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Theory and Applications

Lorenzo Farina
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Preface

The aim of this book is to introduce the reader to the world of positive linear systems, a particular, but important and fascinating, class of linear systems. We have made absolutely no effort to hide our enthusiasm for the topics presented in the hope that this will be enough of an excuse for being "informal" at times. We have divided the subject into three parts. The first part contains the definitions and the basic properties of positive linear systems. In the second part, the main theoretical results are reported. The third part is devoted to the study of some classes of positive linear systems relevant in applications. The reader familiar with linear algebra and linear systems theory should appreciate the way the arguments are treated and the subject is presented. A number of excellent books on these topics are available; nevertheless we have included two appendixes for making the book (reasonably) self-contained. The exposition of all the topics is supported with a number of examples and problems. To facilitate the reader, the theoretical (T) or applicative (A) character of each problem is explicitly pointed out, together with its level of difficulty (I, II, or III). We would like to express gratitude to our colleagues Luca Benvenuti, Luca Ghezzi, Salvatore Monaco, Simona Muratori, and Carlo Piccardi for their support, suggestions and ideas provided during our research work on positive systems and to Ms. Patrizia Valentini for artful LaTeXing.

L.F.

S.R.

Rome, May 2000

Part I

Definitions

1

Introduction

This book is concerned with *positive linear systems*, a remarkable class of linear systems. For the sake of simplicity we will only refer to the case of time-invariant, finite-dimensional single input, single output systems, described by state equations of the form

$$\dot{x}(t) = Ax(t) + bu(t)$$

or

$$x(t+1) = Ax(t) + bu(t)$$

which correspond, respectively, to continuous-time (t is defined over the reals) or discrete-time (t is defined over the integers) systems, and by an output transformation of the form

$$y(t) = c^T x(t)$$

Such systems are identified by the triple (A, b, c^T), which has peculiar features imposed by the positivity conditions highlighted in Chapter 2.

Positive systems are, by definition, systems whose state variables take only nonnegative values. From a general point of view, they should be viewed as very particular. To see this, consider a random generation of the elements a_{ij} of the matrix A and of the elements b_i and c_i of the vectors b and c^T. The probability of obtaining a positive system in this way is very low [$2^{-(n^2+2n)}$ for a n-dimensional discrete-time system]. From a practical point of view, however, such systems are anything but particular since positive systems are often encountered in applications.

Positive systems are, for instance, networks of reservoirs, industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems (memories, warehouses, ...), hierarchical systems, compartmental systems (frequently used when modeling transport and accumulation phenomena of substances in the human bodies), water and atmospheric pollution models, stochastic models where state variables must be nonnegative since they represent probabilities, and many other models commonly used in economy and sociology. One is tempted to assert that positive systems are the most often encountered systems in almost all areas of science and technology, except electro mechanics, where the variables (voltages, currents, forces, positions, velocities) may assume either positive and negative values. However, the existence of positive systems in an electrical or mechanical context cannot be excluded. Consider, as an example, a simple mechanical system composed of a point mass driven along a straight line by an external force. Position and velocity of the mass cannot become negative provided their initial values are nonnegative and the force is unidirectional: This is a positive system. On the other hand, even the simplest electrical circuit, namely, the $R - C$ circuit, is a positive system since the voltage on the capacitor remains nonnegative if initially such.

Positive linear systems, as any other linear system, satisfy the superposition principle and also a peculiar one, that of *comparative dynamics*. Such a principle can be expressed by saying that "*positive perturbations of inputs, states, and parameters cannot produce a decrease of the state and output at any instant of time following the perturbation*". This rule can be quite useful whenever one is interested in a qualitative analysis of the influence of some design parameter (or input) on the system.

Among a number of properties holding for positive systems, the one concerning a dominant mode undoubtedly stands out. It often allows one to dramatically simplify the stability analysis. This property is expressed through a series of results known as the *Frobenius–Perron theorems*, holding for matrices with positive entries. But, even more important is the fact that a number of properties rely only on the structure of the system, that is on the structure of existing influences among all the input, state, and output variables. In other words, it often suffices to know "*who influences who*" in order to give a complete answer to fundamental questions. In fact, if the influence of one variable on another is always positive, the compensation among different paths of influence will not be allowed. Due to this property, *the influence graph*, which shows the direct influences among the variables, becomes a valuable tool (*structural model*) of analysis. For this reason, after the definition of positive systems, we will introduce the notion of the influence graph and will systematically highlight which properties rely on the topology of the graph and which on the "*level of influence*".

