VECTOR INTEGRATION AND STOCHASTIC INTEGRATION IN BANACH SPACES

Nicolae Dinculeanu

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Preface

This book consists of two parts. The first part, Chapter 1, is devoted to vector integration. The rest of the book, chapters 2–10, is devoted to stochastic integration in Banach spaces.

Vector integration of various kinds has been presented in other books. We mention especially the books by N. Dunford and J. Schwartz [D–S], N. Dinculeanu [D.1], J. Diestel and J. J. Uhl Jr. [D–U] and A. U. Kussmaul [Kus.1]. In the text, we refer the reader to these books for the proof of some important theorems that we do not want to repeat.

The core of Chapter 1 is §5, devoted to integration of vector-valued functions with respect to vector measures with finite *semivariation*. This kind of integration is not contained in any other book and was presented first in [B–D.2]. Kussmaul [Kus.1] considers a similar kind of integration but only for real-valued functions.

Among the applications of the integral of §5 we quote: the Riesz representation theorem, the integral representation of continuous linear operators on L^p-spaces, the Stieltjes integral with respect to vector-valued functions with *finite semivariation* (which was not considered before) and, especially, the stochastic integration in Banach spaces.

The reader interested in integration theory only, could use only chapter 1 and the paragraphs 19, 21, 29 and 31.

For the part devoted to stochastic integration we assume familiarity with the definitions and the results of the general theory of stochastic processes, as presented, for example, in the book by C. Dellacherie and P. A. Meyer [D–M] and often we refer the reader to this book.

The classical theory of stochastic integration for real-valued processes, reduces, essentially, to integration with respect to a square integrable martingale. This is done by defining the stochastic integral first for simple processes and then extending it to a larger class of processes, by means of an isometry between certain L^2-type spaces of processes. This method has been also used by Kunita [Ku] for processes with values in Hilbert spaces, by using the existence of the inner product to prove the above-mentioned isometry. But this approach cannot be used for Banach spaces, which lack an inner product. A new approach is needed for Banach-valued processes.

On the other hand, the classical stochastic integral as described above is not a genuine integral, in the sense that it is not an integral with respect to a measure.

It is desirable, as in classical Measure Theory, to have a space of "integrable" processes with a norm on it, for which it is a Banach space, and to have an integral for integrable processes, which would be the stochastic integral. Also desirable would be to have Vitali- and Lebesgue-type convergence theorems. Such a goal is legitimate and many attempts have been made to fulfill it.

Any measure-theoretic approach to stochastic integration has to use an integration theory with respect to a vector measure. Pellaumail [P] was the first to attempt such an approach, but due to the lack of a satisfactory integration theory, this goal was not achieved. Kussmaul [Kus.1] used the idea of Pellaumail and was able to define a consistent, measure theoretic stochastic integral, but only for real-valued processes. He used in [Kus.2] the same method for Hilbert-valued processes, but the goal was only partially fulfilled, again due to the lack of a satisfactory general integration theory. The integration theory presented here in §5 seems to be tailor-made for application to the stochastic integral. It was presented for the first time in [B–D.2].

In order to apply the integration theory to define a stochastic integral with respect to a Banach-valued process X, we associate to it a measure I_X on the ring $\mathcal{R}$ generated by the predictable rectangular sets. The process X is called *summable*, if I_X can be extended to a σ-additive measure with finite semivariation on the σ-algebra $\mathcal{P}$ of predictable sets. Roughly speaking, the stochastic integral $H{\cdot}X$ is the process $(\int_{[0,t]} H dI_X)_{t\geq 0}$ of integrals with respect to I_X.

The summable processes play, in this theory, the role played by the square integrable martingales in the classical theory. It turns out that every Hilbert-valued, square integrable martingale is summable and the processes with integrable variation are also summable. In addition, a new class of summable processes emerges: the processes with integrable *semivariation*. Moreover, the stochastic integral with respect to such a process can be computed pathwise, as a Stieltjes integral (itself a Stieltjes integral with respect to a function of finite *semivariation*, rather than finite variation). This new class of summable processes could not be made evident in the classical case of scalar processes, since for these processes, the variation and the semivariation are equal. It is only for processes with values in an infinite dimensional Banach space that the semivariation is different from the variation.

