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FOURIER ANALYSIS

Eric Stade

Pure and Applied Mathematics: A Wiley-Interscience Series of Texts, Monographs, and Tracts

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Fourier Analysis

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Fourier Analysis

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Contents

List of Figures

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Preface

This book is aimed primarily at upper level undergraduate and early graduate students in mathematics. I hope and believe, though, that it will also be of value to engineering students (particularly those studying electrical engineering, signal processing, and the like); to students in physics and other sciences; to scientists and technicians who use Fourier analysis "in real life"; and to anyone fostering an appreciation of the beauty and power of Fourier analysis, or a desire to acquire such an appreciation.

The prerequisites are few: A good grasp of calculus in one and several variables will suffice. A number of more advanced concepts are encountered in the course of the text—complex numbers, linear algebra, differential equations, and a good truckload or two of ideas from real analysis. But familiarity with these concepts is not required; in fact, I hope that the reader who has not encountered them previously will find herein a good introduction to them.

I've strived for relatively high degrees of mathematical rigor and completeness. But at the same time, I've tried to place the math in scientific and technological (and historical) context and to infuse due detail into discussions of specific applications. (The detail is, though, for the most part limited to that which relates to Fourier analysis. Real-world applications necessarily entail myriad other considerations; I've avoided reflection on these in order to stay on point, and because I understand them only marginally.)

Cover to cover, the book probably amounts to two semesters; however, a variety of different single-semester tracks may be extracted. For example: a robust course in Fourier series and boundary value problems could be constructed around Chapters 1–4, more or less. Here, "more or less" means the following. Sections 1.5, 1.10, and

3.9 could be skipped; they are not relevant to the solution of boundary value problems. Sections 1.6 and 3.10 are of tangential relevance; they might also be skipped or at most skimmed. Additionally, the proofs in Sections 1.4 and 1.7 and throughout Chapter 3 might also be omitted or treated cursorily, depending on one's focus. Omitting these proofs would likely leave time for some discussion of Fourier *integrals* and boundary value problems; the material for this may be found in Chapters 5 and 6 and Sections 7.1–7.3. (Only the first and last sections of Chapter 5 and the first, second, sixth, seventh, and eighth of Chapter 6 are truly essential here, as long as one is willing to glance back at other sections, upon the occasional reference to results therein.)

Alternatively, a course emphasizing aspects of Fourier analysis relevant to technological and scientific issues (other than the issues arising in the context of boundary value problems) could be based on Chapters 1, 3, and 5–8. (Any of the Sections 1.5, 1.10, 3.9, 6.3, 6.9, and 7.1–7.3 might be omitted.) Or one can teach a very "pure" Fourier theory course, omitting or minimizing discussion of the material in Chapters 2, 4, 7, and 8. Or one can, as I do, mix it up. In my own single-semester course, composed mostly of junior and senior math majors and beginning math graduate students, I focus on Chapter 1 and Sections 3.1–3.7, 5.1–5.4, 5.7, and 6.1–6.6 (from all of this I omit a good many proofs, but include a good many too) and throw in a smattering from the remainder. The smattering varies from year to year. I lean toward Sections 2.1, 2.4, 2.7, 2.9, 4.1, 4.2, and 7.4–7.10, and as much of Chapter 8 as there is time left for; usually, there's not much.

Additionally, there are excellent arguments to be made for two-semester (undergraduate and graduate) real analysis/Fourier analysis course sequences. These disciplines grew up together; indeed, real analysis was invented (discovered?) expressly to create the proper framework for investigation and formalization of Fourier's assertions. So a course in Fourier analysis constitutes a logical sequel to one in real analysis—and the converse is also true! That is, questions arising out of Fourier analysis *demand*, and therefore *motivate*, many real analysis constructs whose *raisons d'etre* are, sometimes, not otherwise apparent. So, the two semesters of a real analysis/Fourier analysis sequence might reasonably be presented in either order, depending on how one feels about these things, philosophically speaking. I like to think that this text would serve as well at the tail end of a real analysis/Fourier analysis track as it would at the leading end of a Fourier analysis/real analysis track. Either way, Chapters 1, 3, 5, and 6 should be of particular interest. (Perhaps Chapter 8 too. Local frequency analysis and wavelets certainly admit a cornucopia of applications, but the theory, and especially the real analysis, behind them is also of exceptional beauty. Theorem 8.4.1, in particular, encompasses an impressive array of real analysis BIG IDEAS.)

