 2.1. *Let A be a nonempty subset of $\mathbb{N}$. Then $\langle A\rangle$ is a numerical semigroup if and only if* $\gcd(A)=1$.

Proof. Let $d=\gcd(A)$. Clearly, if s belongs to $\langle A\rangle$, then $d \mid s$. As $\langle A\rangle$ is a numerical semigroup, $\mathbb{N}\setminus\langle A\rangle$ is finite, and thus there exists a positive integer x such that $d \mid x$ and $d \mid x+1$. This forces d to be one.

For the converse, it suffices to prove that $\mathbb{N}\setminus\langle A\rangle$ is finite. Since $1=\gcd(A)$, there exist integers $z_1,\dots,z_n$ and $a_1,\dots,a_n\in A$ such that $z_1a_1+\cdots+z_na_n=1$. By moving those terms with z_i negative to the right-hand side, we can find $i_1,\dots,i_k,j_1,\dots,j_l\in\{1,\dots,n\}$ such that $z_{i_1}a_{i_1}+\cdots+z_{i_k}a_{i_k}=1-z_{j_1}a_{j_1}-\cdots-z_{j_l}a_{j_l}$. Hence there exists $s\in\langle A\rangle$ such that $s+1$ also belongs to $\langle A\rangle$. We prove that if $n\geq(s-1)s+(s-1)$, then $n\in\langle A\rangle$. Let q and r be integers such that $n=qs+r$ with $0\leq r<s$. From $n\geq(s-1)s+(s-1)$, we deduce that $q\geq s-1\geq r$. It follows that $n=(rs+r)+(q-r)s=r(s+1)+(q-r)s\in\langle A\rangle$. □

Numerical semigroups classify, up to isomorphism, the set of submonoids of $\mathbb{N}$.

Proposition 2.2. *Let M be a nontrivial submonoid of $\mathbb{N}$. Then M is isomorphic to a numerical semigroup.*

Proof. Let $d=\gcd(M)$. By Lemma 2.1, we know that $S=\langle\{\frac{m}{d}\mid m\in M\}\rangle$ is a numerical semigroup. The map $f:M\to S$, $f(m)=\frac{m}{d}$ is clearly a monoid isomorphism. □

If A and B are subsets of integer numbers, we write $A+B=\{a+b\mid a\in A, b\in B\}$. Thus for a numerical semigroup S, if we write $S^*=S\setminus\{0\}$, the set S^*+S^* corresponds with those elements in S that are the sum of two nonzero elements in S.

Lemma 2.3. *Let S be a submonoid of $\mathbb{N}$. Then $S^*\setminus(S^*+S^*)$ is a system of generators of S. Moreover, every system of generators of S contains $S^*\setminus(S^*+S^*)$.*

Proof. Let s be an element of S^*. If $s\notin S^*\setminus(S^*+S^*)$, then there exist $x,y\in S^*$ such that $s=x+y$. We repeat this procedure for x and y, and after a finite number of steps ($x,y<s$) we find $s_1,\dots,s_n\in S^*\setminus(S^*+S^*)$ such that $s=s_1+\cdots+s_n$. This proves that $S^*\setminus(S^*+S^*)$ is a system of generators of S.

Now, let A be a system of generators of S. Take $x\in S^*\setminus(S^*+S^*)$. There exist $n\in\mathbb{N}\setminus\{0\}$, $\lambda_1,\dots,\lambda_n\in\mathbb{N}$ and $a_1,\dots,a_n\in A$ such that $x=\lambda_1a_1+\cdots+\lambda_na_n$. As $x\notin S^*+S^*$, we deduce that $x=a_i$ for some $i\in\{1,\dots,n\}$. □

This property also holds for any submonoid S of $\mathbb{N}^r$ for any positive integer r. The idea is that whenever $s=x+y$ with x and y nonzero, then x is strictly less than s with the usual partial order on $\mathbb{N}^r$. And there are only finitely many elements $x\in\mathbb{N}^r$

with $x \leq s$. However the set $S^* \setminus (S^* + S^*)$ needs not be finite for r greater than one (see Exercise 2.15). We are going to see that for $r = 1$ this set is always finite. To this end we introduce what is probably the most versatile tool in numerical semigroup theory.

