LEARNING TO REASON

AN INTRODUCTION TO LOGIC, SETS, AND RELATIONS

NANCY RODGERS

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An Introduction to Logic, Sets, and Relations

Nancy Rodgers

Hanover College

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This book is dedicated

to the memory of

Edith and Neville Rodgers

◈ Contents

To Students

Rarely does one hear an English major say, "I like English, but I don't like to write," yet math students often say, "I like math, but I don't like to write proofs." Some students even tremble at the sound of an approaching proof assignment. The purpose of this book is to demystify the proof process by giving you the necessary reasoning techniques and language tools for constructing well-written arguments. This skill is as essential in mathematics and computer science as in English or any other discipline.

Learning to Reason is designed for a freshman/sophomore level course with no prerequisites except a desire to improve one's reasoning skills and one's ability to read and write mathematics and symbolic languages. The book covers the process of writing proofs, a process similar to writing in other disciplines, but the topics for our themes *(theorems)* will come from three unifying concepts that run through all areas of mathematics: logic, sets, and relations.

We sometimes require prerequisites for math courses in order to ensure a certain level of mathematical maturity – a maturity where one becomes an independent thinker who can figure things out without being told what to do. One of the main goals of this book is to speed up this maturation process by focusing on how we reason with mathematical language, emphasizing those elements of the language that tend to confuse students in advanced courses. Simple-sounding concepts such as substitution are not as simple as they sound. Simple words, such as "and," "or," "not," and "implies," lose their simplicity when we combine them in a sentence. If you are not fluent in how to manipulate these basic terms from which we build our language, you will be severely handicapped when you try to do any type of mathematical reasoning.

Another goal of this book is to help you see the common thread that runs throughout the vast universe of mathematics. Without this connection, you can easily get lost in an endless maze of mathematical concepts and not be able to see the forest for the trees. Many people have the misconception that mathematics is primarily a subject in which you do computations. I must confess that I have never been a fan of doing computations. In my college days, my fellow bridge players always wanted me to keep score because I was a math major. I felt like a chef being asked to wash the dishes. A chef creates dirty dishes in the process of cooking, but the goal is not to create dirty dishes. Similarly, mathematicians often generate computations in the process of doing mathematics, but the goal is not to generate computations. The goal is to create interesting structures and relations that can be supported with logical reasoning. This is the common thread that connects all of mathematics.

Contents

In Chapter 1, we cover the basic elements of mathematical language. Mathematical language is quite simple, which may surprise those who consider mathematics to be difficult and complex. Consider the myriad ways that we can form complex sentences in everyday language. In contrast, mathematical language is constructed from only five connectives and two quantifiers. If you understand how to manipulate these seven terms and how to use substitution, then you have acquired the basic technique on which logical reasoning is based.

In Chapter 2, we examine the reasoning process and how we organize our reasoning into a well-written form that can be classified as a proof. As in any good essay, a written proof contains an introduction, a body, and a conclusion. We will study various templates for writing proofs; however, the ability to construct a proof requires a deeper level of intellectual maturity than merely following an established procedure. To construct a proof, one must explore and question, find the inner structure of the situation, analyze the various parts, and then use logical reasoning to put the different pieces together to create the proof. The sparks that leap across our synapses during this creative process strengthen our powers of reasoning, one of the major benefits of studying mathematics.

In Chapter 3, we look at how we work with sets, the building blocks of mathematical language. Since prehistoric times, when people counted with a set of sticks or stones, sets have been at the foundation of mathematics. When we count, we are counting the number of elements in a set; when we analyze the form of a figure, we are analyzing a set of points; when we look at a function, we see a relation between two sets. Sets provide the basic framework for mathematical discourse.

In Chapter 4, we examine relations, a reasoning concept common to all disciplines. There are relations among pieces of music from the same period, works of art of the same style, and books of the same genre. In no discipline, including mathematics, can we analyze an object by itself; we must compare it to other objects. Relations provide a simple way to describe mathematics: Mathematics is the study of abstract relations.