This book could never have come to be without the assistance and support of a number of people. My colleagues Larry Baggett and Dick Holley at the University of Colorado have been endlessly patient with my persistent emails and have done a monumental job of unconfusing me with regard to a number of questions. Edward Burger of Williams College, a better friend even than he is a teacher, has been an unflagging source of constructive criticism and, even more importantly, encouragement. Martin Hairer of HairerSoft has kindly permitted me to use, in the text, output from his excellent "Amadeus II" sound editing software. Joseph Hornak of Magnetic Resonance Laboratories has generously provided me with the FT-NMR data used in Section 7.10, along with some quite illuminating commentary and explanation regarding the related material.

The crew at John Wiley & Sons has been terrific throughout. I am particularly grateful to Steve Quigley, Laurie Rosatone, Susanne Steitz, and Lisa Van Horn, all of whom had the audacity to believe in this project.

A heartfelt thanks goes to ten years plus of Fourier analysis students, who have greatly inspired and continue to inspire me while also providing free proofreading. Among these students, Marc Lanskey, Tiffany Tasset, and Sonja Wieck have offered especially valuable suggestions regarding the manuscript itself.

Most of all, thanks to my beautiful, exceptional wife Beth and my wonderful, extraordinary boys Jack and Nick. Without you, I am nothing. This book is dedicated to you.

ERIC STADE

Boulder, Colorado

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Introduction

Thus there is no function ... which cannot be expressed by a trigonometric series ... [or] definite integral.

—Joseph Fourier [20]

Fourier analysis is the art and science whereby any reasonable function may be realized as a superposition of sinusoids, each of these sinusoids possessing a distinct frequency.

We explain: First, by *superposition* we mean a summation or similar process of amalgamation. We'll elaborate shortly, but for the moment the notion of a sum will be sufficient to capture the general sense of what's going on. Next, by *sinusoid* we mean a linear combination of the two functions $\cos 2\pi sx$ and $\sin 2\pi sx$, where x denotes a real variable and s some nonnegative, real constant. And by *frequency* of such a sinusoid we mean precisely this constant s. See Figure I.1.

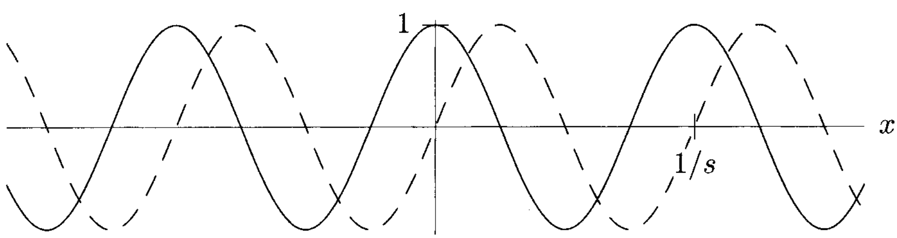

Fig. I.1 The functions $\cos 2\pi sx$ (solid) and $\sin 2\pi sx$ (dashed)

(We'll depict some other sinusoids a bit later: see Figs. I.3 and 1.17.) Note that, in Figure I.1, $s > 0$: A sinusoid of frequency zero is just a constant function, since $\cos 0 = 1$ and $\sin 0 = 0$.

All of this permits the following somewhat more explicit, though still rather rough, description of Fourier analysis: Fourier analysis *is*, roughly, the study of expressions of the form

$$f(x) = \sum_{s \in F_f} \left(A_s(f) \cos 2\pi s x + B_s(f) \sin 2\pi s x\right), \tag{I.1}$$

of conditions (on f) under which such an expression will exist; of the nature of the set F_f, and the coefficients $A_s(f)$ and $B_s(f)$, when it does; of generalizations and abstractions of such expressions; of their applications; and so on.

Fourier analysis is, in fact, often also called "frequency analysis," and the elements

$$A_s(f) \cos 2\pi s x + B_s(f) \sin 2\pi s x \tag{I.2}$$

of a superposition (I.1) the "frequency components" of the given function f. Indeed (I.2) is, for a given s, generally understood as "that part of f having frequency s." Also, the set F_f is frequently (pun intended) called the "frequency domain" of f, and (I.1) itself a "frequency decomposition" (also known as a "sinusoid decomposition") for f.