Let S be a numerical semigroup and let n be one of its nonzero elements. The Apéry set (named so in honour of [2]) of n in S is

$$\mathrm{Ap}(S,n) = \{ s \in S \mid s - n \notin S \}.$$

Lemma 2.4. *Let S be a numerical semigroup and let n be a nonzero element of S. Then* $\mathrm{Ap}(S,n) = \{0 = w(0), w(1), \ldots, w(n-1)\}$*, where $w(i)$ is the least element of S congruent with i modulo n, for all $i \in \{0, \ldots, n-1\}$.*

Proof. It suffices to point out that for every $i \in \{1, \ldots, n-1\}$, there exists $k \in \mathbb{N}$ such that $i + kn \in S$. □

Example 2.5. Let S be the numerical semigroup generated by $\{5,7,9\}$. Then $S = \{0,5,7,9,10,12,14,\rightarrow\}$ (the symbol $\rightarrow$ means that every integer greater than 14 belongs to the set). Hence $\mathrm{Ap}(S,5) = \{0,7,9,16,18\}$.

Observe that the above lemma in particular implies that the cardinality of $\mathrm{Ap}(S,n)$ is n. With this result, we easily deduce the following.

Lemma 2.6. *Let S be a numerical semigroup and let $n \in S \setminus \{0\}$. Then for all $s \in S$, there exists a unique $(k,w) \in \mathbb{N} \times \mathrm{Ap}(S,n)$ such that*

$$s = kn + w.$$

This lemma does not hold for submonoids of $\mathbb{N}^r$ in general. However, there are certain families of submonoids of $\mathbb{N}^r$ for which a similar property holds, and this apparently innocuous result makes it possible to translate some of the known results for numerical results to a more general scope (see [78]).

We say that a system of generators of a numerical semigroup is a *minimal* system of generators if none of its proper subsets generates the numerical semigroup.

Theorem 2.7. *Every numerical semigroup admits a unique minimal system of generators. This minimal system of generators is finite.*

Proof. Lemma 2.3 states that $S^* \setminus (S^* + S^*)$ is the minimal system of generators of S. By Lemma 2.6, we have that for any $n \in S^*$, we get that $S = \langle \mathrm{Ap}(S,n) \cup \{n\} \rangle$. As $\mathrm{Ap}(S,n) \cup \{n\}$ is finite, we deduce that $S^* \setminus (S^* + S^*)$ is finite. □

As every submonoid of $\mathbb{N}$ is isomorphic to a numerical semigroup, this property translates to submonoids of $\mathbb{N}$.

Corollary 2.8. *Let M be a submonoid of $\mathbb{N}$. Then M has a unique minimal system of generators, which in addition is finite.*

Proof. Set $d = \gcd(M)$. Then $T = \left\{ \frac{x}{d} \mid x \in M \right\}$ is a submonoid of $\mathbb{N}$ such that $\gcd(T) = 1$. In view of Lemma 2.1, this means that T is a numerical semigroup. If A is the minimal system of generators of T, then $\{ da \mid a \in A \}$ is the minimal system of generators of M. □

Corollary 2.9. *Let M be a submonoid of $\mathbb{N}$ generated by $\{0 \neq m_1 < m_2 < \cdots < m_p\}$. Then $\{m_1, \ldots, m_p\}$ is a minimal system of generators of M if and only if $m_{i+1} \notin \langle m_1, \ldots, m_i \rangle$.*

Let S be a numerical semigroup and let $\{n_1 < n_2 < \cdots < n_p\}$ be its minimal system of generators. Then n_1 is known as the *multiplicity* of S, denoted by $\mathrm{m}(S)$. The cardinality of the minimal system of generators, p, is called the *embedding dimension* of S and will be denoted by $\mathrm{e}(S)$.

Proposition 2.10. *Let S be a numerical semigroup. Then*

1) $\mathrm{m}(S) = \min(S \setminus \{0\})$,
2) $\mathrm{e}(S) \leq \mathrm{m}(S)$.

Proof. Clearly the multiplicity is the least positive integer in S. The other statement follows from the fact that $\{\mathrm{m}(S)\} \cup \mathrm{Ap}(S, \mathrm{m}(S)) \setminus \{0\}$ is a system of generators of S with cardinality $\mathrm{m}(S)$. □

Observe that $\mathrm{e}(S) = 1$ if and only if $S = \mathbb{N}$. If m is a positive integer, then clearly $S = \{0, m, \rightarrow\}$ is a numerical semigroup with multiplicity m. It is easy to check that a minimal system of generators for S is $\{m, m+1, \ldots, 2m-1\}$. Hence $\mathrm{e}(S) = m = \mathrm{m}(S)$.

