The Language of Mathematics

Utilizing Math in Practice

Robert Laurence Baber

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Robert Laurence Baber
Bad Homburg vor der Höhe, Germany

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This book is dedicated

to those who would like to improve their ability
to apply mathematics effectively to practical problems,

to teachers of mathematics who would like to improve their ability
to convey a better understanding and appreciation of mathematics
to their students,

and

to those who are curious about
the linguistic nature and aspects of mathematics and its notation.

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PREFACE

We live today in a highly technological world built upon science and engineering. These, in turn, are based extensively on mathematics. It is not an exaggeration to state that mathematics is the language of engineering. Thus, to be able to understand science and engineering—and hence, the physical world in which we live—one must have at least a basic understanding of mathematics. This need will increase in time as the world in which we live becomes ever more technological in nature.

Unfortunately, too few people today have a sufficient understanding of mathematics to enable them to understand important technological topics. They are inadequately prepared to contribute substantially to resolving related issues, such as the safe employment of nuclear systems in our society, avoiding or resolving environmental problems, or structuring transportation systems (including vehicles, their energy, roadways, terminal facilities), and especially, to making trade-off decisions among the many aspects of such issues.

An important reason for this widespread lack of familiarity with mathematics and the disciplines based on mathematics is the way in which mathematics is typically introduced and taught. Many people are turned off mathematics early in their school experience. Although current teaching approaches are effective for a relatively small group of pupils already oriented to technical, mathematical, and scientific subjects, they fail to motivate the majority. They do not build on the prior knowledge and interests of the target group. They are typically too late in addressing the ultimate and nontechnical advantages of applying mathematics, doing so only after many students have already lost interest and have turned off their minds to mathematics. A primary goal of this book is to present a view of mathematics that can overcome these shortcomings.

In this book I present a new and unique way of looking at mathematics. In it, mathematics is viewed through the specialized language and notation that mathematicians have developed for communicating among themselves, for recording the results of their work, and perhaps most important, for reasoning and conducting the various analyses involved in their investigations. This view of mathematics differs significantly from that presented in the traditional works on mathematics available in the extensive mathematical literature. It also differs significantly from the ways in which mathematics is taught today. This book will improve and increase the reader's insight into mathematics and how to utilize it in practice.

No particular previous knowledge of mathematics by the reader is required. All readers will, of course, have encountered arithmetic and some mathematics in school, and whatever they remember correctly will make it easier for them to read and understand some of the consequences of the material and concepts presented in this book. Readers with a more extensive prior knowledge of mathematics will be able to read the book faster, but they will still find many ideas to be new and different from their previous views of mathematics and its language. They will find that the material in the book will help them to apply mathematics to practical problems more easily, efficiently, and effectively than they could have earlier.

The book is an introduction to how to apply mathematics to practical problems by translating English statements of a problem to be solved into the Language of Mathematics. We also study some fundamental aspects of mathematics via the language used in mathematics, but that is only a by-product of investigating the Language of Mathematics.

The first step in solving a problem stated in English with the help of mathematics is to reformulate the English text into appropriate mathematical expressions reflecting the essential aspects of the problem and the requirements that its solution must satisfy. Reformulating the English text into such mathematical expressions is often the hardest part of solving a problem. It is often presumed to be part of the mathematical task, but actually, it is a *translation* problem—a *language* problem. Omissions and errors in this step will often be discovered only later, when the final mathematical solution is found to be wrong or inadequate—or found to be a solution to a different problem. Only after a suitable mathematical formulation of the problem and its solution has been completed can one begin to apply mathematics itself to find the desired solution.

As with most large and complex bodies of knowledge made up of a number of different subdisciplines, mathematics can be viewed from many different standpoints and in many different ways. None of these views exclude the validity of the others; rather, they complement each other. Each view typically offers something that the others do not offer. The most appropriate view depends on the viewer's goals, interests, particular purpose at the time, background knowledge, experience, and many other factors. Any person will find it useful to view mathematics from a different viewpoint—and the more, the better. The more able one is to take advantage of many different views, the better one will understand a subject and be able to apply mathematics to it efficiently, effectively, and productively. The approach taken in this book consolidates many of these different viewpoints within a unifying umbrella of language. It builds a bridge between natural languages such as English and mathematics.

