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Water Waves

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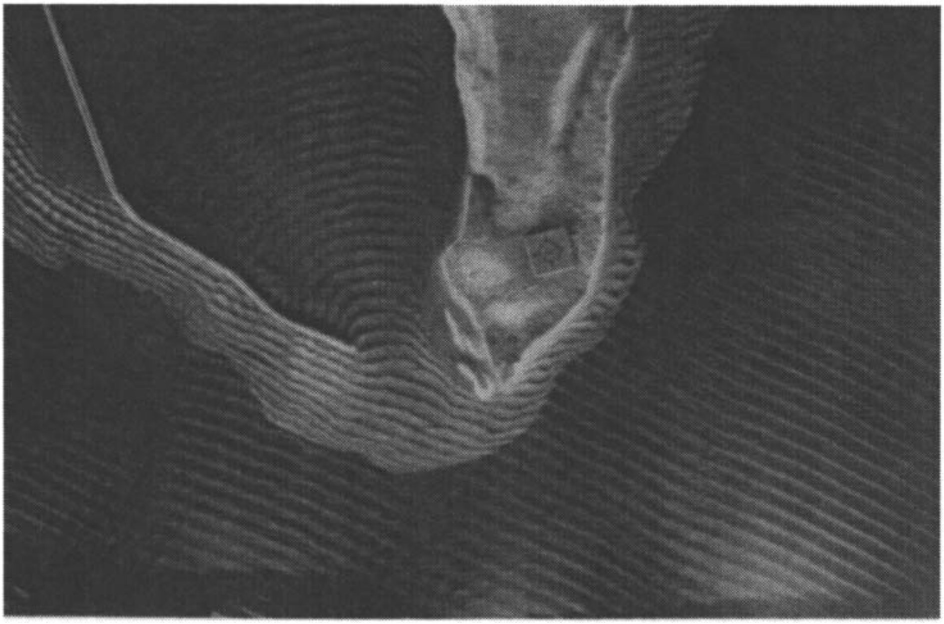

Waves about a harbor

WATER WAVES
The Mathematical Theory with Applications

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Introduction

1. Introduction

The purpose of this book is to present a connected account of the mathematical theory of wave motion in liquids with a free surface and subjected to gravitational and other forces, together with applications to a wide variety of concrete physical problems.

Surface wave problems have interested a considerable number of mathematicians beginning apparently with Lagrange, and continuing with Cauchy and Poisson in France.* Later the British school of mathematical physicists gave the problems a good deal of attention, and notable contributions were made by Airy, Stokes, Kelvin, Rayleigh, and Lamb, to mention only some of the better known. In the latter part of the nineteenth century the French once more took up the subject vigorously, and the work done by St. Venant and Boussinesq in this field has had a lasting effect: to this day the French have remained active and successful in the field, and particularly in that part of it which might be called mathematical hydraulics. Later, Poincaré made outstanding contributions particularly with regard to figures of equilibrium of rotating and gravitating liquids (a subject which will not be discussed in this book); in this same field notable contributions were made even earlier by Liapounoff. One of the most outstanding accomplishments in the field from the purely mathematical point of view — the proof of the existence of progressing waves of finite amplitude — was made by Nekrassov [N.1], [N.1a]† in 1921 and independently by a different means by Levi-Civita [L.7] in 1925.

The literature concerning surface waves in water is very extensive. In addition to a host of memoirs and papers in the scientific journals, there are a number of books which deal with the subject at length. First and foremost, of course, is the book of Lamb [L.3], almost a third of which is concerned with gravity wave problems. There are books by Bouasse [B.15], Thorade [T.4], and Sverdrup [S.39]

* This list would be considerably extended (to include Euler, the Bernoullis, and others) if hydrostatics were to be regarded as an essential part of our subject.

† Numbers in square brackets refer to the bibliography at the end of the book.

devoted exclusively to the subject. The book by Thorade consists almost entirely of relatively brief reviews of the literature up to 1931 — an indication of the extent and volume of the literature on the subject. The book by Sverdrup was written with the special needs of oceanographers in mind. One of the main purposes of the present book is to treat some of the more recent additions to our knowledge in the field of surface wave problems. In fact, a large part of the book deals with problems the solutions of which have been found during and since World War II; this material is not available in the books just now mentioned.