It turns out that Fourier analysis is of great import and utility, in both "abstract" and "concrete" settings: And why should this be? Why should Fourier analysis be central to so many issues in mathematics and the sciences? The short answer is this: It's because sinusoids do two particularly nice things. First, they *differentiate* in extraordinarily simple, fundamental ways; second, they *cycle* in extraordinarily simple, fundamental ways.

Regarding the first of these nice things, we note that the first derivative—and therefore also any higher derivative—of a sinusoid is another one, of the same frequency. In particular, the frequency component (I.2), let's call it $f_s(x)$, is readily seen to satisfy the basic differential equation

$$f_s''(x) = -(2\pi s)^2 f_s(x). \tag{I.3}$$

In fact, as is generally shown in an introductory differential equations course, *any* function f_s satisfying (I.3), for a given $s > 0$, is a sinusoid of frequency s. (If $s = 0$, then the solutions to (I.3) are the first-degree polynomials $f_0(x) = Ax + B$.)

As a consequence of this, a great variety of differential equations may be solved according to the following strategy. First, it's stipulated that the desired solution f have an expression of the form (I.1), for some set F_f dictated by the specifics of the problem at hand and for as yet unknown coefficients $A_s(f)$ and $B_s(f)$ (possibly dependent on other variables). Next, Fourier analysis and other considerations are applied to the explicit determination of these coefficients, and thus the exact nature of f is uncovered.

It was, in fact, precisely to address the solution of certain differential equations that Joseph Fourier (1768–1830) developed Fourier analysis—hence the name. He was specifically concerned, in his landmark 1807 manuscript *Theory of the Propagation of Heat in Solid Bodies* (and subsequent revisions and expansions thereof, the culmination of these being the 1822 book *The Analytical Theory of Heat* [20]), with the differential equations governing heat conduction. He was able not only to determine the nature of these equations, but also, via his sinusoidal analysis of functions and the *separation of variables* technique—which he also developed, and of which we'll make copious use in coming chapters—to solve them under various sets of assumptions. (Strictly speaking, both frequency decompositions and separation-of-variables arguments predated Fourier, in rudimentary forms. However, he was unquestionably the first to make systematic, fruitful use of either.)

Fourier was aware of the relevance of his ideas to other differential equations, besides those modeling heat flow. In particular he was able, using these ideas, to shed considerable light on the "wave equation," which had previously been studied by Leonard Euler and Daniel Bernoulli, among others. Even so, Fourier himself likely would not have predicted the ubiquity presently enjoyed by his ideas in the theory of differential equations.

Concerning the second particularly nice thing about sinusoids, we recall that they do, indeed, cycle. That is, they repeat themselves at regular intervals; that is, they're *periodic*. See Figure I.1 above. This periodicity is familiar, but its extraordinarily simple, fundamental nature should be emphasized. To this end we consider a point moving, with constant angular velocity, around the perimeter of a circle—such motion is, arguably, the simplest kind of periodic motion imaginable. Let's suppose, to be specific, that the motion is counterclockwise in a circle of radius 1, centered at the origin, and that the point sweeps out $2\pi s$ radians per unit of time t. Then, if this point has coordinates $(1, 0)$ at time zero, it will have coordinates $(\cos 2\pi st, \sin 2\pi st)$ at time t. See Figure I.2.

(We note especially that, because each revolution comprises 2π radians, the point just described has "cycling rate" s. That is, it completes s revolutions per unit t. This justifies our application to s of the term "frequency.")

Thus the simple, fundamental nature of sinusoids, from the perspective of periodicity, is evident. So in light of (I.1) and the discussion surrounding it, we can recognize Fourier analysis as a theory whereby quite general phenomena, whether or not they are themselves periodic or exhibit any obvious overall cyclical behavior, can be understood as *amalgamations* of basic periodic elements of definite frequencies. In particular Fourier analysis, in concert with other ideas and techniques from mathematics, science, and engineering, gives us the ability to investigate, identify, and even alter these elements. And this ability has a large number of practical applications.

Thus we've addressed, however briefly, the what and why of Fourier analysis. But let's turn our attention, even more briefly, to the when and how. Specifically, we now ask: Which functions *are* reasonable enough to be expressible as superpositions of sinusoids, and for those that are, what do these expressions look like and how do we find them?

