Beam Structures

Classical and Advanced Theories

Erasmo Carrera, Gaetano Giunta
and Marco Petrolo

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Contents

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After earning two degrees (Aeronautics, 1986, and Aerospace Engineering, 1988) at the Politecnico di Torino, Erasmo Carrera received his PhD degree in Aerospace Engineering jointly at the Politecnico di Milano, Politecnico di Torino, and Università di Pisa in 1991. He began working as a Researcher in the Department of Aerospace for the Politecnico di Torino in 1992 where he held courses on Missiles and Aerospace Structure Design, Plates and Shells, and the Finite Element Method. He became Associate Professor of Aerospace Structures and Computational Aeroelasticity in 2000, and Full Professor at the Politecnico di Torino in 2011. He has visited the Institute für Statik und Dynamik, Universität Stuttgart twice, the first time as a PhD student (six months in 1991) and then as Visiting Scientist under a GKKS Grant (18 months in 1995--1996). In the summers of 1996, 2003 and 2009, he was Visiting Professor at the ESM Department of Virginia Tech, at SUPMECA in Paris (France) and at the CRP TUDOR in Luxembourg, respectively. His main research topics are: inflatable structures, composite materials, finite elements, plates and shells, postbuckling and stability, smart structures, thermal stress, aeroelasticity, multibody dynamics, and the design and analysis of non-classical lifting systems. He is author of more than 300 articles on these topics, many of which have been published in international journals. He serves as referee for international journals, and as a contributing editor for Mechanics of Advanced Materials and Structures, Composite Structures and Journal of Thermal Stress.

Gaetano Giunta

Gaetano Giunta graduated in Aerospace Engineering at the Politecnico di Torino in 2004. In 2007 he defended his PhD thesis on “Deterministic and Stochastic Hierarchical Analysis of Failure and Vibration of Composites Plates and Shells” at the Politecnico di Torino. Dr Giunta carried out his post-doc at the Centre de Recherche Public Henri Tudor in Luxembourg and the Politecnico di Torino from February 2008 to January 2010. Currently he is an R&D Engineer at the Centre de Recherche Public Henri Tudor and is working on the project FNR CORE C09/MS/05 FUNCTIONALLY on “Functionally Graded Materials: Multi-Scale Modelling, Design and Optimisation” funded by the Fonds National de la Recherche Luxembourg (FNR). Dr Giunta would like to acknowledge the FNR for its support. His research covers the formulation of hierarchical analytical and finite element models for the static, free vibration, buckling and failure analysis of beam, plate, and shell structures made of conventional and advanced materials.

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Marco Petrolo is an Aerospace Engineer and a PhD student at the Politecnico di Torino. He has a BSc in Aerospace Engineering at the Politecnico di Torino and an MSc degree in Aerospace Engineering in a joint program between the Politecnico di Torino, TU Delft, and EADS. He is a Fulbright alumnus, and, as a such, he has spent research periods at the San Diego State University and at the University of Michigan. His research activity is focused on the development of refined models for the structural and aeroelastic design of composite and metallic structures. He works in Professor Carrera's research group in Turin on different aerospace applications, such as the structural analysis of composite lifting surfaces, multiscale problems, and nonlinear problems.

Preface

Beam models have made it possible to solve a large number of engineering problems over the last two centuries. Early developments, based on kinematic intuitions (bending theories), by pioneers such as Leonardo da Vinci, Euler, Bernoulli, Navier, and Barre de Saint Venant, have permitted us to consider the most general three-dimensional (3D) problem as a one-dimensional (1D) problem in which the unknowns only depend on the beam-axis position. These early theories are known as engineering beam theories (EBTs) or the Euler--Bernoulli beam theory (EBBT). Recent historical reviews have proposed that these theories should be referred to as the DaVinci--Euler--Bernoulli beam theory (DEBBT). The drawbacks of EBT are due to the intrinsic decoupling of bending and torsion (cross-section warping is not addressed by EBT) as well as to the difficulties involved in evaluating the additional five (normal and shear) stress components that are not provided by the Navier formula. Many torsion-beam theories which are effective for different types of beam sections are known. Many refinements of original EBT kinematics have been proposed. Amongst these, the one attributed to Timoshenko in which transverse shear deformations are included should be mentioned. The other refined theories mentioned herein are those by Vlasov and by Wagner, both of which lead to improved strain/stress field descriptions.