The subject of surface gravity waves has great variety whether regarded from the point of view of the types of physical problems which occur, or from the point of view of the mathematical ideas and methods needed to attack them. The physical problems range from discussion of wave motion over sloping beaches to flood waves in rivers, the motion of ships in a sea-way, free oscillations of enclosed bodies of water such as lakes and harbors, and the propagation of frontal discontinuities in the atmosphere, to mention just a few. The mathematical tools employed comprise just about the whole of the tools developed in the classical linear mathematical physics concerned with partial differential equations, as well as a good part of what has been learned about the nonlinear problems of mathematical physics. Thus potential theory and the theory of the linear wave equation, together with such tools as conformal mapping and complex variable methods in general, the Laplace and Fourier transform techniques, methods employing a Green's function, integral equations, etc. are used. The nonlinear problems are of both elliptic and hyperbolic type.

In spite of the diversity of the material, the book is not a collection of disconnected topics, written for specialists, and lacking unity and coherence. Instead, considerable pains have been taken to supply the fundamental background in hydrodynamics — and also in some of the mathematics needed — and to plan the book in order that it should be as much as possible a self-contained and readable whole. Though the contents of the book are outlined in detail below, it has some point to indicate briefly here its general plan. There are four main parts of the book:

Part I, comprising Chapters 1 and 2, presents the derivation of the basic hydrodynamic theory for non-viscous incompressible fluids, and also describes the two principal approximate theories which form the basis upon which most of the remainder of the book is built.

Part II, made up of Chapters 3 to 9 inclusive, is based on the approximate theory which results when the amplitude of the wave motions considered is small. The result is a linear theory which from the mathematical point of view is a highly interesting chapter in potential theory. On the physical side the problems treated include the propagation of waves from storms at sea, waves on sloping beaches, diffraction of waves around a breakwater, waves on a running stream, the motion of ships as floating rigid bodies in a seaway. Although this theory was known to Lagrange, it is often referred to as the Cauchy-Poisson theory, perhaps because these two mathematicians were the first to solve interesting problems by using it.

Part III, made up of Chapters 10 and 11, is concerned with problems involving waves in shallow water. The approximate theory which results from assuming the water to be shallow is not a linear theory, and wave motions with amplitudes which are not necessarily small can be studied by its aid. The theory is often attributed to Stokes and Airy, but was really known to Lagrange. If linearized by making the additional assumption that the wave amplitudes are small, the theory becomes the same as that employed as the mathematical basis for the theory of the tides in the oceans. In the lowest order of approximation the nonlinear shallow water theory results in a system of hyperbolic partial differential equations, which in important special cases can be treated in a most illuminating way with the aid of the method of characteristics. The mathematical methods are treated in detail in Chapter 10. The physical problems treated in Chapter 10 are quite varied; they include the propagation of unsteady waves due to local disturbances into still water, the breaking of waves, the solitary wave, floating breakwaters in shallow water. A lengthy section on the motions of frontal discontinuities in the atmosphere is included also in Chapter 10. In Chapter 11, entitled Mathematical Hydraulics, the shallow water theory is employed to study wave motions in rivers and other open channels which, unlike the problems of the preceding chapter, are largely conditioned by the necessity to consider resistances to the flow due to the rough sides and bottom of the channel. Steady flows, and steady progressing waves, including the problem of roll waves in steep channels, are first studied. This is followed by a treatment of numerical methods of solving problems concerning flood-waves in rivers, with the object of making flood predictions through the use of modern high speed digital computers. That such methods can be used to furnish accurate predictions has been verified for a flood in a 400-mile stretch of the Ohio River, and for a flood coming down the Ohio River and passing through its junction with the Mississippi River.

Part IV, consisting of Chapter 12, is concerned with problems solved in terms of the exact theory, in particular, with the use of the exact nonlinear free surface conditions. A proof of the existence of periodic waves of finite amplitude, following Levi-Civita in a general way, is included.

The amount of mathematical knowledge needed to read the book varies in different parts. For considerable portions of Part II the elements of the theory of functions of a complex variable are assumed known, together with some of the standard facts in potential theory. On the other hand Part III requires much less in the way of specific knowledge, and, as was mentioned above, the basic theory of the hyperbolic differential equations used there is developed in all detail in the hope that this part would thus be made accessible to engineers, for example, who have an interest in the mathematical treatment of problems concerning flows and wave motions in open channels.