These are big questions. Their answers, and the very pursuit of those answers, have had a profound impact on civilization as we know it. Our considerations of these answers, and this pursuit, will occupy a large portion of this book. For now, though,

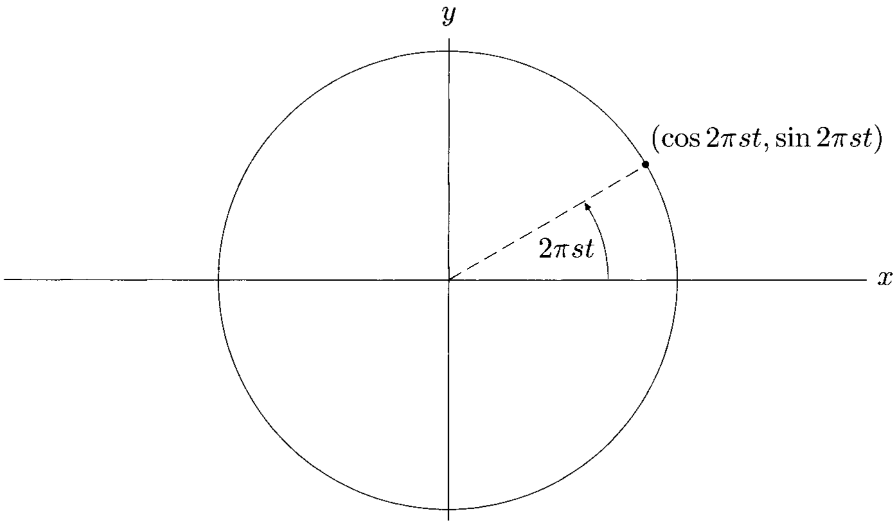

Fig. I.2 The functions $\cos 2\pi st$ and $\sin 2\pi st$

we'll try to say just enough about the how and the when to flesh out our overview.

We do so by addressing some gory details that we've thus far glossed over. Here's an especially gory one: The frequency domain F_f of a given function f is, in fact, usually *infinite*. So generally, for a superposition of sinusoid components, we'll need not a sum *per se*, but some infinite analog thereof. Namely, we'll require either an infinite series, if the set F_f is *discrete* (meaning it comprises isolated points on the real line), as it often is, or a definite integral, if this set is *continuous* (meaning it consists of a continuum of points on the line), as it also often is.

And this gives our discussion a new wrinkle; namely, once infinite processes are thrown into the mix, the issue of *convergence* of these processes must be addressed. That is, if we say a function *has* an expression as a superposition of sinusoids, we really mean this superposition *converges to* the function, but for this to make precise sense, we must be explicit about the meaning of "converges to."

As we'll see in this book, there are various useful notions of convergence of functions. And to each of these corresponds a different notion of reasonable function. That is, to each given sense of convergence will correspond a different set of criteria assuring that a function has a frequency decomposition converging to it.

It should be noted that, at the time of Fourier's work, the concept of convergence of functions—or of convergence at all, for that matter—was not a very well-formed one. Nor was the concept of reasonable function—or of function at all, for that matter. In fact, the first truly systematic studies of functions and convergence grew out of efforts to understand, clarify, elaborate on, generalize, and make rigorous Fourier's decompositions. Such studies then generated distinct mathematical theories of their own; these theories in turn formed the seeds of the vast discipline presently known, in the math world, as "real analysis." So Fourier, in case you were wondering whom to blame, is (indirectly) as responsible as anyone, and more responsible than most, for the development of that discipline.

More specifically, such mathematical quantities, constructs, and results as Riemann sums and integrals; the formal "ε-N" and "ε-δ" definitions of limits; related formal definitions of continuity and differentiability; definitions and theories of pointwise, uniform, and norm convergence of functions; Hilbert spaces; Lebesgue integrals and such theorems regarding them as Fubini's theorem and the Lebesgue dominated convergence theorem; distributions; and so on all arose out of investigations into Fourier's sinusoid decompositions. We'll touch upon each of these results, constructs, and quantities at one place or another in this book, and even so we'll only skim the surface of the vast collection of mathematical ideas developed in response to his work.

Indeed it has been argued, convincingly we believe, that the very standards of precision and rigor that currently prevail in mathematics grew, themselves, out of examination of Fourier's claims.

To summarize the story so far, the idea of a sinusoid decomposition, first espoused (in any generality) by Joseph Fourier in 1807, has wide practical *and* theoretical implications.