We will first discuss the classical properties of dynamical systems, that is, stability, reachability, observability, input–output maps, and minimum phase. Obviously, other properties of peculiar interest for positive systems, such as cyclicity, primitivity, excitability, and transparency will also be considered. These properties will enable us to give a better physical interpretation of the various results presented in the book. Following the exposition of the theory, we will consider a number of applications tied to models widely used by researchers and professionals during the last decades. We will discuss, in particular, the Leontief model used by economists for predicting productions and prices; the Leslie model used by demographers to study age-structured populations; the Markov chains; the compartmental models; and the birth and death processes, relevant to the analysis of queueing systems.

At the end of this book, we will present a detailed guided bibliography and two appendixes concerning linear algebra and linear systems theory in order to make, if needed, the reader familiar with the mathematics used throughout the book.

2

Definitions and Conditions of Positivity

In this chapter, we give two definitions of positivity for linear systems (called *internal* and *external*) and we derive the corresponding necessary and sufficient conditions. Some of these conditions require the notion of positivity of matrices, vectors, and functions.

We will say that a matrix F is *strictly greater* than a matrix G (having the same number of rows and columns) and denote this by $F \gg G$, if and only if *all* the elements f_{ij} of F are *greater* than the corresponding elements g_{ij} of G. If *all* the elements of F are *greater than or equal to* the corresponding elements of G, but *at least one* of the f_{ij} is *greater* than g_{ij}, we will say that F is *greater* than G and denote this by $F > G$. Obviously, both $F \gg G$ and $F > G$ imply $F \neq G$. The notation $F \geq G$, which should be read F *greater than or equal to* G, will mean that *all* the elements of F are *greater than or equal to* the corresponding elements of G. Thus $F \geq G$ is satisfied also when $F = G$. These definitions justify the use of the following notation and terminology:

strictly positive matrix $F : F \gg 0 \quad (f_{ij} > 0 \quad \forall(i,j))$

positive matrix $F : F > 0 \quad (f_{ij} \geq 0 \forall(i,j), \exists(i,j) : f_{ij} > 0)$

nonnegative matrix $F : F \geq 0 \quad (f_{ij} \geq 0 \quad \forall(i,j))$

strictly positive diagonal matrix $F : F \gg 0 \quad (f_{ij} = 0 \quad \forall i \neq j, f_{ii} > 0 \quad \forall i)$

The last notation, though not rigorous, will be used in the sequel for the sake of simplicity. Analogous definitions and notations can be given also for n-dimensional vectors with $n \geq 2$. When dealing with scalars, however, strict positivity $(a \gg 0)$

coincides with positivity ($a > 0$). It is also worth noting that the following rules hold:

$$
\begin{aligned}
&F \gg 0, G > 0 \Longrightarrow FG > 0\\
&F > 0, G \gg 0 \Longrightarrow FG > 0\\
&F \gg 0, x > 0 \Longrightarrow Fx \gg 0\\
&F > 0, x \gg 0 \Longrightarrow Fx > 0\\
&x \gg 0, y > 0 \Longrightarrow x^T y = y^T x > 0
\end{aligned}
$$

Finally, it is important to note that there is no trivial link between positive matrices (in the above mentioned sense) and positive definite matrices.

Analogous to what was previously stated for matrices and vectors, we will say that a real function $u(\cdot)$ of a real variable is strictly positive in an interval and denote this by $u(\cdot) \gg 0$ provided that $u(\xi) > 0$ at every point in the interval. Similarly, we will say that a real function is positive, provided that $u(\xi) \geq 0$ at every point in the interval and $u(\xi) > 0$ at least at one point. Finally, we will call nonnegative the functions $u(\cdot)$ for which $u(\xi)$ is nonnegative at every point in the interval. Obviously, in the case of real functions of integer variables strict positivity (or positivity, or nonnegativity) of the function $u(\cdot)$ in an interval $[0, t-1]$ coincides with strict positivity (or positivity, or nonnegativity) of the t-dimensional vector

$$u_0{}^{t-1} = (u(t-1)u(t-2)\cdots u(0))^T$$

We are now able to give the first definition of positivity of a linear system.