My own experience learning, utilizing, and teaching mathematics has led me to the conclusion that mathematics should be introduced by examining the basics of the Language of Mathematics. I believe that learning mathematics in this way will help—even enable—many people to understand mathematics who would otherwise be turned off the subject by the current and traditional approaches to learning mathematics. Unfortunately, there are many such people in today's world whose work would benefit through simple applications of mathematics. This conclusion is based on my experience learning mathematics, learning how to apply it to a variety of technical, business, and economic problems, utilizing it extensively in practice in these areas, as well as teaching certain areas of mathematics and how to apply them both to university students and to people working in various technical, business, and management positions.

This language-oriented approach will make mathematics more accessible to those who like language and languages, but who have until now avoided—even disliked—mathematics. In my experience as a pupil in primary school through to teaching university courses involving applying mathematics to various types of problems, I have repeatedly observed that students at all levels and people on the job have considerable trouble solving word problems using mathematics. They have as much, usually more trouble coping with translating the English statement of the problem into mathematical notation as they do with solving the resulting mathematical expressions for the answers desired—if they ever get that far. It is my considered opinion that this difficulty is due to an inappropriate approach to teaching this material. The normal teaching approach presents word problems within the context of mathematics and as mathematics problems. In reality, they are, as mentioned above, translation and language problems, not mathematics problems. The mathematics comes later, after the word problem has been translated into the Language of Mathematics.

I believe that presenting word problems as language problems will draw students' conscious attention to the real issues involved in applying mathematics and will make learning this material easier. It will give them a broader and deeper basic understanding of mathematics, link mathematics with their previous knowledge of language, and provide them with a better foundation upon which specific skills in applying mathematics can then be developed. Instead of learning mathematics as something different, new, and unrelated to their previous experience and knowledge, they will learn mathematics as an extension of their already accumulated experience with and knowledge of language.

Viewing mathematics, mathematical models and mathematical expressions from a language standpoint can, in my experience, facilitate communication between people with different areas of expertise working on specific problems to which mathematics is applied. A language viewpoint diverts attention away from explaining mathematics to the less mathematically literate experts working on a problem. Instead, it directs attention to the real need to translate between the language of the application domain and the mathematical model and expressions representing an application problem and its solution. Secondarily, it can help those working on and affected by the application to improve their ability to read, at least passively, the mathematical model and expressions.

I also believe that many people who already understand mathematics well will find the new view presented in this book beneficial and that conscious awareness of and familiarity with it will help them when applying mathematics to practical problems and when explaining mathematics to others. At least that was my experience after I began to consider, first subconsciously, then consciously, the linguistic aspects of mathematics and to view mathematics from the standpoint of the Language of Mathematics as presented in this book.

While the explicitly language-oriented view of mathematics presented in this book is atypical and new, the mathematical material itself is old, having been developed over five or more millenia. This development has been uneven and sporadic, with flurries of creative phases interspersed between longer intervals of slow or no improvement. In the last few centuries, the development of mathematics has tended to become somewhat more regular, continuous, and productive. Whereas some aspects of mathematics are millenia old (e.g., numbers and arithmetic operations on numbers), other important features have been introduced comparatively recently: variable names to represent numbers or other values, functions and functional notation, compact standard forms for writing mathematical expressions, and symbolic logic.

The idea of viewing mathematics (or a part thereof) as a language is not at all widespread, nor is it completely new. To the best of my knowledge, however, the particular approach taken in this book is new. Whereas other works nominally dealing with linguistic aspects of mathematics tend to view the topic from the standpoint of *mathematics*, this book quite intentionally views the Language of Mathematics from the opposite side: from the standpoint of *language*. Whereas other works tend to concentrate on defining and understanding mathematical concepts and terms in English, this book deals explicitly and extensively with translating English statements into the Language of Mathematics, pointing out grammatical clues useful as guidelines. Ways of modeling dynamic, temporal processes described in English using the static, tenseless Language of Mathematics are dealt with in this book. Also new in this book is the observation that all verbs implicit in expressions in the Language of Mathematics are stative in nature (timeless, tenseless verbs of state or being), a characteristic that has significant implications for translating from English to the Language of Mathematics. In particular, many sentences in English cannot be translated directly into the Language of Mathematics, but must first be substantially reformulated.