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Learning a Language

As you begin your study of the language of reason, please remember that people do not learn a language through memorizing a list of words but through hearing the words used many times in various ways. The compactness of the language of mathematics with its attendant density of meaning requires that we read mathematics at a slow but contemplative pace. More than likely, we will not grasp its full import from one reading, and even if we do grasp it, we probably will not remember it all, for human memory needs a great deal of repetition to build enough bridges for the easy retrieval of stored information. So, it is important not only to read the sections, but also to reread them and ask questions about the content until you have a deep understanding of the material in both a verbal and a visual form. Anyone who is a lover of poetry knows that each rereading of a poem can bring new insights. The same is true in mathematics.

Working Out

Anyone can develop their reasoning skills if they are willing to invest the necessary time to work out with the exercises and the concepts. To become a good athlete or a good musician requires long hours of practice, so it is not surprising that learning how to reason also requires a similar investment of time. The exercises at the end of each section are an essential component of the learning process. To develop your reasoning skills, you should work out with the exercises on a daily basis. As you work through the discussions in the text, you should also write your own questions and observations. Through this process, you will build your understanding and personally internalize the meaning of the various concepts.

Throughout this text you will find activities that introduce you to concepts in the sections following them. If you work on the activities before you read the section, you will have the opportunity to discover relationships on your own. What you discover for yourself burns an indelible image in your memory and helps you to become a creative thinker, which is one of the most important skills needed in a changing society. Problems are easy when we have examples to guide us, but the creative thinkers are those who can blaze a path and create examples for others to follow. To be a logical thinker, we must develop our ability beyond merely copying procedures from examples provided by others.

When you take the extra time to figure out a problem on your own, you are building mental bridges that you can use in the future. The long hours of work that you do in building these bridges makes a deep impression that is firmly secured in your memory bank. On the other hand, when someone shows you how to do a problem, you are learning how to run across a bridge that someone else has built, which is not the same as learning how to build a bridge on your own. Computers are very adept at running across bridges that others have built, but they lack the human creativity to build new bridges for thought processes. To develop our reasoning powers beyond the mechanistic circuits of a computer, we must learn how to be creative thinkers.

To enliven your journey into the abstract world of reasoning, you may want to get into the gamesmanship of it by considering the exercises as a highly sophisticated game of mental prowess, or, for the more physically inclined, you may want to view them as aerobic exercises for the mind. The time that you spend will be a wise investment, for whatever path you take in life, the study of the topics in this book will help you to become an independent thinker who can reason in a logical manner.

Nancy Rodgers

To Teachers

Mathematics is simpler than other disciplines – physics or history, for example – because mathematics is concerned with such a very limited aspect of reality. Why, then, does such a simple subject seem so hard to so many people? I have come to believe that it is primarily a language problem. I became painfully aware of this problem in my first abstract algebra course when I ran head-on into a brick wall of mathematical language. I remember long hours of mental labor interrupted by a recurring question: why on earth did I major in math?

The next year I had a topology teacher, Professor John Seldon, who gave us a collection of theorems to prove from *Eléments de mathématique* by Bourbaki. As I worked though Bourbaki's organization of the foundations of mathematics, I began, for the first time, to understand the beautiful simplicity of mathematical language. After that experience, my studies became much easier because I now knew how to use mathematical language to structure my thinking.

Years later, while contemplating pedagogical methods that I might use to help my students over the same hurdle, I decided to write this text. The first version was used in an Algebraic Structures class. Because of student inquiries as to why they did not have this class earlier – since it would have helped them with the proofs they struggled with in other classes – the course was moved to the freshman/sophomore level. Through their many questions over the years, I began to understand the source of the great difficulty students have in writing proofs in upper division courses. The rules of syntax that seem so obvious after we subconsciously master them through long years of study are a huge language barrier to those on the other side of the fence. Some students have a great ear for the subtleties and nuances of languages and can easily learn a foreign language; a very small percentage of students have a similar gift for learning the language of mathematics. Granted, young children learn their native tongue by listening to those around them, but as we get older, most of us can benefit greatly by understanding the basic structure and syntax of a new language we are learning.

Organization

The initial goal in developing this text was to make Bourbaki's organization of the foundations of mathematics understandable and relevant at the freshman level. In addition, the book presents a lively discussion of the reasoning process, with a primary focus on deductive reasoning, but also including inductive reasoning, visual reasoning, and translations from everyday language to pictures and symbolic representations.