In general, the author has made considerable efforts to try to achieve a reasonable balance between the mathematics and the mechanics of the problems treated. Usually a discussion of the physical factors and of the reasons for making simplified assumptions in each new type of concrete problem precedes the precise formulation of the mathematical problems. On the other hand, it is hoped that a clear distinction between physical assumptions and mathematical deductions — so often shadowy and vague in the literature concerned with the mechanics of continuous media — has always been maintained. Efforts also have been made to present important portions of the book in such a way that they can be read to a large extent independently of the rest of the book; this was done in some cases at the expense of a certain amount of repetition, but it seemed to the author more reasonable to save the time and efforts of the reader than to save paper. Thus the portion of Chapter 10 concerned with the dynamics of the motion of fronts in meteorology is largely self-contained. The same is true of Chapter 11 on mathematical hydraulics, and of Chapter 9 on the motion of ships.

Originally this book had been planned as a brief general introduction to the subject, but in the course of writing it many gaps and inadequacies in the literature were noticed and some of them have been filled in; thus a fair share of the material presented represents the result of researches carried out quite recently. A few topics which are even rather speculative have been dealt with at some length (the theory of the motion of fronts in dynamic meteorology, given in Chapter 10.12, for example); others (like the theory of waves on sloping beaches) have been treated at some length as much because the author had a special fondness for the material as for their intrinsic mathematical interest. Thus the author has written a book which is rather personal in character, and which contains a selection of material chosen, very often, simply because it interested him, and he has allowed his predilections and tastes free rein. In addition, the book has a personal flavor from still another point of view since a quite large proportion of the material presented is based on the work of individual members of the Institute of Mathematical Sciences of New York University, and on theses and reports written by students attending the Institute. No attempt at completeness in citing the literature, even the more recent literature, was made by the author; on the other hand, a glance at the Bibliography (which includes only works actually cited in the book) will indicate that the recent literature has not by any means been neglected.

In early youth by good luck the author came upon the writings of scientists of the British school of the latter half of the nineteenth century. The works of Tyndall, Huxley, and Darwin, in particular, made a lasting impression on him. This could happen, of course, only because the books were written in an understandable way and also in such a way as to create interest and enthusiasm: — but this was one of the principal objects of this school of British scientists. Naturally it is easier to write books on biological subjects for non-specialists than it is to write them on subjects concerned with the mathematical sciences — just because the time and effort needed to acquire a knowledge of modern mathematical tools is very great. That the task is not entirely hopeless, however, is indicated by John Tyndall's book on sound, which should be regarded as a great classic of scientific exposition. On the whole, the British school of popularizers of science wrote for people presumed to have little or no foreknow-ledge of the subjects treated. Now-a-days there exists a quite large potential audience for books on subjects requiring some knowledge of mathematics and physics, since a large number of specialists of all kinds must have a basic training in these disciplines. The author hopes that this book, which deals with so many phenomena of every day occurrence in nature, might perhaps be found interesting, and understandable in some parts at least, by readers who have some mathematical training but lack specific knowledge of hydrodynamics.* For example, the introductory discussion of waves on sloping beaches in Chapter 5, the purely geometrical discussion of the wave patterns created by moving ships in Chapter 8, great parts of Chapters 10 and 11 on waves in shallow water and flood waves in rivers, as well as the general discussion in Chapter 10 concerning the motion of fronts in the atmosphere, are in this category.

2. Outline of contents

It has already been stated that this book is planned as a coherent and unified whole in spite of the variety and diversity of its contents on both the mathematical and the physical sides. The possibility of achieving such a purpose lies in the fortunate fact that the material can be classified rather readily in terms of the types of mathematical problems which occur, and this classification also leads to a reasonably consistent ordering of the material with respect to the various types of physical problems. The book is divided into four main parts.

Part I begins with a brief, but it is hoped adequate, development of the hydrodynamics of perfect incompressible fluids in irrotational flow without viscosity, with emphasis on those aspects of the subject relevant to flows with a free surface. Unfortunately, the basic general theory is unmanageable for the most part as a basis for the solution of concrete problems because the nonlinear free surface conditions make for insurmountable difficulties from the mathematical point of view. It is therefore necessary to make restrictive assumptions which have the effect of yielding more tractable mathematical formulations. Fortunately there are at least two possibilities in this respect which are not so restrictive as to limit too drastically the physical interest, while at the same time they are such as to lead to mathematical problems about which a great deal of knowledge is available.