In composing the presentation of the Language of Mathematics in this book, I have followed an old, common, and very successful strategy for formulating a mathematical model to be used as the basis for solving a given problem:

- Generalize.
- Identify the essential aspects of the problem and the corresponding mathematical model.
- Simplify, retaining the essentials but eliminating nonessentials where helpful.

Nonessential details often confuse both a model's developers and its readers by distracting their attention from the essentials. Nonessential details also make a mathematical model larger, more complex, and therefore more complicated. The resulting structure is more difficult to understand and use than one including only the essential details would be.

In introductory articles, lectures, and so on, one often encounters an apology for mathematical formulas and a statement that the reader or listener does not really have to understand the formulas in detail, only generally what they are about, and even that not really seriously or deeply. In this book, the reader will find no such apology or excuse. Such false rationalization is like telling the audience attending a play by Shakespeare that they need listen only to the poetic, musical flow of the voices—that the actual meaning of the words is unimportant. In this book, the meaning of each mathematical expression (formula) is important; the meaning, not the poetic style, is the message. The style can help or hinder the reader to understand the meaning, but appreciating the style is not enough; the meaning must be understood. If you read a play by Shakespeare but do not understand the language used, you will not get the message. The same applies to expressions in the Language of Mathematics.

The Language of Mathematics has evolved to facilitate reasoning logically about things. It has been developed to make it easy to make exact, precise, unambiguous logical statements and to make it difficult—even impossible—to make vague, ambiguous statements. One should take advantage of these characteristics of the Language of Mathematics and use it, not English, when reasoning about things. Therefore, convert from English to the Language of Mathematics at as early a stage in the reasoning process as possible.

When asked about their motivation for writing a book, authors often state that they wrote the book that they would have liked to have read earlier themselves. That was definitely an important reason for my writing this book. I would have liked very much to have had a copy of it when I was in high school and during my early undergraduate years. It would have given me a view of mathematics and mathematical notation that would have helped me to learn mathematics better and faster and to understand it more thoroughly and deeply. It would not have replaced any of the other books from which I learned mathematics, but it would have been a very helpful adjunct and introduction to them. I hope that you find reading this book as useful and as enjoyable as I would have so many years ago, and as I did conceiving and writing it.

Acknowledgments

I would like to thank the many people who have contributed directly and indirectly to this book and to my ability to write it. They begin with my teachers in primary, secondary, and tertiary schools, especially those who taught me mathematics, languages, and related courses. Authors of the books and other works I have read over the years have also contributed unknowingly, indirectly, but surely. Many of my work colleagues, consulting clients, and other friends contributed by posing questions and helping me to answer them. Others, by sharing their ideas on many different subjects with me over the years, also contributed indirectly but importantly to the book. Most recently, the reviewers of partial drafts of this book contributed valuable suggestions and advice.

Over the last five or more millenia, numerous mathematicians have made many significant contributions, both large and small, to the development of mathematics, its notation, its language, and its practical application. Through their efforts they have guided the evolution of mathematics from very basic beginnings to its present state. That evolution started with counting, went through stages of arithmetic and reasoning about numerical quantities and measures, and led to reasoning about nonnumerical entities, qualities, properties, and attributes. This book is built upon the cumulative results of their work. As the author of this book, I owe a great debt to all of them.

Last but not least, I thank my many students. The best students, who easily grasp the concepts presented to them and then build upon them with only a little help, are always a joy to have. But the most valuable ones to any teacher are those students who ask at first seemingly simple questions and who have difficulty coming to grips with the material. Instructors who dismiss their questions and difficulties lightly not only fail to rise to the challenges of ~~teaching~~ helping them to learn, but also pass up many opportunities to develop the material further and to consolidate and structure it better. They pass up opportunities for a research paper or even a book such as this one.

ROBERT LAURENCE BABER

Bad Homburg, Germany
http://Language-of-Mathematics.eu
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PART A
Introductory Overview

1 Introduction

Welcome to mathematics and particularly to its language. You will find it to be a simple language, with only a little grammar and a limited vocabulary, but quite different from the other languages you know. Unlike natural languages such as English, its semantics are precisely defined and unambiguous. In particular, its complete lack of ambiguity enables exact reasoning, probably its greatest advantage. On the negative side, one cannot express such a wide variety of things in the Language of Mathematics as in English, and intentional vagueness, so important in English poetry and much prose, cannot be expressed directly. Nonetheless, it is often surprising what one can express in and with the help of the Language of Mathematics, especially when combined appropriately with English or some other natural language.