Starting with the foundations of logic in Chapter 1, the text explains how to analyze and logically manipulate individual sentences. In Chapter 2, the focus is on how to structure our thinking so that we can put sentences together to form a well-reasoned proof. The text illustrates the concepts with an elementary chain of ideas concerning integers, rational numbers, and real numbers. This connected series of examples and exercises helps students learn how to structure their thinking while also developing their understanding of numbers. The techniques learned here are reinforced as we examine sets, the basic building blocks of mathematics, in Chapter 3, and relations, where the action is in mathematics, in Chapter 4. This organizational structure gives students a meaningful overview of the vast subject of mathematics, while building their reasoning skills and their understanding of the basic concepts used throughout mathematics.

Special Features

The study of logical skeletons is fleshed out in mathematical settings with overviews of the structures they support and exercises that get students actively involved in and intrigued by the intellectual game of logical reasoning. Each section is preceded with a set of activities that give students the opportunity to discover for themselves important concepts from the next section. The activities encourage independent thinking and initiative, as well as help to raise the student's curiosity and interest in the upcoming material. After each section is a finely crafted set of exercises designed to help students develop their reasoning skills as they build a personal understanding of the language and notation. The exercises focus on those areas of mathematical language that tend to confuse students in upper division courses. They have been class-tested for several years and revised to maximize their benefit. Each chapter has a review section with related definitions grouped together. The definitions are alphabetized in a comprehensive glossary at the end of the book, followed by a symbol list.

The easy-going style of the book makes it accessible to a wide range of students. The concepts are carefully developed in a conversational writing style that speaks with a gentle authority, offering students motivation and encouragement along the way. It moves along at a brisk pace with careful analyses at points most likely to cause problems. The examples are cogent and thoughtfully presented, set off by lines that clearly separate them from the discussion. There is an energy in the conciseness of the writing and layout that makes it easy for students to read and remember what they have read.

Layout

In response to the first question in the book, one of my students, Becky Cantonwine, gave the following description of the difference between mathematical language and everyday language: "Mathematical language differs from everyday language in the same way that poetry differs from prose; every word or symbol is important and necessary, and their position is important to their meanings." Albert Einstein saw the same connection in his eloquent description of pure mathematics as "the poetry of logical ideas." Like written poetry, mathematical language is enhanced through the use of poetic lineation. Gestalt holistic patterns are easier to retain in the mind's eye, so poetic lineation is used in the text to highlight featured ideas and to assist the reader in working through dense notation and the thought processes involved in the reading of a proof. Great attention has been paid to the visual tone set by the geometric form of text layout, with white space generously used to minimize the denseness of the subject matter and to feature key thoughts and signposts in the reading. The overriding issue in all layout decisions was the presentation that would make it easiest to remember. Block text with its dense wrap-around lines is not as easy to assimilate and retain as text that incorporates active white space. I have tried to make the text as simple as possible, using a minimal but sufficient amount of words in explaining the concepts.

Audience

The text is designed as a bridge course for mathematics and computer science majors at the lower or upper division level. Any student who wants to learn how to structure their thinking and develop their reasoning skills will find it easy to use as a self-study text. Teachers of upper division math courses may want to use it as a supplementary text.

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◈ Chapter 1

Logical Reasoning

Logical reasoning is a form of discourse that is distinguished from other forms by its complete objectivity. In order to attain a pure state of objectivity with no room for ambiguities, the language of logic had to be developed with great precision and clearly defined rules. Personal interpretations of a story, a painting, or an historical event may vary considerably, but any two people who understand the language of logic will interpret a logical argument in essentially the same way. Unlike the tangled web of rules that we use subconsciously in our everyday discourse, the rules for logical reasoning are very exact with no exceptions to the rule.

When we reason within a logical framework, words must be manipulated according to the rules of the game. Fortunately, the rules are fairly simple because the language of logic is built from only seven basic terms: two quantifiers, *for all* and *for some*, and five operators for building compound sentences, *not*, *and*, *or*, *implies*, and *is equivalent to*. The first stage in mastering the art of logical reasoning is to learn how to manipulate these seven terms. Each of these terms is simple by itself, but the meaning can easily be misconstrued when two or more are used in the same sentence, especially since we do not always use them in a consistent way in our everyday language. Once you master the basic rules, called the laws of logic, for using these seven terms, this stage of the reasoning process will be as easy as driving a car.

