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The Elements of Integration and Lebesgue Measure

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Preface

This book consists of two separate, but closely related, parts. The first part (Chapters 1-10) is subtitled *The Elements of Integration*; the second part (Chapters 11-17) is subtitled *The Elements of Lebesgue measure*. It is possible to read these two parts in either order, with only a bit of repetition.

The Elements of Integration is essentially a corrected reprint of a book with that title, originally published in 1966, designed to present the chief results of the Lebesgue theory of integration to a reader having only a modest mathematical background. This book developed from my lectures at the University of Illinois, Urbana-Champaign, and it was subsequently used there and elsewhere with considerable success. Its only prerequisites are a understanding of elementary real analysis and the ability to comprehend "ε-δ arguments". We suppose that the reader has some familarity with the Riemann integral so that it is not necessary to provide motivation and detailed discussion, but we do not assume that the reader has a mastery of the subtleties of that theory. A solid course in "advanced calculus", an understanding of the first third of my book *The Elements of Real Analysis*, or of most of my book *Introduction to Real Analysis* with D. R. Sherbert provides an adequate background. In preparing this new edition, I have seized the opportunity to correct certain errors, but I have resisted the temptation to insert additional material, since I believe that one of the features of this book that is most appreciated is its brevity.

The Elements of Lebesgue Measure is descended from class notes written to acquaint the reader with the theory of Lebesgue measure in the space R^p. While it is easy to find good treatments of the case $p = 1$, the case $p > 1$ is not quite as simple and is much less frequently discussed. The main ideas of Lebesgue measure are presented in detail in Chapters 10-15, although some relatively easy remarks are left to the reader as exercises. The final two chapters venture into the topic of nonmeasurable sets and round out the subject.

There are many expositions of the Lebesgue integral from various points of view, but I believe that the abstract measure space approach used here strikes directly towards the most important results: the convergence theorems. Further, this approach is particularly well-suited for students of probability and statistics, as well as students of analysis. Since the book is intended as an introduction, I do not follow all of the avenues that are encountered. However, I take pains not to attain brevity by leaving out important details, or assigning them to the reader.

Readers who complete this book are certainly not through, but if this book helps to speed them on their way, it has accomplished its purpose. In the References, I give some books that I believe readers can profitably explore, as well as works cited in the body of the text.

I am indebted to a number of colleagues, past and present, for their comments and suggestions; I particularly wish to mention N. T. Hamilton, G. H. Orland, C. W. Mullins, A. L. Peressini, and J. J. Uhl, Jr. I also wish to thank Professor Roy O. Davies of Leicester University for pointing out a number of errors and possible improvements.

ROBERT G. BARTLE

Ypsilanti and Urbana
November 20, 1994

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CHAPTER 1

Introduction

The theory of integration has its ancient and honorable roots in the "method of exhaustion" that was invented by Eudoxos and greatly developed by Archimedes for the purpose of calculating the areas and volumes of geometric figures. The later work of Newton and Leibniz enabled this method to grow into a systematic tool for such calculations.

As this theory developed, it has become less concerned with applications to geometry and elementary mechanics, for which it is entirely adequate, and more concerned with purely analytic questions, for which the classical theory of integration is not always sufficient. Thus a present-day mathematician is apt to be interested in the convergence of orthogonal expansions, or in applications to differential equations or probability. For him the classical theory of integration which culminated in the Riemann integral has been largely replaced by the theory which has grown from the pioneering work of Henri Lebesgue at the beginning of this century. The reason for this is very simple: the powerful convergence theorems associated with the Lebesgue theory of integration lead to more general, more complete, and more elegant results than the Riemann integral admits.