Vagueness cannot be expressed directly in the Language of Mathematics, but it can be modeled—precisely and unambiguously—with mathematics. Expressed differently, the Language of Mathematics enables one to make precise and unambiguous statements about vaguely determined things. Probability theory, statistics, and, more recently, fuzzy theory are the mathematical subdisciplines that enable one to talk and write about uncertainty and vagueness—but with precision and without ambiguity.

Although the Language of Mathematics is quite limited in the range of things that can be expressed in it directly, many things outside the Language of Mathematics can be related to mathematical objects as needed for specific applications. Thus, the Language of Mathematics is, in effect, a template language for such applications. Adapting it to the needs of a particular application extends its usefulness greatly and poses the main challenge in its application. This challenge is primarily linguistic, not mathematical, in nature. Helping the reader to meet this challenge is an important goal of this book and underlies essentially all of the material in it.

One of the limitations in the Language of Mathematics is the fact that the notion of time is absent from it completely. This fact is mentioned here, at the very beginning, because the lack of conscious awareness of it has led to many students becoming (and sometimes remaining) very confused without realizing this source of confusion. Time and dynamic processes can easily be and often are modeled mathematically, but this is part of the adaptation of the template Language of Mathematics to the particular application in question. How this can be done is covered in several places in the book, in particular in Section 7.5.

The Language of Mathematics: Utilizing Math in Practice, First Edition. Robert Laurence Baber.

1.1 WHAT IS LANGUAGE?

A *language* is a medium for:

- Expressing or communicating:
  - Verbally or visually (e.g., in written form)
  - Facts, opinions, thoughts, ideas, feelings, desires, or commands
  - At one time or from one time to another
  - Between different people or from one person to her/himself at a different time
- Thinking
- Analyzing or reasoning

Every language employs abstract symbols—verbal, visual, and sometimes using other senses, such as touch—to represent things. In many natural languages, the visual form was developed to represent the verbal form, so that there is a close relationship between the spoken and written forms. Other languages, however, have developed spoken and written forms which are not directly related. Originally, their symbols were often pictorial in nature, albeit often rather abstractly. One can think of such a language as two distinct languages, a spoken language and a written language. In the case of Sumerian (the earliest known written language), written symbols (in cuneiform) represented what we think of today as words, so that there was no direct connection between the written and spoken forms of the language. Later, the cuneiform symbols were taken over by other languages (e.g., Akkadian) to represent syllables in the spoken language, establishing a direct connection between the spoken and written forms of the language. Still later, other languages introduced symbols for parts of a syllable, leading to the abstract symbols that we now call letters.

Mathematics exhibits the characteristics of a language described above. The range and distribution of topics communicated in natural languages such as English and those communicated in the Language of Mathematics differ in some significant ways, however. Feelings and emotions are rarely expressed in mathematical terms. Vague (i.e., imprecisely defined) terms are not permitted in the Language of Mathematics. Otherwise, all of the characteristics of a language mentioned above are found in the Language of Mathematics, albeit with different emphasis and importance.

Scientists and historians believe that language began by our distant ancestors communicating with one another via sounds made by using the vocal chords, the mouth, the lips, and the tongue (hence our term *language*, from *lingua*, Latin for “tongue”). This form of language was useful for communicating between individuals at one time and when they were physically close to one another. Sounds made in other ways (e.g., by drums) were used for communication over greater distances, but still between people at essentially one time. Visual signals of various kinds were also employed in much the same way.

Marks on bones apparently representing numbers are believed to be an early (perhaps the earliest) form of record keeping: communicating from one time to another. Gradually, this idea was extended to symbols for various things, ideas, concepts, and so on, leading in a long sequence of developmental steps to language as we know it today. It is noteworthy that even precursors to writing apparently included numbers, the basic objects of arithmetic, and hence of mathematics. The earliest forms of writing known to us today certainly included numbers. Thus, the development of languages included elements of mathematics from very early times.