Our space of integrable processes with respect to a summable process is a Lebesgue-type space, endowed with a seminorm, for which it is complete and in which the Vitali and the Lebesgue convergence theorems are valid. The legitimate goal mentioned above is thus fulfilled.

It is worth mentioning the following summability criterion: X is summable iff I_X is bounded and has finite semivariation on the ring $\mathcal{R}$. It is quite unexpected that the mere boundedness of I_X on $\mathcal{R}$ implies not only that I_X is $\sigma-$additive on $\mathcal{R}$, but that I_X can be extended to a $\sigma-$additive measure on $\mathcal{P}$.

Using the same measure-theoretic approach, we extend the theory of Stochastic integration for vector valued, two-parameter processes, in Chapters 7-10.

The same measure-theoretic approach can be used to extend the theory of stochastic integration for process measures and martingale measures in Banach spaces ([D.9], [D.10], [Di–Mu]). This extension, which is not included in this book, has applications in the theory of stochastic partial differential equations [W.3].

Each chapter is divided into paragraphs, numbered in continuation from one chapter to the last one, from 1 to 32. Each paragraph is divided into several sections indicated anew, in each paragraph, by capital letters. The numbering of definitions and theorems starts anew in each paragraph. The quotation, in the text, of definitions and theorems, is done in the following way: if we refer in a paragraph to a theorem from the same paragraph, then we quote it by its number in that paragraph; if we refer to a theorem from a different paragraph, then we quote it by a pair of numbers, in the form Theorem $a.b$, the first number a indicating the paragraph, and the second number b indicating the number of the theorem in paragraph a.

Gainesville, Florida
February 26, 1999

N. Dinculeanu

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Chapter 1

Vector Integration

§1. PRELIMINARIES

In this paragraph we establish the notations used in the book and present the immediate integral, the classical integral, and the Bochner integral. Special attention is given to measurability of vector-valued functions.

A. Banach spaces

1. Throughout the book, E, F, G, D will denote Banach spaces. All Banach spaces are over the real field $\mathbb{R}$.

Numbers $\alpha \geq 0$ are called *positive* (rather than nonnegative). A sequence (α_n) of numbers such that $\alpha_n \leq \alpha_{n+1}$ for every n is called *increasing* (rather than nondecreasing).

For any Banach space M, the norm of an element $x \in M$ is denoted by $|x|$ or $|x|_M$; the dual of M is denoted by M^* and the unit ball of M by M_1. The duality between M and M^* is denoted by $\langle x, x^* \rangle$ or x^*x, $\langle x^*, x \rangle$ or even xx^*.

If M is a Hilbert space, the inner product of two elements $x, y \in M$ is denoted by $\langle x, y \rangle$, or $\langle x, y \rangle_M$ or even xy.

The space of bounded linear operators from F into G is denoted by $L(F, G)$. We write $E \subset L(F, G)$ to mean that E is continuously embedded into $L(F, G)$, that is, $|xy| \leq |x|\,|y|$, for $x \in E$ and $y \in F$. Special mention will be made in case the embedding is an isometry.

Examples of such isometries are: $E = L(\mathbb{R}, E)$; $E \subset L(E^*, \mathbb{R}) = E^{**}$; $L \subset L(F, E\hat{\otimes}_\pi F)$; $E \subset L(F, F\hat{\otimes}_\pi E)$; if E is a Hilbert space, $E = L(E, \mathbb{R})$;

if E and F are Hilbert spaces, $E \subset L(F, E\hat{\otimes}_{HS}F)$, where HS denotes the Hilbert–Schmidt norm.

We write $c_0 \not\subset M$ to mean that M does not contain a copy of c_0, that is, M does not contain a subspace which is isomorphic to the Banach space c_0.

If M is a Banach space, a subspace $Z \subset M^*$ is said to be a *norming* space for M, if for every $x \in M$ we have

$$|x| = \sup\{|\langle x, z\rangle| : z \in Z_1\}.$$

Obviously, M^* is norming for M and if we consider $M \subset M^{**}$, then M is norming for M^*.