Lebesgue's definition of the integral enlarges the collection of functions for which the integral is defined. Although this enlargement is useful in itself, its main virtue is that the theorems relating to the interchange of the limit and the integral are valid under less stringent assumptions than are required for the Riemann integral. Since one frequently needs to make such interchanges, the Lebesgue integral is more convenient to deal with than the Riemann integral. To exemplify these remarks, let the sequence (f_n) of functions be defined for $x > 0$ by $f_n(x) = e^{-nx}/\sqrt{x}$. It is readily seen that the (improper) Riemann integrals

$$I_n = \int_0^{+\infty} \frac{e^{-nx}}{\sqrt{x}}\, dx$$

exist and that $\lim_{n\to\infty} f_n(x) = 0$ for all $x > 0$. However, since $\lim_{x\to 0} f_n(x) = +\infty$ for each n, the convergence of the sequence is certainly not uniform for $x > 0$. Although it is hoped that the reader can supply the estimates required to show that $\lim I_n = 0$, we prefer to obtain this conclusion as an immediate consequence of the Lebesgue Dominated Convergence Theorem which will be proved later. As another example, consider the function F defined for $t > 0$ by the (improper) Riemann integral

$$F(t) = \int_0^{+\infty} x^2 e^{-tx}\, dx.$$

With a little effort one can show that F is continuous and that its derivative exists and is given by

$$F'(t) = -\int_0^{+\infty} x^3 e^{-tx}\, dx,$$

which is obtained by differentiating under the integral sign. Once again, this inference follows easily from the Lebesgue Dominated Convergence Theorem.

At the risk of oversimplification, we shall try to indicate the crucial difference between the Riemann and the Lebesgue definitions of the integral. Recall that an **interval** in the set $\boldsymbol{R}$ of real numbers is a set which has one of the following four forms:

$$[a, b] = \{x \in \boldsymbol{R} : a \leqslant x \leqslant b\}, \qquad (a, b) = \{x \in \boldsymbol{R} : a < x < b\},$$
$$[a, b) = \{x \in \boldsymbol{R} : a \leqslant x < b\}, \qquad (a, b] = \{x \in \boldsymbol{R} : a < x \leqslant b\}.$$

In each of these cases we refer to a and b as the **endpoints** and prescribe

$b - a$ as the **length** of the interval. Recall further that if E is a set, then the **characteristic function** of E is the function χ_E defined by

$$\begin{aligned} \chi_E(x) &= 1, \qquad \text{if } x \in E, \\ &= 0, \qquad \text{if } x \notin E. \end{aligned}$$

A **step function** is a function φ which is a finite linear combination of characteristic functions of intervals; thus

$$\varphi = \sum_{j=1}^{n} c_j \chi_{E_j}.$$

If the endpoints of the interval E_j are a_j, b_j, we define the **integral** of φ to be

$$\int \varphi = \sum_{j=1}^{n} c_j(b_j - a_j).$$

If f is a bounded function defined on an interval $[a, b]$ and if f is not too discontinuous, then the **Riemann integral** of f is defined to be the limit (in an appropriate sense) of the integrals of step functions which approximate f. In particular, the **lower Riemann integral** of f may be defined to be the supremum of the integrals of all step functions φ such that $\varphi(x) \leqslant f(x)$ for all x in $[a, b]$, and $\varphi(x) = 0$ for x not in $[a, b]$.

The Lebesgue integral can be obtained by a similar process, except that the collection of step functions is replaced by a larger class of functions. In somewhat more detail, the notion of length is generalized to a suitable collection $\boldsymbol{X}$ of subsets of $\boldsymbol{R}$. Once this is done, the step functions are replaced by **simple functions**, which are finite linear combinations of characteristic functions of sets belonging to $\boldsymbol{X}$. If

$$\varphi = \sum_{j=1}^{n} c_j \chi_{E_j}$$

is such a simple function and if $\mu(E)$ denotes the "measure" or "generalized length" of the set E in $\boldsymbol{X}$, we define the integral of φ to be

$$\int \varphi = \sum_{j=1}^{n} c_j \mu(E_j).$$

If f is a nonnegative function defined on $\boldsymbol{R}$ which is suitably restricted, we shall define the **(Lebesgue) integral** of f to be the supremum of the integrals of all simple functions φ such that $\varphi(x) \leqslant f(x)$ for all x in $\boldsymbol{R}$. The integral can then be extended to certain functions that take both signs.