The next stage is a bit more challenging, for we must learn how to 1) translate sentences phrased within the complex structure of everyday language into the simplified language of logic, 2) use the powerful tool of substitution to convert abstract knowledge into various forms, and 3) translate visual reasoning to a verbal form and vice-versa. In this chapter, we will cover the basic elements of logical reasoning, including quantifiers, logical operators, substitutions, and translations.

Activity 1.1

1. Reasoning is mentally performed within the context of a language, which provides the medium through which we organize and present our thoughts. To speak or think in any language, we must be aware of the basic structure of the language.
   a. How does mathematical language differ from everyday language?
   b. Compare the way that you learn mathematical language with the way that you learned to communicate with others in your preschool days.
   c. Compare the use of pronouns in everyday language with the use of variables in abstract languages. Do they serve the same role in the following two sentences?
      He is taller than 5 feet. $x > 5$
   d. What does "complete thought" mean to you? What elements of language are needed to express a complete thought?
   e. Make a list of nouns and a list of verb phrases that you have used in mathematics. Which have you used the most?
   f. What is a sentence? Do any of the following expressions form sentences? $1 < 2$ $1 + 2$ $1 + 2 = 3$
2. Let p and q represent sentences.
   Let $\sim p$ represent the negation of p.
   a. Does $\sim(p \textit{ and } q)$ mean the same as $(\sim p \textit{ and } \sim q)$?
      This question is very abstract.
      How should you start thinking about it?
   b. What is an abstraction? Is a number an abstraction? Is the color blue an abstraction?

≡ 1.1 Symbolic Language ≡

All written languages are based on symbols. The English language is written in terms of phonetic symbols that give pronunciation information. We can symbolically represent the addition concept with the phonetic symbol "plus" or with the ideographic symbol "+" which does not give pronunciation information. They both represent the same concept. However, in the process of logical reasoning, phonetic words can bog down our thought processes. For example, consider the following question from an algebra textbook by Al-Khowarizmi in the 9th century.

> The importance of an easily manipulated symbolism is that it enables those who are not great mathematicians in their generation to do without effort mathematics which would have baffled the greatest of their predecessors.
>
> *E. T. Bell*, 1945

> What must be the amount of a square, which, when twenty-one units are added to it, becomes equal to the equivalent of ten roots of that square?

Al-Khowarizmi's question, which would have challenged the great thinkers of the Middle Ages, can be answered by most high school students today who understand symbolic manipulations. Of course, the question would have to be posed in a symbolic form or they, too, might become entangled in the phonetic words:

> Find a solution to the equation $x^2 + 21 = 10x$.

> The function f assigns to each number in the domain the value that is the square of the number obtained by multiplying the original number by three and then adding one.

Take a moment and contemplate the adjacent sentence. How long did it take you to decipher its meaning? If you know function notation, you can comprehend the same sentence in symbolic form almost instantly: $f(x) = (3x+1)^2$

The great power of mathematical symbols is the ease with which the brain can process the information. Without the pronunciation baggage, the brain manipulates the symbols with great speed, thereby enabling us to focus on deeper questions. At the other extreme, though, too many ideographic symbols tend to shorten our attention span. A page full of nothing but symbols is not as inviting as a page where symbols are interwoven with words, so we try to find a delicate balance between the two, as illustrated in the above translation.

Unfortunately, mathematical symbols pose a language barrier to those who have not taken the time to learn their meaning, leaving many people with the impression that they are viewing a foreign language. However, it is not as difficult as it appears. All it requires is that we take the time to build a personal meaning for the various symbols.

Using Symbols

In order to use symbols in the reasoning process, we must know how the symbols can be manipulated. Even more importantly, though, we need to have a personal understanding of what the symbols represent. For example, we may be able to compute $145 \div 3$ with an algorithm, but we will not be able to use the answer in a meaningful way if we do not understand the meaning of dividing a set into subsets of equal size. If we do not build a personal meaning for symbols, we lose the base for our reasoning powers and become nothing more than a computer performing mechanical processes.

Learning a Language

When learning a foreign language, we may know the meaning of a word one week but forget it the next week. The same thing happens when we learn a symbolic language. Each symbol represents a concept, and to understand the concept, we need to think about what it represents and what it does not represent. We should work through examples for which the concept applies as well as examples for which the concept does not apply. As we use a new symbol in different examples and exercises, we will slowly build our personal understanding of it until we are comfortable using it. The more we use a concept, the deeper we implant it in our memory.