One of the two approximate theories results from the assumption that the wave amplitudes are small, the other from the assumption

* The book by Rachel Carson [C.16] should be referred to here. This book is entirely nonmathematical, but it is highly recommended for supplementary reading. Parts of it are particularly relevant to some of the material in Chapter 6 of the present book.

that it is the depth of the liquid which is small — in both cases, of course, the relevant quantities are supposed small in relation to some other significant length, such as a wave length, for example. Both of these approximate theories are derived as the lowest order terms of formal developments with respect to an appropriate small dimensionless parameter; by proceeding in this way, however, it can be seen how the approximations could be carried out to include higher order terms. The remainder of the book is largely devoted to the working out of consequences of these two theories, based on concrete physical problems: Part II is based on the small amplitude theory, and Part III deals with applications of the shallow water theory. In addition, there is a final chapter (Chapter 12) which makes up Part IV, in which a few problems are solved in terms of the basic general theory and the nonlinear boundary conditions are satisfied exactly; this includes a proof along lines due to Levi-Civita, of the existence, from the rigorous mathematical point of view, of progressing waves of finite amplitude.

Part II, which is concerned with the first of the possibilities, might be called the linearized exact theory, since it can be obtained from the basic exact theory simply by linearizing the free surface conditions on the assumption that the wave motions studied constitute a small deviation from a constant flow with a horizontal free surface. Since we deal only with irrotational flows, the result is a theory based on the determination of a velocity potential in the space variables (containing the time as a parameter, however) as a solution of the Laplace equation satisfying certain linear boundary and initial conditions. This linear theory thus belongs, generally speaking, to potential theory.

There is such a variety of material to be treated in Part II, which comprises Chapters 3 to 9, that a further division of it into subdivisions is useful, as follows: 1) subdivision A, dealing with wave motions that are simple harmonic oscillations in the time; 2) subdivision B, dealing with unsteady, or transient, motions that arise from initial disturbances starting from rest; and 3) subdivision C, dealing with waves created in various ways on a running stream, in contrast with subdivisions A and B in which all motions are assumed to be small oscillations near the rest position of equilibrium of the fluid.

Subdivision A is made up of Chapters 3, 4, and 5. In Chapter 3 the basically important standing and progressing waves in liquids of uniform depth and infinite lateral extent are treated; the important fact that these waves are subject to dispersion comes to light, and the notion of group velocity thus arises. The problem of the uniqueness of the solutions is considered — in fact, uniqueness questions are intentionally stressed throughout Part II because they are interesting mathematically and because they have been neglected for the most part until rather recently. It might seem strange that there could be any interesting unresolved uniqueness questions left in potential theory at this late date; the reason for it is that the boundary condition at a free surface is of the mixed type, i.e. it involves a linear combination of the potential function and its normal derivative, and this combination is such as to lead to the occurrence of non-trivial solutions of the homogeneous problems in cases which would in the more conventional problems of potential theory possess only identically constant solutions. In fact, it is this mixed boundary condition at a free surface which makes Part II a highly interesting chapter in potential theory — quite apart from the interest of the problems on the physical side. Chapter 4 goes on to treat certain simple harmonic forced oscillations, in contrast with the free oscillations treated in Chapter 3. Chapter 5 is a long chapter which deals with simple harmonic waves in cases in which the depth of the water is not constant. A large part of the chapter concerns the propagation of progressing waves over a uniformly sloping beach; various methods of treating the problem are explained — in part with the object of illustrating recently developed techniques useful for solving boundary problems (both for harmonic functions and functions satisfying the reduced wave equation) in which mixed boundary conditions occur. Another problem treated (in Chapter 5.5) is the diffraction of waves around a vertical wedge. This leads to a problem identical with the classical diffraction problem first solved by Sommerfeld [S.12] for the special case of a rigid half-plane barrier. Here again the uniqueness question comes to the fore, and, as in many of the problems of Part II, it involves consideration of so-called radiation conditions at infinity. A uniqueness theorem is derived and also a new, and quite simple and elementary, solution for Sommerfeld's diffraction problem is given. It is a curious fact that these gravity wave problems, the solutions of which are given in terms of functions satisfying the Laplace equation, nevertheless require for the uniqueness of the solutions that conditions at infinity of the radiation type, just as in the more familiar problems based on the linear wave equation, be imposed;

ordinarily in potential theory it is sufficient to require only boundedness conditions at infinity to ensure uniqueness.