Although the generalization of the notion of length to certain sets in $\boldsymbol{R}$ which are not necessarily intervals has great interest, it was observed in 1915 by Maurice Fréchet that the convergence properties of the Lebesgue integral are valid in considerable generality. Indeed, let X be any set in which there is a collection $\boldsymbol{X}$ of subsets containing the empty set $\emptyset$ and X and closed under complementation and countable unions. Suppose that there is a nonnegative measure function μ defined on $\boldsymbol{X}$ such that $\mu(\emptyset) = 0$ and which is **countably additive** in the sense that

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)$$

for each sequence (E_j) of sets in $\boldsymbol{X}$ which are mutually disjoint. In this case an integral can be defined for a suitable class of real-valued functions on X, and this integral possesses strong convergence properties.

As we have stressed, we are particularly interested in these convergence theorems. Therefore we wish to advance directly toward them in this abstract setting, since it is more general and, we believe, conceptually simpler than the special cases of integration on the line or in $\boldsymbol{R}^n$. However, it does require that the reader temporarily accept the fact that interesting special cases are subsumed by the general theory. Specifically, it requires that he accept the assertion that there exists a countably additive measure function that extends the notion of the length of an interval. The proof of this assertion is in Chapter 9 and can be read after completing Chapter 3 by those for whom the suspense is too great.

In this introductory chapter we have attempted to provide motivation and to set the stage for the detailed discussion which follows. Some of our remarks here have been a bit vague and none of them has been proved. These defects will be remedied. However, since we shall have occasion to refer to the system of extended real numbers, we now append a brief description of this system.

In integration theory it is frequently convenient to adjoin the two symbols $-\infty$, $+\infty$ to the real number system $\boldsymbol{R}$. (It is stressed that these symbols are not real numbers.) We also introduce the convention that $-\infty < x < +\infty$ for any $x \in \boldsymbol{R}$. The collection $\bar{\boldsymbol{R}}$ consisting of the set $\boldsymbol{R} \cup \{-\infty, +\infty\}$ is called the **extended real number system.**

One reason we wish to consider $\bar{\boldsymbol{R}}$ is that it is convenient to say that the length of the real line is equal to $+\infty$. Another reason is that we will frequently be taking the supremum (= least upper bound) of a set of real numbers. We know that a nonempty set A of real numbers which has an upper bound also has a supremum (in $\boldsymbol{R}$). If we define the supremum of a nonempty set which does not have an upper bound to be $+\infty$, then every nonempty subset of $\boldsymbol{R}$ (or $\bar{\boldsymbol{R}}$) has a unique supremum in $\bar{\boldsymbol{R}}$. Similarly, every nonempty subset of $\boldsymbol{R}$ (or $\bar{\boldsymbol{R}}$) has a unique infimum (= greatest lower bound) in $\bar{\boldsymbol{R}}$. (Some authors introduce the conventions that $\inf \emptyset = +\infty$, $\sup \emptyset = -\infty$, but we shall not employ them.)

If (x_n) is a sequence of extended real numbers, we define the **limit superior** and the **limit inferior** of this sequence by

$$\limsup x_n = \inf_m \left(\sup_{n \geq m} x_n \right),$$
$$\liminf x_n = \sup_m \left(\inf_{n \geq m} x_n \right).$$

If the limit inferior and the limit superior are equal, then their value is called the **limit** of the sequence. It is clear that this agrees with the conventional definition when the sequence and the limit belong to $\boldsymbol{R}$.

Finally, we introduce the following algebraic operations between the symbols $\pm\infty$ and elements $x \in \boldsymbol{R}$:

$$
\begin{aligned}
(\pm\infty) + (\pm\infty) = x + (\pm\infty) = (\pm\infty) + x &= \pm\infty, \\
(\pm\infty)(\pm\infty) = +\infty,\ (\pm\infty)(\mp\infty) &= -\infty, \\
x(\pm\infty) = (\pm\infty)x &= \pm\infty \qquad \text{if } x > 0, \\
&= 0 \qquad\ \ \ \text{if } x = 0, \\
&= \mp\infty \qquad \text{if } x < 0.
\end{aligned}
$$

It should be noticed that we do not define $(+\infty) + (-\infty)$ or $(-\infty) + (+\infty)$, nor do we define quotients when the denominator is $\pm\infty$.