3 Frobenius number and genus

Frobenius in his lectures proposed the problem of giving a formula for the largest integer that is not representable as a linear combination with nonnegative integer coefficients of a given set of positive integers whose greatest common divisor is one. He also threw the question of determining how many positive integers do not have such a representation. By using our terminology, the first problem is equivalent to give a formula, in terms of the elements in a minimal system of generators of a numerical semigroup S, for the greatest integer not in S. This element is usually known as the *Frobenius number* of S, though in the literature it is sometimes replaced by the *conductor* of S, which is the least integer x such that $x + n \in S$ for all $n \in \mathbb{N}$. The Frobenius number of S is denoted here by $\mathrm{F}(S)$ and it is the conductor of S minus one. As for the second problem, the set of elements in $\mathrm{G}(S) = \mathbb{N} \setminus S$ is known as the set of *gaps* of S. Its cardinality is the *genus* of S, $\mathrm{g}(S)$, which is sometimes referred to as the *degree of singularity* of S.

Example 2.11. Let $S = \langle 5, 7, 9 \rangle$. We know that $S = \{0, 5, 7, 9, 10, 12, 14, \rightarrow\}$ and thus $\mathrm{F}(S) = 13$, $\mathrm{G}(S) = \{1, 2, 3, 4, 6, 8, 11, 13\}$ and $\mathrm{g}(S) = 8$.

There is no known general formula for the Frobenius number nor for the genus for numerical semigroups with embedding dimension greater than two (see [21] where it is shown that no polynomial formula can be found in this setting or the monograph [52] for a state of the art of the problem). However, if the Apéry set of any nonzero element of the semigroup is known, then both invariants are easy to compute.

Proposition 2.12 ([101]). *Let S be a numerical semigroup and let n be a nonzero element of S. Then*

1) $\mathrm{F}(S) = (\max \mathrm{Ap}\,(S,n)) - n$,
2) $\mathrm{g}(S) = \frac{1}{n}(\sum_{w \in \mathrm{Ap}(S,n)} w) - \frac{n-1}{2}$.

Proof. Note that by the definition of the elements in the Apéry set, $(\max \mathrm{Ap}\,(S,n)) - n \notin S$. If $x > (\max \mathrm{Ap}\,(S,n)) - n$, then $x+n > \max \mathrm{Ap}\,(S,n)$. Let $w \in \mathrm{Ap}\,(S,n)$ be such that w and $x+n$ are congruent modulo n. As $w < x+n$, this implies that $x = w+kn$ for some positive integer k, and consequently $x-n = w+(k-1)n$ belongs to S.

Observe that for every $w \in \mathrm{Ap}\,(S,n)$, if w is congruent with i modulo n and $i \in \{0,\dots,n-1\}$, then there exists a nonnegative integer k_i such that $w = k_i n + i$. Thus, by using the notation of Lemma 2.4,

$$\mathrm{Ap}\,(S,n) = \{0, w(1) = k_1 n + 1, w(2) = k_2 n + 2, \dots, w(n-1) = k_{n-1} n + n - 1\}.$$

An integer x congruent with $w(i)$ modulo n belongs to S if and only if $w(i) \le x$. Thus

$$\begin{aligned} \mathrm{g}(S) &= k_1 + \cdots + k_{n-1} \\ &= \frac{1}{n}((k_1 n + 1) + \cdots + (k_{n-1} n + n - 1)) - \frac{n-1}{2} \\ &= \frac{1}{n} \sum_{w \in \mathrm{Ap}(S,n)} w - \frac{n-1}{2}. \end{aligned}$$ □

If S is a numerical semigroup minimally generated by $\langle a,b \rangle$, then

$$\mathrm{Ap}\,(S,a) = \{0, b, 2b, \dots, (a-1)b\}$$

and Proposition 2.12 tells us the following result that goes back to the end of the 19th century.

Proposition 2.13 ([102]). *Let a and b be positive integers with* $\gcd(a,b) = 1$.

1) $\mathrm{F}(\langle a,b \rangle) = ab - a - b$,
2) $\mathrm{g}(\langle a,b \rangle) = \frac{ab-a-b+1}{2}$.

Observe that for numerical semigroups of embedding dimension two $\mathrm{g}(S) = (\mathrm{F}(S)+1)/2$ (and thus $\mathrm{F}(S)$ is always an odd integer). This is not in general the case for higher embedding dimensions, though this property characterizes a very interesting class of numerical semigroups as we will see later.

If S is a numerical semigroup and $s \in S$, then $\mathrm{F}(S) - s$ cannot be in S. From this we obtain that the above equality is just an inequality in general.