A useful example of a norming space, which will be used in the sequel, is the following one:

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $1 \leq p \leq +\infty$. Denote $L^p_E = L^p_E(P)$, the space of Bochner-integrable functions $f : \Omega \to E$ with respect to P (see Section J on the Bochner integral). If $\frac{1}{p} + \frac{1}{q} = 1$, then $L^q_{E^*}$ is isometrically contained in $(L^p_E)^*$. If E^* has the Radon–Nikodym Property (RNP) and if $p < \infty$, then $(L^p_E)^* = L^q_{E^*}$. But even if E^* does not have the RNP and even if $p = \infty$, $L^q_{E^*}$ is a norming space for L^p_E. Moreover, if $\mathcal{R}$ is a ring generating the σ-algebra $\mathcal{F}$, the subspace $\mathcal{S}_{E^*}(\mathcal{R})$ of E^*-valued, $\mathcal{R}$-step functions is a norming space for L^p_E.

B. Classes of sets

2. Throughout the first chapter, S is a set and $\mathcal{P}, \mathcal{R}, \mathcal{A}, \mathcal{D}, \mathcal{S}, \Sigma$ are respectively a semiring, a ring, an algebra, a δ-ring, a σ-ring and a σ-algebra of subsets of S.

A *semiring* $\mathcal{P}$ is a class of subsets of S, closed under intersection $A \cap B$ and satisfying the following condition: for any pair (A, B) of sets from $\mathcal{P}$ such that $A \subset B$, there is a finite family $(C_i)_{0 \leq i \leq n}$ of sets from P with

$$A = C_0 \subset C_1 \subset \ldots \subset C_n = B$$

and

$$C_i - C_{i-1} \in \mathcal{P}, \text{ for } i = 1, 2, \ldots, n.$$

An important example of semiring is the class of the intervals of the form $(a, b]$.

A *ring* is a class of subsets of S closed under union $A \cup B$ and difference $A - B$.

Any ring is a semiring.

An *algebra* is a ring containing S.

A *δ-ring* is a ring closed under countable intersections.

A *σ-ring* is a ring closed under countable unions.

A *σ-algebra* is a σ-ring containing S.

For any class $\mathcal{C}$ of subsets of S we denote by $r(\mathcal{C})$, $a(\mathcal{C})$, $\delta r(\mathcal{C})$, $\sigma r(\mathcal{C})$, $\sigma a(\mathcal{C})$ respectively the ring, the algebra, the δ-ring, the σ-ring, and the σ-algebra generated by $\mathcal{C}$.

If $\mathcal{P}$ is a semiring, the ring $r(\mathcal{P})$ generated by $\mathcal{P}$ consists of all the finite, disjoint unions of sets from $\mathcal{P}$.

The σ-ring $\sigma r(\mathcal{D})$ generated by a δ-ring $\mathcal{D}$ consists of all countable unions of disjoint sets from $\mathcal{D}$.

For any class $\mathcal{C}$ of subsets of S we denote by $\mathcal{C}_{\text{loc}}$ the class of sets $A \subset S$ that are "locally" in $\mathcal{C}$, that is, such that $A \cap B \in \mathcal{C}$ for every $B \in \mathcal{C}$.

If $\mathcal{R}$ is a ring, then $\mathcal{R}_{\text{loc}}$ is an algebra; if $\mathcal{D}$ is a δ-ring and $\mathcal{S}$ is a σ-ring, then $\mathcal{D}_{\text{loc}}$ and $\mathcal{S}_{\text{loc}}$ are σ-algebras.

For any class $\mathcal{C}$ of subsets of S and any set $A \subset S$ we denote

$$\mathcal{C} \cap A = \{B \cap A : B \in \mathcal{C}\}.$$

If $\mathcal{C}$ is a ring, δ-ring, σ-ring, then so is $\mathcal{C} \cap A$.

The characteristic function of a set $A \subset S$ is denoted by φ_A, 1_A or I_A .