Some students pick up the symbolic language of mathematics or computer science faster than others do. Similarly, some people can sit down and play the piano by ear, while others have to struggle with years of practice. Those who learned how to play through hard work, though, often end up playing far superior to those blessed with an ear for music. It is not how fast you learn a language but how hard you work to develop a deep understanding of it.

Variables

A *variable* is a letter used to represent an arbitrary element of a given set; that set is called the *domain* of the variable.

Variables are an essential component of a symbolic language. As its name implies, a variable can vary and represent a variety of elements. Instead of talking about specific numbers, we usually talk about a generic number that is symbolized by a variable, such as x. Like pronouns in everyday language, variables serve as a place holder for substituting specific elements.

The set of elements that may be substituted for a variable is called its *domain*. In the following example, the domain for x is the set of integers:

$$\textit{For every integer } x,\ x < x + 1.$$

In computer science, a variable represents a storage space in the computer's memory where a number or a string of characters can be stored. Each variable is assigned a type that represents its domain. If a variable is assigned an integer type, then only integers can be stored in that variable.

We can use any letter as a variable, but we cannot use a letter to represent two different things within the same discussion. For example, an even number is any number that can be represented in the form $2k$ where k is an integer. However, if we apply this definition to two different even numbers within the same discussion, we cannot use "k" both times, for that would imply the two numbers are equal. Instead, we use another letter:

> **Theorem:** The sum of two even numbers is even.
> **Proof:**
> Let m and n be even numbers.
> Then $m = 2k$ for some integer k.
> Also, $n = 2j$ for some integer j.
> So, $n + m = 2k + 2j = 2(k + j)$.
> Since $k + j$ is an integer, by the definition of even, $n + m$ is even.

Let m and n be even numbers.
Then $m = 2k$ for some integer k.
Also, $n = 2j$ for some integer j.

In the adjacent proof, notice how the use of variables gives us a tangible way to work with even numbers, enabling us to make logical deductions about the abstract concept of even.

Sentences

Most communications in everyday language are phrased in terms of sentences, so it is not surprising that the same is true in mathematics. To express a complete thought, we use a sentence. Conversely, sentences require complete thoughts. If we are working with incomplete thoughts, either in our head or on paper, we cannot hope to make much progress in the reasoning process.

> Sentences require complete thoughts.

Our work in this chapter will focus on how we logically manipulate sentences. When we reason, the steps in our reasoning process are built from sentences, so it is essential that we know how to recognize sentences, especially those that are written in symbolic form.

◈ *Example*

Which of the following are sentences? $5 < 8$ $5 + 8$ $5 + 8 = 13$

1. "$5 < 8$" is a sentence. 5 is the subject and $<$ is the verb.
2. "$5 + 8$" is not a sentence because it does not have a verb.
3. "$5 + 8 = 13$" is a sentence. The subject is "$5 + 8$" and the verb is "$=$."

Relations & Operations

When we place the < symbol between two numbers, we get a sentence. These types of symbols represent *relations*. However, when we place the + symbol between two numbers, we get a number, not a sentence. The + symbol operates on two numbers and produces a new number, such as $5+8$.

Relations	Operations
$=$ $\neq$	$+$ $-$
$\approx$ $\cong$	$\times$ $\div$
$<$ $\leqslant$	$\cup$ $\cap$
$\subset$ $\subseteq$	$\vee$ $\wedge$

A relation gives a connection between two objects, whereas a binary *operation* operates on two objects and produces a third object. Relations produce sentences, but operations produce objects, such as a number or a set. In order to write well-formed mathematical sentences, we must be able to distinguish between relations and operations. Since they are different components of mathematical language, most word processors organize their equation editor with all relations grouped under one menu and all operations grouped under another menu, as illustrated on the left.

Fragments

$$\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

We may sometimes jot down fragments of sentences, such as the adjacent fragment from the famous quadratic formula, but we cannot use fragments in a proof. To complete the thought, we must add a subject and a verb. Students who do not carry along the beginning of the sentence, "$x=$," often do not know what the answer represents when they finish the computation. When we do not write in complete sentences, it is easy to get confused and lose track of what we are doing.