Lemma 2.14. *Let S be a numerical semigroup. Then*

$$\mathrm{g}(S) \geq \frac{\mathrm{F}(S)+1}{2}.$$

Thus numerical semigroups for which the equality holds are numerical semigroups with the “least” possible number of gaps.

Remark 2.15. If one fixes a positive integer f, then it is not true in general that there are more numerical semigroups with Frobenius number $f+1$ than numerical semigroups with Frobenius number f. The following table can be found in [91] ($\mathrm{ns}(F)$ stands for the number of numerical semigroups with Frobenius number F).

F	$\mathrm{ns}(F)$	F	$\mathrm{ns}(F)$	F	$\mathrm{ns}(F)$
1	1	14	103	27	16132
2	1	15	200	28	16267
3	2	16	205	29	34903
4	2	17	465	30	31822
5	5	18	405	31	70854
6	4	19	961	32	68681
7	11	20	900	33	137391
8	10	21	1828	34	140661
9	21	22	1913	35	292081
10	22	23	4096	36	270258
11	51	24	3578	37	591443
12	40	25	8273	38	582453
13	106	26	8175	39	1156012

Bras-Amorós in [10] has computed the number of numerical semigroups with genus g for $g \in \{0,\ldots,50\}$, and her computations show a Fibonacci like behavior on the number of numerical semigroups with fixed genus less than or equal to 50. However it is still not known in general if for a fixed positive integer g there are more numerical semigroups with genus $g+1$ than numerical semigroups with genus g. We reproduce in Table 1 the results obtained by Bras-Amorós (in the table n_g stands for the number of numerical semigroups with genus g).

Lemma 2.16. *Let S be a numerical semigroup generated by* $\{n_1, n_2, \ldots, n_p\}$. *Let* $d = \gcd\{n_1, \ldots, n_{p-1}\}$ *and set* $T = \langle n_1/d, \ldots, n_{p-1}/d, n_p \rangle$. *Then*

$$\mathrm{Ap}(S, n_p) = d(\mathrm{Ap}(T, n_p)).$$

Table 1 Number of numerical semigroups of genus up 50.

g	n_g	$n_{g-1}+n_{g-2}$	$(n_{g-1}+n_{g-2})/n_g$	n_g/n_{g-1}
0	1			
1	1			1
2	2	2	1	2
3	4	3	0.75	2
4	7	6	0.857143	1.75
5	12	11	0.916667	1.71429
6	23	19	0.826087	1.91667
7	39	35	0.897436	1.69565
8	67	62	0.925373	1.71795
9	118	106	0.898305	1.76119
10	204	185	0.906863	1.72881
11	343	322	0.938776	1.68137
12	592	547	0.923986	1.72595
13	1001	935	0.934066	1.69088
14	1693	1593	0.940933	1.69131
15	2857	2694	0.942947	1.68754
16	4806	4550	0.946733	1.68218
17	8045	7663	0.952517	1.67395
18	13467	12851	0.954259	1.67396
19	22464	21512	0.957621	1.66808
20	37396	35931	0.960825	1.66471
21	62194	59860	0.962472	1.66312
22	103246	99590	0.964589	1.66006
23	170963	165440	0.967695	1.65588
24	282828	274209	0.969526	1.65432
25	467224	453791	0.971249	1.65197
26	770832	750052	0.973042	1.64981
27	1270267	1238056	0.974642	1.64792
28	2091030	2041099	0.976121	1.64613
29	3437839	3361297	0.977735	1.64409
30	5646773	5528869	0.97912	1.64254
31	9266788	9084612	0.980341	1.64108
32	15195070	14913561	0.981474	1.63973
33	24896206	24461858	0.982554	1.63844
34	40761087	40091276	0.983567	1.63724
35	66687201	65657293	0.984556	1.63605
36	109032500	107448288	0.98547	1.63498
37	178158289	175719701	0.986312	1.63399
38	290939807	287190789	0.987114	1.63304
39	474851445	469098096	0.987884	1.63213
40	774614284	765791252	0.98861	1.63128
41	1262992840	1249465729	0.98929	1.63048
42	2058356522	2037607124	0.989919	1.62975
43	3353191846	3321349362	0.990504	1.62906
44	5460401576	5411548368	0.991053	1.62842
45	8888486816	8813593422	0.991574	1.62781
46	14463633648	14348888392	0.992067	1.62723
47	23527845502	23352120464	0.992531	1.62669
48	38260496374	37991479150	0.992969	1.62618
49	62200036752	61788341876	0.993381	1.6257
50	101090300128	100460533126	0.99377	1.62525