If $\mathcal{R}$ is a ring (or any other class), we denote by $\mathcal{S}_F(\mathcal{R})$, the set of $\mathcal{R}$-step functions (or $\mathcal{R}$-simple functions) $f : S \to F$ of the form

$$f = \sum_{1 \le i \le n} \varphi_{A_i} x_i,$$

with $A_i \in \mathcal{R}$ and $x_i \in F$. If $\mathcal{R}$ is a ring, the sets A_i can be taken mutually disjoint. In this case

$$|f| = \sum_{1 \le i \le n} \varphi_{A_i} |x_i|.$$

If $\mathcal{P}$ is a semiring and $\mathcal{R} = r(\mathcal{P})$, then $\mathcal{S}_F(\mathcal{P}) = \mathcal{S}_F(\mathcal{R})$.

C. Measurable functions

Measurability will be defined with respect to a σ-algebra.

Let Σ be a σ-algebra of subsets of S.

We start with the usual definition of measurability of numerical functions.

3. Definition. *A function $f : S \to \overline{\mathbb{R}}$ is Σ-measurable if $f^{-1}(B) \in \Sigma$ for every Borel set $B \subset \overline{\mathbb{R}}$.*

The Σ-step functions are Σ-measurable.

We state the following characterization of measurability in terms of step functions:

4. Theorem. a) *A function $f : S \to \overline{\mathbb{R}}$ is Σ-measurable iff there is a sequence (f_n) of finite, real-valued, Σ-step functions such that $f_n \to f$ pointwise.*

b) *If $f : S \to \overline{\mathbb{R}}$ is Σ-measurable there is a sequence (f_n) of finite, real-valued, Σ-step functions such that $f_n \to f$ pointwise and $|f_n| \le |f|$, for each n.*

If $f \geq 0$, the sequence (f_n) can be chosen to be increasing and $f_n \geq 0$.

If f is bounded, one can choose the sequence (f_n) to converge uniformly to f.

For vector-valued functions we take the statement in Theorem 4 a) as a definition of measurability:

5. Definition. *A function $f : S \to F$ is said to be Σ-measurable if there is a sequence (f_n) of F-valued, Σ-step functions such that $f_n \to f$ pointwise.*

In particular, the Σ-step functions are Σ-measurable. It follows that if $f : S \to F$ is Σ-measurable, then $|f|$ is also Σ-measurable.

Since the range of a step function is finite, it follows that the range of a Σ-measurable function is separable.

Assertion b) in Theorem 4 remains valid for vector-valued, measurable functions:

6. Theorem. *If $f : S \to F$ is Σ-measurable, then there is a sequence (f_n) of F-valued, Σ-step functions, such that $f_n \to f$ pointwise and $|f_n| \leq |f|$ for every n.*

Proof. Let (g_n) be a sequence of F-valued, Σ-step functions such that $g_n \to f$ pointwise. Then $|g_n| \to |f|$ pointwise. Since $|g_n|$ are positive Σ-step functions, by Theorem 4 a), the function $|f|$ is Σ-measurable. By Theorem 4 b), there is an increasing sequence (h_n) of positive, finite, Σ-step functions such that $h_n \to |f|$ pointwise. Then $|g_n| - h_n \to 0$ pointwise.

For each n, we can represent g_n and h_n using the same sets of Σ:

$$g_n = \sum_{i \in I(n)} \varphi_{A_i} x_i \text{ and } h_n = \sum_{i \in I(n)} \varphi_{A_i} \alpha_i$$

with $(A_i)_{i \in I(n)}$ a finite family of mutually disjoint sets from Σ, $x_i \in F$ and $\alpha_i \geq 0$. For each n we define

$$f_n = \sum_{i \in I(n)} \varphi_{A_i} x_i |x_i|^{-1} \alpha_i,$$

where we set $x_i |x_i|^{-1} \alpha_i = 0$ if $x_i = 0$. Then

$$|f_n| \leq \sum_{i \in I(n)} \varphi_{A_i} \alpha_i = h_n \leq |f|$$

and

$$|f_n - g_n| = \sum_{i \in I(n)} \varphi_{A_i} \left| x_i |x_i|^{-1} \alpha_i - x_i \right| \leq$$

$$\leq \sum_{i \in I(n)} \varphi_{A_i} \big| \alpha_i - |x_i| \big| = \big| h_n - |g_n| \big| \to 0.$$

As $g_n \to f$ pointwise, we deduce that $f_n \to f$ pointwise. ■

The property used in Definition 3 is preserved under pointwise limits for real-valued functions, as well as for vector-valued functions.