In subdivision B of Part II, comprised of Chapter 6, a variety of problems involving transient motions is treated. Here initial conditions at the time $t = 0$ are imposed. The technique of the Fourier transform is explained and used to obtain solutions in the form of integral representations. The important classical cases (treated first by Cauchy and Poisson) of the circular waves due to disturbances at a point of the free surface in an infinite ocean are studied in detail. For this purpose it is very useful to discuss the integral representations by using an asymptotic approximation due to Kelvin (and, indeed, developed by him for the purpose of discussing the solutions of just such surface wave problems) and called the principle, or method, of stationary phase. These results then can be interpreted in a striking way in terms of the notion of group velocity. Recently there have been important applications of these results in oceanography: one of them concerns the type of waves called tsunamis, which are destructive waves in the ocean caused by earthquakes, another concerns the location of storms at sea by analyzing wave records on shore in the light of the theory at present under discussion. The question of uniqueness of the transient solutions — again a problem solved only recently — is treated in the final section of Chapter 6. An opportunity is also afforded for a discussion of radiation conditions (for simple harmonic waves) as limits as $t \to \infty$ in appropriate problems concerning transients, in which boundedness conditions at infinity suffice to ensure uniqueness.

The final subdivision of Part II, subdivision C, deals with small disturbances created in a stream flowing initially with uniform velocity and with a horizontal free surface. Chapter 7 treats waves in streams having a uniform depth. Again, in the case of steady motions, the question of appropriate conditions of the radiation type arises; the matter is made especially interesting here because the circumstances with respect to radiation conditions depend radically on the parameter U^2/gh, with U and h the velocity and depth at infinity, respectively. Thus if $U^2/gh > 1$, no radiation conditions need be imposed, if $U^2/gh < 1$ they are needed, while if $U^2/gh = 1$ something quite exceptional occurs. These matters are studied, and their physical interpretations are discussed in Chapter 7.3 and 7.4. In Chapter 8 Kelvin's theory of ship waves for the idealized case of a ship regarded as a point disturbance moving over the surface of the water is treated in considerable detail. The principle of stationary phase leads to a beautiful and elegant treatment of the nature of ship waves that is purely geometrical in character. The cases of curved as well as straight courses are considered, and photographs of ship waves taken from airplanes are reproduced to indicate the good accord with observations. Finally, in Chapter 9 a general theory (once more the result of quite recent investigations) for the motion of ships, regarded as floating rigid bodies, is presented. In this theory no restrictive assumptions — regarding, for example, the coupling (or lack of coupling, as in an old theory due to Krylov [K.20] between the motion of the sea and the motion of the ship, or between the various degrees of freedom of the ship — are made other than those needed to linearize the problem. This means essentially that the ship must be regarded as a thin disk so that it can slice its way through the water (or glide over the surface, perhaps) with a finite velocity and still create waves which do not have large amplitudes; in addition, it is necessary to suppose that the motion of the ship is a small oscillation relative to a motion of translation with uniform velocity. The theory is obtained by making a formal development of all conditions of the complete nonlinear boundary problem with respect to a parameter which is a thickness-length ratio of the ship. The resulting theory contains the classical Michell-Havelock theory for the wave resistance of a ship in terms of the shape of its hull as the simplest special case.

We turn next to Part III, which deals with applications of the approximate theory which results from the assumption that it is the depth of the liquid which is small, rather than the amplitude of the surface waves as in Part II. The theory, called here the shallow water theory, leads to a system of nonlinear partial differential equations which are analogous to the differential equations for the motion of compressible gases in certain cases. We proceed to outline the contents of Part III, which is composed of two long chapters.

In Chapter 10 the mathematical methods based on the theory of characteristics are developed in detail since they furnish the basis for the discussion of practically all problems in Part III; it is hoped that this preparatory discussion of the mathematical tools will make Part III of the book accessible to engineers and others who have not had advanced training in mathematical analysis and in the methods of mathematical physics. In preparing this part of the book the author's task was made relatively easy because of the existence of the book by Courant and Friedrichs [C.9], which deals with gas dynamics; the presentation of the basic theory given here is largely modeled on the presentation given in that book. The concrete problems dealt with in Chapter 10 are quite varied in character, including the propagation of disturbances into still water, conditions for the occurrence of a bore and a hydraulic jump (phenomena analogous to the occurrence of shock waves in gas dynamics), the motion resulting from the breaking of a dam, steady two dimensional motions at supercritical velocity, and the breaking of waves in shallow water. The famous problem of the solitary wave is discussed along the lines used recently by Friedrichs and Hyers [F.13] to prove rigorously the existence of the solitary wave from the mathematical point of view; this problem requires carrying the perturbation series which formulate the shallow water theory to terms of higher order. The problem of the motion of frontal discontinuities in the atmosphere, which lead to the development of cyclonic disturbances in middle latitudes, is given a formulation — on the basis of hypotheses which simplify the physical situation — which brings it within the scope of a more general "shallow water theory". Admittedly (as has already been noted earlier) this theory is somewhat speculative, but it is nevertheless believed to have potentialities for clarifying some of the mysteries concerning the dynamical causes for the development and deepening of frontal disturbances in the atmosphere, especially if modern high speed digital computing machines are used as an aid in solving concrete problems numerically.