DEFINITION 1 *(externally positive linear system)*

A linear system (A, b, c^T) is said to be *externally positive* if and only if its forced output (*i.e.*, the output corresponding to a zero initial state) is nonnegative for every nonnegative input function.

External positivity is a property that is often easy to check, since, in most cases, for physical reasons input and output variables are necessarily positive. For example, a hydrological system composed of a series of lakes in which the input is the inflow into the upstream lake and the output is the outflow from the downstream lake is an externally positive system. Other obvious examples are chemical systems composed of a set of reactors in which the input is the feed concentration and the output is the product concentration and educational systems in which input and output are the annual number of freshmen and graduates.

Before we proceed, it is worth making two remarks. The first is that positivity is not independent of the basis used for representing inputs and outputs. For example, in the case of the hydrological system we could decide to assign a positive sign to the inflow into the upstream lake and a negative sign to the outflows of the downstream lake. Thus, the system would have positive inputs and negative outputs and would not be an externally positive system. It is clear, however, that this problem could be easily avoided by giving a slightly more general definition of external positivity. This extension has not been made here since in most relevant applications the natural choice of the basis is also the correct one.

The second, and more important, remark is that there exist externally positive systems in which the input and output variables can also take negative values. For example, in the electrical network depicted in *Fig. 2.1*, which is externally positive (see *Problem 1*) for appropriate values of the electrical parameters, the current u and the voltage y could also be negative. In such cases, it is not possible to easily deduce the external positivity of the system.

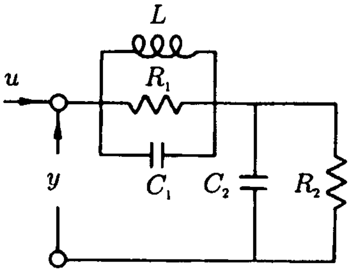


Figure 2.1 An electrical network that is externally positive for appropriate values of the parameters.

The following theorem is therefore of interest:

THEOREM 1 *(condition for external positivity)*

A linear system is externally positive if and only if its impulse response is nonnegative.

Proof. Consider a continuous-time linear system with zero initial state. The output is the convolution integral of the input and the impulse response, namely,

$$y(t) = \int_0^t h(t - \xi)u(\xi)d\xi$$

with

$$h(t) = c^T e^{At} b \quad t \geq 0$$

Therefore, if the impulse response $h(t)$ is nonnegative, the output $y(t)$ is nonnegative for every nonnegative input $u(t)$, so that the system is externally positive. On the other hand, if the system is externally positive, then $h(t)$ must be nonnegative. In fact, if this were not the case, $h(t)$ would be negative at least at one point and by continuity in a whole interval $[t_1, t_2]$. Thus, the output $y(t)$ would be negative for $t > t_2$ for every input function $u(t)$, which is strictly positive in $[(t - t_2), (t - t_1)]$

and zero elsewhere. This would contradict the external positivity of the system. A similar proof can be given for discrete-time systems.

◊

EXAMPLE 1 (externally positive electrical network)
Consider the electrical network in *Fig. 2.2* with positive parameters R, L, and C.

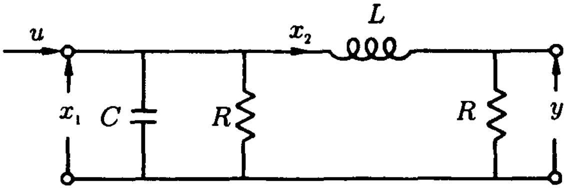


Figure 2.2 An electrical network that is externally positive for $L/R + RC \geq \sqrt{8LC}$.