7. Proposition. *Let (f_n) be a sequence of functions $f_n : S \to F$ (or $\overline{\mathbb{R}}$) converging pointwise to a function $f : S \to F$ (or $\overline{\mathbb{R}}$).*

Assume that for each n and each Borel set $B \subset F$ (or $\overline{\mathbb{R}}$) we have $f_n^{-1}(B) \in \Sigma$. Then $f^{-1}(B) \in \Sigma$ for each Borel set $B \subset F$ (or $\overline{\mathbb{R}}$).

Proof. Let $G \subset F$ be an open set and for each $k \in \mathbb{N}$ let G_k be the set of all points $x \in F$ with distance $d(x, G^c) > \frac{1}{k}$. Then G_k is open, $\overline{G}_k \subset G$ and $\bigcup_{k\in\mathbb{N}} G_k = G$ and we have

$$f^{-1}(G) = \bigcup_{k\geq 1} \bigcup_{n\geq 1} \bigcap_{p\geq 1} f_{n+p}^{-1}(G_k) \in \Sigma.$$

It follows that $f^{-1}(B) \in \Sigma$ for every Borel set $B \subset F$.

The above proof remains valid for functions with values in $\overline{\mathbb{R}}$, if we take a distance d on $\overline{\mathbb{R}}$ compatible with its topology. ■

The property in Definition 3 can now be used to characterize Σ-measurability of vector-valued functions (cf. [N.1], p. 101):

8. Theorem. *A function $f : S \to F$ is Σ-measurable iff it has separable range and $f^{-1}(B) \in \Sigma$ for every Borel set $B \subset F$.*

Proof. Assume first that f is Σ-measurable and let (f_n) be a sequence of F-valued, Σ-step functions such that $f_n \to f$ pointwise.

For each step function f_n we have $f_n^{-1}(B) \in \Sigma$ for every Borel set $B \subset F$. By Proposition 7, we have also $f^{-1}(B) \in \Sigma$ for every Borel set $B \subset F$. We already mentioned above that a Σ-measurable function has separable range. The first implication is proved.

To prove the converse implication, assume that f has separable range and $f^{-1}(B) \in \Sigma$ for every Borel set $B \subset F$. Let F_0 be a separable subspace of F containing the range of f.

Let $(y_n)_{n\geq 0}$ be a sequence dense in F_0 with $y_0 = 0$. For each $n \in \mathbb{N}$ define $\varphi_n : F_0 \to \{y_0, y_1, \ldots, y_n\}$ for each $x \in F_0$ as the first y_k with $0 \leq k \leq n$ for which the minimum $\min_{0\leq m\leq n} |x - y_m|$ is attained; that is, for $k \leq n$ we have $\varphi(x) = y_k$ if $|x - y_k| < |x - y_m|$, for $m = 0, 1, \ldots, k-1$ and $|x - y_k| \leq |x - y_m|$, for $m = k+1, \ldots, n$. Then $\varphi_n : F_0 \to F$ is a Borel function, since the mapping $x \to |x - y_m|$ from F_0 into $\mathbb{R}_+$ is continuous and for each $i \leq n$ we have

$$\begin{aligned}
&\varphi_n^{-1}\{y_i\} = \{x \in F_0; \varphi_n(x) = y_i\} \\
&= \bigcap_{m<i} \{x \in F_0 : |x - y_m| < |x - y_i|\} \cap \bigcap_{m>i} \{x \in F_0 : |x - y_m| \leq |x - y_i|\},
\end{aligned}$$

which is a finite intersection of open or closed sets.

On the other hand, $\lim_n \varphi_n(x) = x$ for $x \in F_0$, since

$$|x - \varphi_n(x)| = \min_{0 \le m \le n} |x - y_m| \downarrow 0 \text{ as } n \to \infty, \text{ for each } x \in F_0,$$

because (y_n) is dense in F_0. For each n set $f_n = \varphi_n \circ f : S \to F_0$. For each $i \le n$, the set $\varphi_n^{-1}\{y_i\}$ is a Borel set, hence $f_n^{-1}(y_i) = f^{-1}(\varphi_n^{-1}(y_i)) \in \Sigma$, hence the function f_n is a Σ-step function and we have $f_n \to f$ pointwise. ■

For functions with values in a separable space, the property in Definition 3 is a complete characterization of measurability.