Symbols and signs recorded physically in visually observable form constitute records that store information for later use. They are a major source of our knowledge of early civilizations. Their durability seriously limits our knowledge of those early civilizations. The most durable records known to date are clay tablets inscribed with cuneiform characters and inscriptions on stone monuments. Records of old societies that used less durable forms of writing have decomposed in the meantime and are no longer available. Those potentially interesting historical records are lost forever.

Recently, humans have begun to communicate with symbols they cannot observe visually but only with the help of technical equipment. The symbols are in the form of electrically and magnetically recorded analog signals and, still more recently, digital symbols. In some cases, these signals and symbols are direct representations of previous forms of human language. In other cases, they are not; rather, they represent new linguistic structures and forms.

Natural languages such as English, Chinese, and Arabic have evolved to enable people to communicate about all the kinds of things they encounter in everyday life. Therefore, the universes of discourse of natural languages overlap considerably. The Language of Mathematics has, however, evolved to fulfill quite different, very specific, and comparatively quite limited goals. It is a language dealing only with abstract things and concepts and having only a rather limited scope. Therefore, for the purposes of applications to real-world situations, the Language of Mathematics is not a finished language but, instead, a *template language*. When applying mathematics, the Language of Mathematics must be adapted for each application. This is done by specifying how the elements of the mathematical description are to be *interpreted* in the terminology of the application area. A new *interpretation* must normally be given for each application, or at least for each group of closely related applications.

1.2 WHAT IS MATHEMATICS?

The archaeological record suggests that mathematics probably originated with counting and measuring things and recording those quantities. Soon, however, people began to pose and answer questions about the quantities of the things counted or needed for some purpose; that is, they began to reason about quantities and to solve related problems. As early as about 4000 years ago, mathematics included the study of geometrical figures: in particular, of relationships between their parts and between numerical measures of their parts. Later, mathematicians turned their attention to ever more abstract things and concepts, including ones not necessarily of a numerical or geometrical nature.

The description of language at the beginning of Section 1.1. also applies to mathematics. The relative emphasis on communication on the one hand and on reasoning and analysis on the other hand is perhaps different, but there is much common ground. Some would say that mathematics itself, in the narrow sense, concentrates on concepts and techniques for reasoning and analyzing, and that mathematics is therefore not itself a language. However, mathematics does use extensively a particular language that has evolved to facilitate reasoning and analyzing. Such reasoning and analyzing is performed primarily by manipulating the symbols of mathematical language mechanistically, according to precise rules. The Language of Mathematics is also used extensively for expressing and communicating both over time and between people. It is also used for thinking.

What is mathematics today? Someone once answered that question with "What mathematicians do." That, of course, begs the question "What do mathematicians do?" Answered most succinctly, they reason logically about things—artificial, abstract things—not just about quantities, numbers, or numerical properties of various objects.

Many of those things, although artificial and abstract, are useful in modeling actual things in the real world; for example:

- Structures of buildings, dams, bridges, and other engineering artifacts
- Materials of all kinds and their properties
- Mechanical devices and equipment
- Machines, engines, and all kinds of energy conversion devices and systems
- Vehicles of all types: land, underwater, water surface, air, space
- Electrical circuits and systems composed of them
- Communication systems—wired and wireless—and their components
- Systems for cryptography
- Molecules, atoms, nuclei, and subatomic particles
- Chemical reactions and chemical reactors
- Systems for generating and distributing electrical power
- Nuclear decay and interaction processes and nuclear reactors
- Heating and cooling systems
- Computer software
- Prices in financial markets
- Sales and markets
- Order processing and billing systems
- Inventory control systems
- Various business assets and liabilities
- Data and information of all types, including names and addresses
- Social, economic, business and technical systems
- Relationships among objects, properties, and values of all (not just numerical) types
- Structural aspects of languages, natural and artificial

Such models better enable us to describe, to understand, and to predict things in the real world—to our considerable benefit.

It is important that the reader always be consciously aware that mathematics today consists of much more than numbers and arithmetic. Important as these are, they constitute only a part of mathematics. The main goal of mathematics is not to work with numbers but to reason about objects, properties, values, and so on, of all types. Mathematicians work mostly with relationships between these things. Mathematicians actually spend very little of their time calculating with numbers. They spend most of their time reasoning about abstract things. Logic is an important part of that work.