The network is described by the triple

$$A = \begin{pmatrix} -1/RC & -1/C \\ 1/L & -R/L \end{pmatrix} \quad b = \begin{pmatrix} 1/C \\ 0 \end{pmatrix}$$
$$c^T = \begin{pmatrix} 0 & R \end{pmatrix}$$

and its eigenvalues $\lambda_{1,2}$ have negative real parts since the system is asymptotically stable. Moreover, it is easy to verify that they are a complex conjugate pair ($\lambda_{1,2} = a \pm ib$) if $L/R + RC < \sqrt{8LC}$ and real otherwise. In the first case, the system cannot be externally positive since its impulse response is oscillatory. In the second case, the impulse response is the sum of two exponentials

$$h(t) = \alpha e^{\lambda_1 t} + \beta e^{\lambda_2, t}$$

Since $h(0) = c^T b = 0$, it follows that $\alpha = -\beta$, and then

$$\dot{h}(t) = \alpha(\lambda_1 e^{\lambda_1, t} - \lambda_2 e^{\lambda_2 t})$$

The impulse response is therefore stationary (maximal or minimal) as a function of time only once at

$$t^* = \frac{\log(\lambda_1) - \log(-\lambda_2)}{\lambda_2 - \lambda_1}$$

This means that the impulse response is positive if and only if $\dot{h}(0) > (0)$. Since

$$\dot{h}(0) = c^T A b = \frac{R}{LC}$$

we can conclude that the system is externally positive if and only if $L/R + RC \geq \sqrt{8LC}$.

♣

The step response of an externally positive system (starting from a zero state) is non-decreasing since it is the integral of the impulse response that is nonnegative. Therefore, when a constant input is applied to the system, its output tends toward an equilibrium without *overshooting* it. This is true also when there are oscillations in the impulse response of the system.

Both *Definition 1* and *Theorem 1* make clear that external positivity of a system is a property of its input–output relationship for a zero initial state. This implies that the knowledge of the *ARMA model* of a discrete-time system

$$y(t) = \sum_{i=1}^{n} (-\alpha_i) y(t-i) + \sum_{i=1}^{n} \beta_i u(t-i)$$

or that of a continuous-time system

$$y^{(n)}(t) = \sum_{i=1}^{n} (-\alpha_i) y^{(n-i)}(t) + \sum_{i=1}^{n} \beta_i u^{(n-i)}(t)$$

is sufficient for checking external positivity of the system. In other words, given the $2n$ parameters $\alpha_i, \beta_i, i = 1, 2, \ldots, n$, it must be possible to derive whether the system is externally positive or not. Unfortunately, a simple algorithm that performs this task is not yet known. There exist, however, useful sufficient conditions for external positivity. A trivial one is the following

$$\alpha_i \leq 0 \quad \beta_i \geq 0 \quad i = 1, 2, \ldots, n$$

which can be easily proved for discrete-time systems. The proof of this property for a continuous-time system is proposed to the reader as an exercise in *Problem 3*. In some cases, the check of the external positivity seems to be possible only by a brute force approach, namely, by a numerical computation of the impulse response. Practically, this means that the state equations have to be solved for $x = (0) = b$ and that nonnegativity of the impulse response $h(t)$ must be checked for a long interval of time in order to be reasonably sure that $h(t)$ also continues to be nonnegative outside of such an interval.

PROBLEM 1 *(A-III) Determine the values of the parameters R_1, R_2, L, C_1, C_2 for which the electrical network in Fig. 2.1 is externally positive. If this problem is too difficult, prove (using arguments related with the dynamics of the system with slow and fast components) that the network is externally positive for $R_1 = L = C_1 = C_2 = 1$ and $R_2 = 10$ and that it is not externally positive for $R_1 = R_2 = L = C_1 = C_2 = 1$. In case this problem is still too difficult, give an empirical proof with the aid of a computer.*

Now we could give the second definition of positivity, which could be called *internal*. We do not do this for the sake of brevity and for consistency with current terminology.

DEFINITION 2 *(positive linear system)*

A linear system (A, b, c^T) is said to be *positive* if and only if for every nonnegative initial state and for every nonnegative input its state and output are nonnegative.