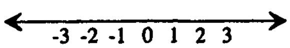
Subjects

A well-formed sentence must have both a subject and a verb. The most frequently used subjects in mathematical sentences are sets and numbers. We will now briefly review the different types of real numbers and examine sets later on in Chapter 3.

- Questions about "how many" elements in a finite set can be answered in terms of the *natural numbers*:

$$1, 2, 3, 4, 5, 6, \ldots$$

The Real Numbers



- To answer questions about "how much," such as how much length or how much area, we need a more extensive set of numbers, called the *real numbers*. We visualize the real numbers as coordinates of points on a number line, as illustrated on the left. In symbolic form, a *real number* is any number that can be represented as a decimal with a finite or infinite number of places.

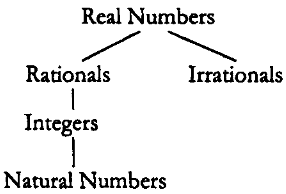
- The *integers* consist of the natural numbers, their negatives, and 0:

$$\ldots -3, -2, -1, 0, 1, 2, 3, \ldots$$

The positive integers are the natural numbers.
0 is neither positive nor negative.

- The *rational numbers* are numbers that can be represented as the quotient of two integers, such as $\frac{23}{5}$. The number .35 is a rational number because we can write it in fraction form: $\frac{35}{100}$. Using variables, we can define a rational number as follows: x is a rational number if and only if $x = \frac{a}{b}$ for some integers a and b with $b \neq 0$.

- Real numbers that are not rational, such as $\sqrt{2}$ or π, are called *irrational numbers*. Every real number is either rational or irrational.



The hierarchy of real numbers is given in the adjacent sketch. Each set is a subset of those sets that are chained above it.

Verbs

The action in everyday language comes from verbs. The same is true in mathematical language. However, most verbs in mathematics require objects, such as $x<y$ or $x=y$ or $X \subseteq Y$. In everyday language, we could have "x sings," but in mathematical language, x would have to sing to somebody, such as y. If x is a loner, we could have "x sings to x," but not just "x sings." Most mathematical verbs, such as those listed on the left, give relations between two objects.

Verbs

$\Rightarrow$
$\Leftrightarrow$
$\in$
$=$
$\approx$
$\cong$
$<$
$\leq$
$\subset$
$\subseteq$

One of the most important verbs is the implication verb, which we will examine in great detail in this chapter. This verb, which lies at the very foundation of logical reasoning, sets the structure for what we mean by a logical deduction. We use the implication to define a valid argument, which gives us the basic method for reasoning in a logical manner. We also use the implication verb to define other important verb phrases, such as "is equal to" and "is a subset of."

The most frequently used verb in mathematics is "*equals.*" In arithmetic and elementary algebra, this little verb provides the main action, with occasional help from the inequality verbs, $<$, $\leq$, $>$, $\geq$. The equals verb is used with both numbers and sets, whereas $\leq$ is used only with numbers.

The analogue of $\leq$ in set language is the subset verb, which gives a relation between two sets. *A is a subset of B*, notated as $A \subseteq B$, means that every element in A is also an element in B. This definition depends on another important verb phrase, *is an element of*, notated as $\in$.

$3 \in A$ means that 3 is an element of the set A.

Verbs that have properties similar to the equals relation, such as $\approx$, and $\cong$, are called *equivalence relations*. Verbs that impart some type of order on objects, such as $<$, $\leq$, $\subset$, and $\subseteq$ are called *order relations*. We will examine both equivalence relations and order relations in Chapter 4.

Statements

A *statement* is a sentence that is either true or false, but not both.

Some sentences, such as "7 is a lucky number," may be considered true by some people and false by others. We do not deal with this type of sentence in mathematics; instead, we restrict our discourse to sentences whose truth values are not debatable. We will use the term *statement* to denote a sentence that is either true or false, but not both. If a statement is true, then it cannot be false.

◈ *Example*

Which of the following sentences are statements?

$$3+2=5 \qquad 3+2=6 \qquad x+2=6$$

1. "$3+2=5$" is a true sentence, so it is a statement.
2. "$3+2=6$" is a false sentence, so it is a statement.
3. "$x+2=6$" is a sentence; however, it is neither true nor false, so it is *not* a statement.