Over the last few decades, computational methods, in particular the finite element method, have made the use of classical beam theories much more successful and attractive. The possibility of solving complex framed structures with very different boundary conditions (mechanical and geometrical) has made it possible to analyze many complex problems involving thousands of degrees of freedom (DOFs) with acceptable accuracy. However, the difficulty of obtaining a complete stress/strain field in those sections with complex geometries or thin walls still remains an open question which can be addressed by refined and advanced beam theories.

During the last decade, the first author of this book proposed the Carrera Unified Formulation (CUF), which was first applied to plates and shells and then recently extended to beams. The CUF permits one to develop a large number of beam theories with a variable number of displacement unknowns by means of a concise notation and by referring to a few fundamental nuclei. Higher-order beam theories can easily be implemented on the basis of the CUF, and the accuracy of a large variety of beam theories can be established in a hierarchical and/or axiomatic vs. asymptotic sense. A modern form of beam theories can therefore be constructed in a hierarchical manner. The number of unknown variables is a free parameter of the problem. A 3D stress/strain field can be obtained by an appropriate choice of these variables for any type of beam problem: compact sections, thin-walled sections, bending, torsion, shear, localized loadings, static and dynamic problems.

This book details classical and modern beam theories. Accuracy of the known theories is established by using the modern technique in the CUF. Various beam problems, in particular beam sections from civil to aerospace applications (wing airfoils), are considered in static and dynamic problems. Numerical results are obtained using the MUL2 software, which is available on the web site www.mul2.com.

www.wiley.com/go/carrera

Introduction

A brief introduction to the contents of the book is given here together with an overview of the milestone contributions to beam structure analysis.

Why another book on beams?

There is no need for another book on beam theories. Many books are, in fact, available, which have been written by some of the most eminent and talented scientists in the theory of elasticity and structures. It would be extremely difficult to write a better book. So, *why a new book on beam theories*? The reason is the following: this book presents a method to deal with beam theories that has never been considered before. As will be explained in the following chapters, the method introduced by the first author over the last decade for plates and shells is applied here to beams to build a large class of 1D (beam) hierarchical (variable kinematic) theories, which are based on automatic techniques to build governing equations and/or finite element matrices. The resulting theories permit one to deal with any section geometries subjected to any loading conditions and, at the same time, to reach quasi-3D solutions. Such results make the present book unique.

Review of historical contributions

Beam theories are extensively used to analyze the structural behavior of slender bodies, such as columns, arches, blades, aircraft wings, and bridges. The main advantage of beam models is that they reduce the 3D problem to a set of variables that only depends on the beam-axis coordinate. The 1D structural elements obtained are simpler and computationally more efficient than 2D (plate/shell) and 3D (solid) elements. This feature makes beam theories very attractive for the static and dynamic analysis of structures.

The classical, most frequently employed theories are those by Euler–Bernoulli (Bernoulli, 1751; Euler, 1744), de Saint-Venant (1856a,b), and Timoshenko (1921, 1922). The first two do not account for transverse shear deformations. The Timoshenko model considers a uniform shear distribution along the cross-section of the beam. A comprehensive comparison of Euler–Bernoulli and Timoshenko theories was made by Mucichescu (1984). However, none of these theories can detect non-classical effects such as warping, out- and in-plane deformations, torsion–bending coupling, or localized boundary conditions, whether geometrical or mechanical. These effects are usually due to small slenderness ratios, thin walls, and the anisotropy of the materials.