Chapter 10 concludes with the discussion of a few applications of the linearized version of the shallow water theory. Such a linearization results from assuming that the amplitude of the waves is small. The most famous application of this theory is to the tides in the oceans (and also in the atmosphere, for that matter); strange though it seems at first sight, the oceans can be treated as shallow for this phenomenon since the wave lengths of the motions are very long because of the large periods of the disturbances caused by the moon and the sun. This theory, as applied to the tides, is dealt with only very summarily, since an extended treatment is given by Lamb [L.3]. Instead, some problems connected with the design of floating breakwaters in shallow water are discussed, together with brief treatments of the oscillations in certain lakes (the lake at Geneva in Switzerland, for example) called seiches, and oscillations in harbors.

Finally, Part III concludes with Chapter 11 on the subject of mathematical hydraulics, which is to be understood here as referring to flows and wave motions in rivers and other open channels with rough sides. The problems of this chapter are not essentially different, as far as mathematical formulations go, from the problems treated in the preceding Chapter 10. They differ, however, on the physical side because of the inclusion of a force which is just as important as gravity, namely a force of resistance caused by the rough sides and bottom of the channels. This force is dealt with empirically by adding a term to the equation expressing the law of conservation of momentum that is proportional to the square of the velocity and with a coefficient depending on the roughness and the so-called hydraulic radius of the channel. The differential equations remain of the same type as those dealt with in Chapter 10, and the same underlying theory based on the notion of the characteristics applies.

Steady motions in inclined channels are first dealt with. In particular, a method of solving the problem of the occurrence of roll waves in steep channels is given; this is done by constructing a progressing wave by piecing together continuous solutions through bores spaced at periodic intervals. This is followed by the solution of a problem of steady motion which is typical for the propagation of a flood down a long river; in fact, data were chosen in such a way as to approximate the case of a flood in the Ohio River. A treatment is next given for a flood problem so formulated as to correspond approximately to the case of a flood wave moving down the Ohio to its junction with the Mississippi, and with the result that disturbances are propagated both upstream and downstream in the Mississippi and a backwater effect is noticeable up the Ohio. In these problems it is necessary to solve the differential equations numerically (in contrast with most of the problems treated in Chapter 10, in which interesting explicit solutions could be given), and methods of doing so are explained in detail. In fact, a part of the elements of numerical analysis as applied to solving hyperbolic partial differential equations by the method of finite differences is developed. The results of a numerical prediction of a flood over a stretch of 400 miles in the Ohio River as it actually exists are given. The flood in question was the 1945 flood — one of the largest on record — and the predictions made (starting with the initial state of the river and using the known flows into it from tributaries and local drainage) by numerical integration on a high speed digital computer (the Univac) check quite closely with the actually observed flood. Numerical predictions were also made for the case of a flood (the 1947 flood in this case) coming down the Ohio and passing through its junction with the Mississippi; the accuracy of the prediction was good. This is a case in which the simplified methods of the civil engineers do not work well. These results, of course, have important implications for the practical applications.

Finally Part IV, made up of Chapter 12, closes the book with a few solutions based on the exact nonlinear theory. One class of problems is solved by assuming a solution in the form of power series in the time, which implies that initial motions and motions for a short time only can be determined in general. Nevertheless, some interesting cases can be dealt with, even rather easily, by using the so-called Lagrange representation, rather than the Euler representation which is used otherwise throughout the book. The problem of the breaking of a dam, and, more generally, problems of the collapse of columns of a liquid resting on a rigid horizontal plane can be treated in this way. The book ends with an exposition of the theory due to Levi-Civita concerning the problem of the existence of progressing waves of finite amplitude in water of infinite depth which satisfy exactly the nonlinear free surface conditions.

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J. J. STOKER

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PART I

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