True	False
T	F
1	0
On	Off

The *truth value* of a statement is either true or false, which we will represent with T and F. In computer science, we use 1 for true and 0 for false. A computer transmits information along an electronic highway in terms of electric circuits which are either on or off. We identify the ON-state, defined as 1, with "true" and the OFF-state, defined as 0, with "false."

Statements severely limit the scope of our discourse because the truth value of many sentences is somewhere between 0 and 1. For example, the weatherman's assertion that it will be "partly cloudy" may be true only 80% of the day. These types of sentences can be analyzed with a more general type of logic known as fuzzy logic (page 60), which was developed to program artificial intelligence into computers.

Open Statements

The sentence $x+2=6$ is not a statement, but it does become a statement when we substitute an element for x.

Substitute 4 for x: $4+2=6$ (True)

Substitute 3 for x: $3+2=6$ (False)

An *open statement* is a sentence with variables that is not a statement but becomes a statement when substitutions are made for the variables.

A sentence of this type is called an *open statement*. We might be tempted to say that an open statement is any statement that has a variable. However, this is not true for we can quantify the variables by prefixing the sentence with a quantifier, as illustrated in the following example.

♦ *Example*

The domain for x is the set of real numbers. Are any of the adjacent sentences open statements?

$2x + 3 = 5$

"$2x + 3 = 5$" is an open statement. It is neither true nor false, but each time we substitute a number for x, the sentence is either true or false.

For all x, $2x + 3 = 5$.

The second sentence is false, so it is not open.

There exists an x such that $2x + 3 = 5$.

The last sentence is true, so it is not open.

Even though the last two sentences in the above example have variables, they are not open statements because the variable is fixed (or bound) by the quantifier. Quantifiers are extremely important components of the reasoning process. We will examine them in detail in Section 1.2.

Solution Set

The *solution set* of an open statement in x is the set of elements from the domain of x that convert it to a true statement. To find the solution set of an equation, we solve the equation and then place the answers in a set. The solution set depends on the domain, as illustrated in the following examples.

♦ *Example*

1. What is the solution set of the open statement, $x + 2 = 0$?

   Before we can answer this question, we must know the domain for x. If the domain is the set of integers, the solution set is the set whose only element is -2, which we represent with set braces as $\{-2\}$.

   If the domain is the set of natural numbers, though, the solution set is empty, which we represent with either the symbol $\{\ \}$ or ϕ.

2. What is the solution set of the open statement, $x^2 = -1$?

   Before we can answer this question, we must know the domain for x. Both i and $-i$ are solutions to the above equation: $i^2 = -1$ and $(-i)^2 = -1$. So, if the domain is the set of complex numbers (page 14), the solution set consists of i and $-i$: $\{i, -i\}$.

However, if the domain is the set of real numbers, the solution set is the empty set.

When it is not possible to list all the elements in the solution set of an open statement $p(x)$, we can represent the solution set with the following set notation:

$$\{\, x \mid p(x) \,\}$$

We will examine set notation in more detail in Chapter 3.

Example

1. The domain for x is the set of real numbers. What is the solution set of the open statement, $x > 2$?

$\{\, x \mid x > 2 \,\}$

Since we cannot list the elements in the solution set nor give a pattern that indicates all the members of the set, we use the adjacent set notation to express the solution set. This notation is read as "the set of all x such that $x > 2$." If the reader does not know that the domain is the set of real numbers, then we should include it in the set description:

$$\{\, x \mid x > 2 \text{ and } x \text{ is a real number} \}$$

If the reader does know the domain of x, the shorter form gives a simpler image for focusing our thinking.

2. The domain for x is the set of real numbers and the domain for y is the set of real numbers. What is the solution set of the open statement, $x + 3y = 7$?

$\{\, (x, y) \mid x + 3y = 7 \}$

We cannot list all the elements in this set, so we use the adjacent set notation. Since we have two variables, the elements of the solution set are ordered pairs.

$(1, 2)$ is a member of this set since $1 + 3(2) = 7$.

$(2, 1)$ is a not a member of this set since $2 + 3(1) \neq 7$.

Compound Sentences

When we link two sentences with a connective like *and*, we create a compound sentence. For example, we can use *and* to connect the sentence $2 + 3 = 5$ with the sentence $4 + 5 = 9$:

$$2 + 3 = 5 \textit{ and } 4 + 5 = 9$$

"$2 + 3 = 5$" is called a *component sentence* of the compound sentence. In logic, we use only four connectives for building compound sentences: *and, or, implies, is equivalent to.* These terms are called *logical operators.*

Addition operates on 2 numbers and produces a new number.