But what makes Fourier analysis particularly nifty is the richness of the *interplay* between its theory and its practice. We know of no mathematical discipline where this interplay is deeper, and this includes even disciplines about which we know something.

We'll try, in what follows, to convey not only how the theory makes the practice fly, but also how the practice elucidates and illuminates the theory.

We conclude this section with three remarks. First, considerable gains in mathematical convenience and elegance may be reaped by reformulating (I.1) in terms of the *complex exponentials* $e^{2\pi isx}$ and $e^{-2\pi isx}$ (Fig. I.3). Here i denotes the usual imaginary square root of -1, and

$$e^{\pm i\theta} = \cos\theta \pm i\sin\theta \qquad \text{(I.4)}$$

for $\theta \in \mathbb{R}$ (cf. the Appendix).

The advantages of the complex exponential perspective will manifest themselves amply in the course of things.

Our second remark is that the big picture is actually quite a bit bigger than has been described so far: One may also consider the expression of reasonable functions as superpositions of other, not necessarily sinusoidal, "building blocks." And a bit of investigation reveals not only that decompositions of this sort do exist, but that they're abundant; not only that they're important, but that they're abundantly so. We'll investigate such alternative decompositions a bit later: see especially Chapter 4, Section 7.1, and Chapter 8.

Thirdly, we note that the few details provided to this point concern, strictly speaking, *one-dimensional* Fourier analysis—that is, they apply to functions of a *single* real variable. Certainly *multidimensional* Fourier analysis, wherein functions of several variables $x_1, x_2, \ldots, x_m$ are analyzed (into several-variable analogs of the sinusoid (I.2), or of complex exponentials, or of other sorts of functions), is also of significant interest, and will be given the requisite attention in appropriate contexts. But we take

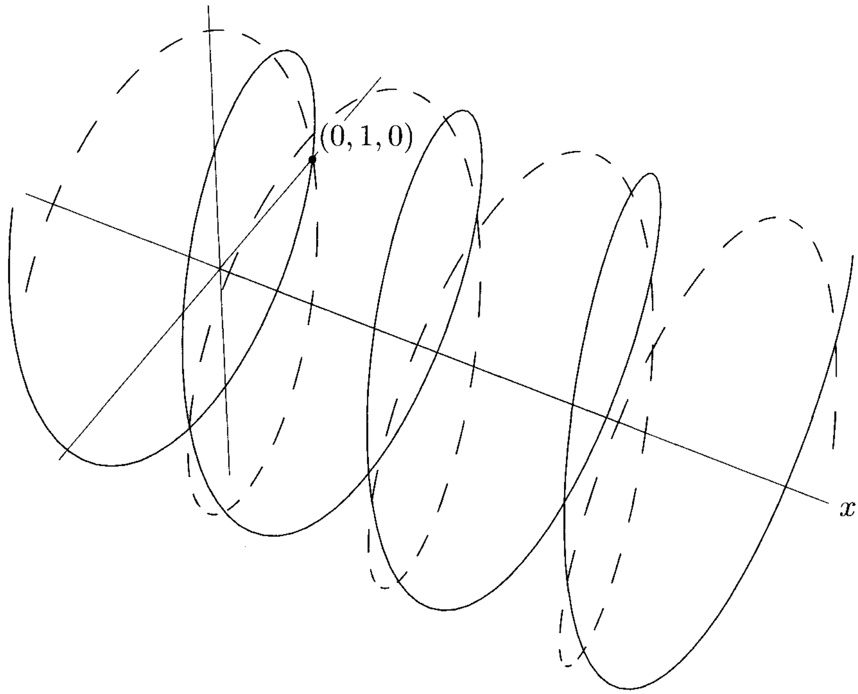

Fig. I.3 The complex exponentials $e^{2\pi isx}$ (solid) and $e^{-2\pi isx}$ (dashed)

the point of view here, as we will throughout, that the major issues are best introduced in the one-dimensional setting. From there, the generalization to several dimensions will usually be quite straightforward.

We now proceed to shade in some elements, already outlined, of the big picture. Warning: Our shadings won't always stay within neat, well defined lines. Which is okay because, in fact, Fourier analysis is like fingerpainting: Its various parts and patches are not disjoint or sharply delineated but blend, swirl, and fade gently into each other.

If we're going to fingerpaint, we're necessarily going to get our hands dirty. And there's no point in trying to wash this stuff off: It's Fourier analysis, it's indelible. So ye who enter here, abandon all soap.

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