Many methods have been proposed to overcome the limitations of classical theories and to allow the application of 1D models to any geometry or boundary condition. Many examples of these models can be found in many well-known books on the theory of elasticity, for example, the book by Novozhilov (1961). Recent developments in beam models have been obtained by means of different approaches: the introduction of shear correction factors, the use of warping functions based on the de Saint-Venant's solution, the variational asymptotic solution (VABS), generalized beam theories (GBTs), and higher-order beam models. Some of the most relevant contributions are discussed below.

A considerable amount of work has been done to try to improve the global response of classical beam theories through the use of appropriate shear correction factors, as in the books by Timoshenko and Goodier (1970) and by Sokolnikoff (1956). Amongst the many available articles on this issue, the papers by Cowper (1966), Krishna Murty (1985), Pai and Schulz (1999), and Mechab *et al.* (2008) are of particular interest. An extensive effort was made by Gruttmann and his co-workers (Gruttmann *et al.*, 1999; Gruttmann and Wagner, 2001; Wagner and Gruttmann, 2002) to compute shear correction factors for several structural cases: torsional and flexural shearing stresses in prismatic beams; arbitrary shaped cross-sections; wide, thin-walled, and bridge-like structures.

El Fatmi (El Fatmi, 2002, 2007a,b,c; El Fatmi and Zenzri, 2004) introduced improvements to the displacement models over the beam section by introducing a warping function, φ, to enhance the description of the normal and shear stress of the beam. End-effects due to boundary conditions have been investigated by means of this model, as in the work by Krayterman and Krayterman (1987).

The de Saint-Venant solution has been the theoretical base of many advanced beam models. The 3D elasticity equations were reduced to beam-like structures by Ladevéze and his co-workers (Ladéveze and Simmonds, 1996, 1998; Ladéveze *et al.*, 2004). The resulting solution was modeled as the sum of a de Saint-Venant part and a residual part and applied to high-aspect-ratio beams with thin-walled sections. Other beam theories have been based on the displacement field proposed by Ieşan (1986) and solved by means of a semi-analytical finite element by Dong and his co-workers (Dong *et al.*, 2001; Kosmatka *et al.*, 2001; Lin *et al.*, 2001; Lin and Dong, 2006).

Asymptotic-type expansions have been proposed by Berdichevsky *et al.* (1992) on the basis of variational methods. This work represents the starting point of an alternative approach to constructing refined beam theories where a characteristic parameter (e.g., the cross-sectional thickness of a beam) is exploited to build an asymptotic series. Those terms that exhibit the same order of magnitude as the parameter when it vanishes are retained. Some valuable contributions on asymptotic methods are those related to VABS models built by Volovoi *et al.* (1999), Volovoi and Hodges (2000), Popescu and Hodges (2000), Yu *et al.* (2002a,b) and Yu and Hodges (2004, 2005).

GBTs have been derived from Schardt’s work (Schardt, 1966, 1989, 1994). GBTs enhance classical theories by exploiting piecewise beam descriptions of thin-walled sections. GBT has been extensively employed and extended, in various forms, by Silvetre and Camotim and their co-workers (Dinis *et al.*, 2006; Silvestre, 2002, 2003, 2007; Silvestre and Camotim, 2002). Many other higher-order theories which are based on enhanced displacement fields over the beam cross-section have been introduced to include non-classical effects. Some considerations of higher-order beam theories were made by Washizu (1968). An advanced model was proposed by Kanok-Nukulchai and Shik Shin (1984); these authors improved classical finite beam elements by introducing new degrees of freedom to describe cross-section behavior. Other refined beam models can be found in the excellent review by Kapania and Raciti (1989a,b) which focused on bending, vibration, wave propagations, buckling, and post-buckling. Aeroelastic problems of thin-walled structures were examined by means of higher-order beams by Librescu and Song (1992) and Qin and Librescu (2002).

The aforementioned literature overview clearly shows the interest in further developments of refined theories for beams.