And operates on 2 sentences and produces a new sentence.

In the adjacent box, notice the similarity between the addition operation on numbers and the *and* operation on sentences. Adding two numbers and combining two sentences are very different types of activities, but at the base level, the structure of what they do is the same. They are both binary operations, which is why we call *and* a logical operator.

Another important logical operator is the negation. Given a sentence, like $2+3=6$, we can make a new sentence by taking its negation:

It is not true that $2+3=6$.

$2+3=5$ and $4+5=9$.

$x<2$ or $x>5$.

$x<2$ implies that $x<3$.

$x<2$ is equivalent to $-x>-2$.

It is not true that $2+3=6$.

Negation is a *unary* logical operator, whereas the other four connectives are *binary* logical operators. As you probably know, "unary" means "one" and "binary" means "two." Negation forms a new sentence from a given sentence; the other four connectives form a new sentence from two given sentences, as illustrated on the left. It is rather surprising how much of our reasoning depends on these five logical operators. When we examine them in detail in Section 1.3, we will work with them in an *abstract* form, similar to abstract algebra.

Symbolic Sentences

In elementary algebra, we use letters to represent numbers and ideographic symbols to represent operations on numbers.

$$a+b=b+a$$
$$a\times(b+c)=a\times b+a\times c$$

Like an x-ray machine, this symbolic representation reveals the inner structure of arithmetic, making it easy to recognize and remember general rules for working with operations on numbers.

5 Logical Operators

$\sim p$:	not p
$p\wedge q$:	p and q
$p\vee q$:	p or q
$p\Rightarrow q$:	p implies q
$p\Leftrightarrow q$:	p is equivalent to q

To find general rules for reasoning with compound sentences, we do a similar type of abstraction. Instead of working with specific sentences, we will use the variables p and q to represent arbitrary sentences and the adjacent symbols to represent the five operations on sentences.

Using this abstract representation of compound sentences, we can formulate basic rules for manipulating the five logical operators. These rules enable us to automate our reasoning about the logical operators so that we have more time to ponder deeper questions. However, to apply the rules to specific sentences, we must be able to see the abstract structure of a compound sentence.

◈ *Example*

What is the structure of the following compound sentence?

$$(2+3=5) \text{ and } (4+5 \neq 7)$$

1. Let p and q represent the following sentences.
   p: $2+3=5$ $\quad$ q: $4+5=7$
   Then $p \wedge \sim q$: $(2+3=5)$ and $(4+5) \neq 7$
2. We could also let p: $2+3=5$ and q: $4+5 \neq 7$
   Then $p \wedge q$: $(2+3=5)$ and $(4+5) \neq 7$

We can view the above compound sentence as having either the structure $p \wedge q$ or the structure $p \wedge \sim q$, depending on whether we want to focus on the outside structure of the sentence or look deeper into its internal structure. The different views of the structure of a sentence are similar to viewing the outside structure of the human body or taking an x-ray view of its skeletal structure.

$p(x)$ notation

We will use the function notation $p(x)$, read as "p of x," to represent an open statement in the variable x. For example, we could let $p(x)$ represent "$x^2 + 4x - 1 = 5$." The notation $p(x)$ has two layers of variables: p is a variable that represents a sentence and x is a variable that represents a number. Whenever a new notation seems a little strange, we should work with examples and before long it will seem like a perfectly natural way to communicate. Function notation is based on the substitution principle. To translate $p(3)$, we substitute 3 for each occurrence of x.

$$p(x)\text{: } x^2 + 4x - 1 = 5$$

$$p(3)\text{: } 3^2 + 4(3) - 1 = 5$$

Formal Logic

In formal logic, a statement is called a *proposition*. Since the logical operators operate on propositions, the study of the rules for manipulating logical operators is called *propositional logic*. Open statements are called *predicates*, and the study of predicates is called *predicate logic*. Symbolic sentences are called *well-formed formulas*, sometimes abbreviated as wffs. Like the rules for grammar in everyday language, formal logic systems have syntax rules that govern how symbols can be strung together. For example, we cannot juxtapose two logical