This definition says that all trajectories emanating from any point in the positive orthant $\mathbb{R}^n_+$ (boundary included) of the state space $\mathbb{R}^n$ obtained by applying a nonnegative input to the system remain in the positive orthant and yield a nonnegative output. *Figure 2.3* shows three trajectories obtained by applying nonnegative inputs to a second-order continuous-time positive system. As expected, the trajectories *a* and *b* which start in the first quadrant, remain confined in it. The trajectory denoted by *c* is not contradictory since it enters the first quadrant but does not leave it.

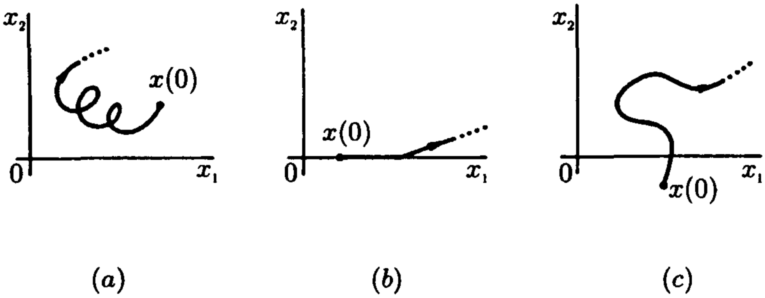
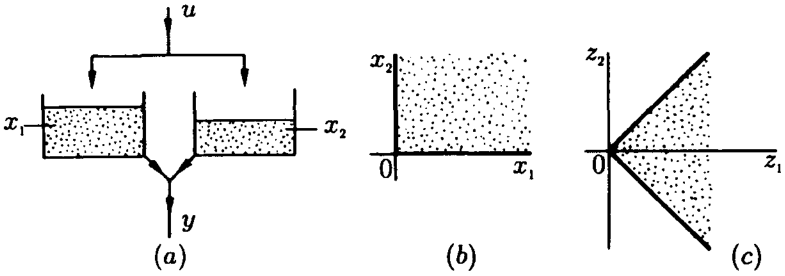


Figure 2.3 Three trajectories $(a), (b)$, and (c) obtained by applying a nonnegative input to a second-order continuous-time system.

The positive orthant is therefore a "trap" in which the trajectories generated by nonnegative inputs can possibily enter but from which they cannot get out. Formally, this can be expressed by saying that the positive orthant $\mathbb{R}^n_+$ is a *nonnegative invariant set*. This is the reason some authors define positive systems by requiring the existence of an invariant set without requiring, however, that such an invariant set be the positive orthant $\mathbb{R}^n_+$.

It is important to note that positivity, besides depending on the basis of the input and output spaces, depends also on the basis of the state space. For example, the hydrological system of *Fig. 2.4(a)* , which is obviously positive from the external point of view, is also positive if the state variables are the two storages x_1 and x_2, but is not such if the state variables are the sum z_1 and the difference z_2 of the two storages. In fact, in the first case [*Fig. 2.4(b)*], the invariant set is the first quadrant while, in the second case [*Fig. 2.4(c)*], the invariant set is not the first quadrant.



(a) (b) (c)

Figure 2.4 A positive system composed of two reservoirs (*a*) and its invariant sets (*b*) and (*c*).

In general, it is not easy to check if there exists a basis for which a given linear system is positive. Nevertheless, in the majority of the applications when such a basis exists, it is the natural one (*i.e.*, the state variables are reservoir storages, concentrations of chemical compounds, voltages on capacitors, etc.). This means that it is often easy to know whether a system admits a basis for which it is positive.