Classical and modern approaches: variational

methods and CUF

This book focuses on refined theories, with only generalized displacement variables, for the static and dynamic analysis of 1D structures, “beams,” with compact and thin-walled sections. Higher-order models are obtained in the framework of the CUF. This formulation was developed over the last decade for plate/shell models (Carrera, 1995, 2002, 2003; Carrera *et al.*, 2008) and it has recently been extended to beam modeling (Carrera and Giunta, 2010). The present formulation has been exploited for the static analysis of compact and thin-walled structures (Carrera *et al.*, 2010a). Free-vibration analyses have been carried out on hollow cylindrical and wing models (Carrera *et al.*, 2011, 2011). A beam model with only displacement degrees of freedom has been developed (Carrera and Petrolo, 2010) and asymptotic-like results were obtained in Carrera and Petrolo (2011).

CUF is a hierarchical formulation which considers the order of the model as a free parameter (i.e., as input) of the analysis; in other words, refined models are obtained with no need for ad hoc formulations. Beam theories are obtained on the basis of Taylor-type expansions. Euler–Bernoulli and Timoshenko beam theories are obtained as particular cases. The finite element method is used to handle arbitrary geometries as well as geometrical and loading conditions.

Outline of the contents

A brief description of the book's layout is given here to provide a brief overview of what will be discussed. Chapter 1 presents the basic equations that the structural analysis is based on: equilibrium equations, strain–displacement geometrical relations, and constitutive equations. The principle of virtual displacements is also introduced in strong and weak forms.

Chapter 2 focuses on the description of classical beam theories: namely, the Euler–Bernoulli and Timoshenko models. The kinematics model of these theories is introduced and then strains, stresses, stress resultants, and elastica equations are derived. Numerical examples are given in order to highlight the differences between these two models.

The first refined model of this book is given in Chapter 3, where the complete linear expansion case is presented. Particular attention is given to the importance of the in-plane stretching terms that characterize this model. Examples are provided in order to underline the importance of these terms and the ineffectiveness of classical models to deal with in-plane stretching.

A first attempt toward the construction of a unified theory is made in Chapter 4, where the aforementioned classical and linear models are represented in a unified manner. The presented procedure represents the first fundamental step in obtaining the Carrera Unified Formulation (CUF). The chapter closes with a discussion on the Poisson locking phenomenon and its correction.

The CUF is introduced in Chapter 5. A detailed description of all the keypoints related to the CUF is provided. Strong and weak forms are provided. All the fundamental matrices are derived by means of the fundamental nuclei assembly system, which represents the core of the formulation.

Chapter 6 presents a number of higher-order theories, in terms of displacement and strain fields. Formulas for obtaining any-order models are provided in order to stress the hierarchical capabilities of the CUF.

A wide and comprehensive implementation guideline report is provided in Chapter 7 where all the main issues related to the finite element implementation of the CUF are addressed, including the assembly procedure, numerical integrations, and the main pre- and post-processing steps. A large number of numerical benchmarks are given for comparison purposes. Several chapters that deal with applications of CUF beam models then follow.

Chapter 8 presents the so-called shell-like capabilities of the model. Several structural problems are considered with particular attention being paid to thin-walled structures and point loads. Comparisons to shell and solid models are provided in terms of both accuracy and computational costs.

A buckling analysis is carried out in Chapter 9, whereas Chapter 10 shows the extension of CUF models to FGM (Functionally Graded Materials) made structures.

Chapter 11 presents the Arlequin method and its application to CUF beam models. Structures are analyzed by means of multiple-order beam theories, that is, the order of the beam model is locally tuned in order to optimize the computational costs of the analysis.

An extensive analysis of the effectiveness of higher-order theories is carried out in Chapter 12, where the so-called axiomatic–asymptotic method is presented and exploited to build reduced refined models on the basis of a given accuracy which is given as an input of the analysis. The effect of different characteristic parameters, such as slenderness, boundary conditions, and output variables, is considered.

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