Different subdisciplines of mathematics have been created in the course of time. Numbers, counting, geometric figures, and quantitative analyses constituted the first subdisciplines. Among the more recent is logic. Unfortunately, especially for the novice learning mathematics, logic used its own terminology and symbols, and this distinction is still evident in the ways in which mathematical logic is often taught today. This leads many beginners to believe that logic is somehow fundamentally different from the other subdisciplines and that a different notation and point of view must be learned. The linguistic approach presented in this book integrates these views and notational schemes, so that the beginner need learn only one mathematical language. Although this integrating view is already present in some mathematical work, it is not really widespread yet, especially not in teaching mathematics.

1.3 WHY USE MATHEMATICS?

Among the several reasons for using mathematics in practice, the two most important are:

- To find a solution to a problem. The statement of a problem or the requirements that a solution must fulfill can often be transformed into the solution itself.
- To understand something better and more thoroughly: for example, to identify all possibilities that must be considered when defining a problem and solving it.

Examples are given in Chapter 2, in Section 6.13.2, and in Chapter 8.

The author and many others have found in the course of their work that mathematics frequently enables them to think effectively about and solve problems they could not have come to grips with in any other way. As long as a problem is expressed in English, one can reason about the problem and deduce its solution only when one constantly keeps the precise meaning of the words, phrases, and sentences consciously in mind. If the text is at all long, this becomes unworkable and very subject to error. It is likely that some important detail will be overlooked. After formulating the problem in mathematical language, the expressions can be transformed in ways reflecting and representing the reasoning about the corresponding English sentences. However, the expressions can be transformed mechanistically according to generally applicable rules without regard to the meaning of the expressions. This effectively reduces reasoning to transformations independent of the interpretation of the expressions being transformed, simplifying the process considerably and enabling much more complicated problems to be considered and solved. People who are specialists in transforming mathematical expressions but who are not specialists in the application area can find solutions. In this way, the mental work of reasoning can be largely reduced to the mechanistic manipulation of symbols. In the words of Edsger W. Dijkstra, a well-known computer scientist whose areas of special interest included mathematics and logic, one can and should "let the symbols do the work."

For the reasons cited above, one should convert from English to the Language of Mathematics at as early a stage as possible when reasoning about anything. Transforming the mathematically formulated statement of a problem into its solution sounds easy. Although it is, in principle, straightforward, it can be computationally intensive and tedious to do manually. For a great many applications, algorithms for solving the problem and computer programs for calculating the numerical solutions exist. Where such solutions do not already exist, mathematicians can often develop them.

The usefulness of the Language of Mathematics for the purposes listed above derives from its precision, the absence of ambiguity, and rules for transforming mathematical expressions into various equivalent forms while preserving meaning. These characteristics are unique to the Language of Mathematics. Natural languages, lacking these characteristics, are much less satisfactory and useful for the purposes noted above.

1.4 MATHEMATICS AND ITS LANGUAGE

In order to reason logically about things, mathematicians have developed a particular language with particular characteristics. That language—the Language of Mathematics—and other languages developed by human societies—such as English—are similar in some respects and different in some ways.

Distinguishing characteristics of the Language of Mathematics are its precision of expression and total lack of ambiguity. These characteristics make it particularly useful for exact reasoning. They also make it useful for specifying technical things. The Language of Mathematics is a language of *uninterpreted* expressions, which are described in Section 3.4. This does not imply that mathematical expressions are uninterpretable. They can be and often are interpreted when applying mathematics in the real world: when associating mathematical values, variables, and expressions with entities in the application area (see Chapters 6 and 7, especially Section 6.13).

Within the Language of Mathematics, however, expressions are never interpreted. When transforming mathematical expressions in order to reason or analyze, one should be very careful not to interpret them, as doing so takes one out of the Language of Mathematics and into English. This can result in the loss of precision and the introduction of ambiguity—the loss of the very reasons for using mathematics—without one being aware that the loss is occurring. Reasoning must be conducted only and strictly within the abstract world of uninterpreted expressions, applying only mathematically valid transformations to the expressions without interpreting them. In