The definition of positivity (*Definition 2*) requires the nonnegativity of the output for every nonnegative initial state $x(0)$, rather than only for $x(0) = 0$ as for external positivity (*Definition 1*). This means that positivity implies external positivity while the opposite is not true, namely an externally positive system can be not positive. Moreover, it can be said that there exist systems that are externally positive and cannot be made positive through any change of basis of the state space. To see this, consider a continuous-time third-order system. Suppose that such a system is asymptotically stable and has a complex conjugate pair of eigenvalues. This implies that in the state space there exists a plane X (corresponding to the complex eigenvalues) on which the trajectories for $u = 0$ spiral toward the origin and a straight line r (corresponding to the real eigenvalue), which coincides with a trajectory tending toward the origin. *Figure 2.5*(*a*) depicts this situation. If the modulus of the real part of the complex eigenvalues is smaller than the modulus of the real eigenvalue (*i.e.*, if the complex eigenvalues are dominant), the trajectory representing the impulse response starts from the point corresponding to the vector b, rapidly approaches the plane X, and then slowly spirals around the origin, as shown in *Figure 2.5*(*b*). The trajectory is on one side of the plane X and is tangent to it as $t \to \infty$. If the vector c, which identifies the output transformation $y(t) = c^T x(t)$, is orthogonal to X the impulse response is positive and, in view of *Theorem 1*, the system is externally positive. It is clear, however, that it is not possible to draw three different straight lines through the origin so that the trajectory lies entirely in the positive orthant determined by these three straight lines. In conclusion, the system cannot be made positive by means of any choice of the basis of the state space.

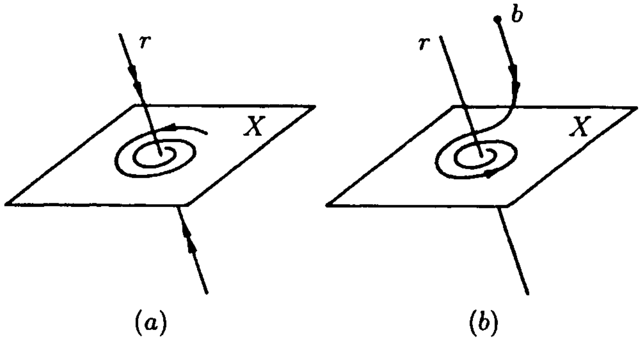


Figure 2.5 The free motion of a third-order system with complex dominant eigenvalues: (a) invariant subspaces; (b) the impulse response (the double arrow indicates fast motion).

We show now how it is formally possible to determine if a given linear system is positive.

THEOREM 2 *(condition for positivity)*

> A discrete-time linear system (A, b, c^T) is positive if and only if $A \geq 0, b \geq 0, c^T \geq 0^T$. A continuous-time linear system (A, b, c^T) is positive if and only if the matrix A is a Metzler matrix, that is, its nondiagonal elements are nonnegative $[a_{ij} \geq 0, \forall (i, j), i \neq j)]$ and $b \geq 0, c^T \geq 0^T$.

Proof. We report here the proof for continuous-time linear systems, leaving to the reader its extension to the case of discrete-time systems.
Necessity. Letting $x(0) = 0$, positivity implies $\dot{x}(0) = bu(0) \geq 0$ for every $u(0) \geq 0$, that is, $b \geq 0$. Moreover, $y(0) = c^T x(0)$ so that positivity $[y(0) \geq 0$ for every $x(0) \geq 0$] implies $c^T \geq 0^T$. Finally, letting $x(0) = e_j$ (unit vector of the x_j axis) it follows $\dot{x}(0) = Ae_j = j$-th column of A. But the trajectory of a positive system cannot leave the positive orthant $\mathbb{R}^n_+$, so that $\dot{x}_i(0) \geq \forall i \neq j$. Therefore, the elements of A that are not on the diagonal must be positive or zero, that is, the matrix A must be a Metzler matrix.

Sufficiency. It is clear that $c^T \geq 0^T$ and $x(t) \geq 0$ imply $y(t) = c^T x(t) \geq 0$. On the other hand, in order to prove that $x(t) \geq 0$, it is sufficient to check that the vector $\dot{x}(t)$ does not point toward the outside of $\mathbb{R}^n_+$ whenever $x(t)$ is on the boundary of $\mathbb{R}^n_+$. This is equivalent to verifying that the components of the vector $\dot{x}(t) = Ax(t) + bu(t)$ corresponding to the zero components of $x(t) \geq 0$ are nonnegative. Denoting by I the set of indices of such components, [*i.e.* $x_i(t) = 0$ for $i \in I$], we can write

$$\dot{x}_i(t) = \sum_{j \notin I} a_{ij} x_j(t) + b_i u(t) \qquad i \in I$$

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