9. Corollary. *If F is separable, then a function $f : S \to F$ is Σ-measurable iff $f^{-1}(B) \in \Sigma$ for every Borel set $B \subset F$.*

From Proposition 7 and Theorem 8 we deduce that Σ-measurability is preserved under pointwise convergence:

10. Theorem. *If (f_n) is a sequence of F (or $\overline{\mathbb{R}}$)-valued, Σ-measurable functions, converging pointwise to a function f, then the limit f is also Σ-measurable.*

As stated in Theorem 4 b), a real-valued, bounded, Σ-measurable function is the uniform limit of a sequence of Σ-step functions. This is still true for vector-valued functions $f : S \to F$ with *relatively compact range*, but not necessarily for bounded functions. However, an *arbitrary* Σ-measurable function $f : S \to F$ is the *uniform limit* of a sequence of Σ-measurable functions with *countable range.*

A Σ-measurable function $g : S \to F$ with countable range is called a Σ-measurable, σ-step function, or simply, a σ-step function, if the σ-algebra Σ is understood. It is of the form

$$g = \sum_{1 \le n < \infty} \varphi_{A_n} x_n$$

with $A_n \in \Sigma$ mutually disjoint and $x_n \in F$.

11. Proposition. *If $f : S \to F$ is Σ-measurable, then there is a sequence (f_n) of F-valued, Σ-measurable, σ-step functions, converging to f uniformly on S.*

Proof. Let $\varepsilon > 0$ and let (x_k) be a sequence dense in the range of f. For each k let $B_k = f^{-1}(S_\varepsilon(x_k)) \in \Sigma$, where $S_\varepsilon(x_k)$ is the ball centered at x_k with radius ε. We have $\bigcup_{1 \le k < \infty} B_k = S$. Denote $A_1 = B_1$ and $A_k = B_k - \bigcup_{i<k} B_i$ for $k > 1$. The sets A_k are mutually disjoint, belong to Σ and their union is S. Define the function $g_\varepsilon : S \to F$ by

$$g_\varepsilon = \sum_{1 \le k < \infty} \varphi_{A_k} x_k.$$

Then g_ε is a Σ-measurable, σ-step function and

$$|f(x) - g_\varepsilon(s)| \leq \varepsilon, \text{ for } s \in S.$$

Taking $\varepsilon = \frac{1}{n}$ and $f_n = g_{\frac{1}{n}}$, we obtain the desired sequence (f_n). ∎

D. Simple measurability of operator-valued functions

12. Definition. *A function $U : S \to E \subset L(F,G)$ is said to be simply Σ-measurable, if for every $x \in F$, the function $Ux : S \to G$ is Σ-measurable.*

It is clear that if U is measurable, then it is simply measurable. We state below a few useful properties of simply measurable functions.

13. Proposition. *If $U : S \to E \subset L(F,G)$ is simply Σ-measurable and $f : S \to F$ is Σ-measurable, then the function $Uf : S \to G$ is Σ-measurable.*

Proof. The proposition is true first, for Σ-step functions f and then, by taking limits, for any Σ-measurable function f. ∎

A simply measurable function U is not necessarily measurable; and even $|U|$ is not necessarily measurable. We give below sufficient condition for $|U|$ or U to be measurable.

14. Proposition. *If $U : S \to E \subset L(F,G)$ is such that $|Ux|$ is Σ-measurable for every $x \in F$ and if F is separable, then the function $|U|$ is Σ-measurable.*

Proof. Let (x_n) be a sequence dense in F with $x_n \neq 0$. For each n, the functions $|Ux_n|$ and $\frac{|Ux_n|}{|x_n|}$ are Σ-measurable. Since

$$|U(s)| = \sup_n \frac{|U(s)x_n|}{|x_n|}, \text{ for } s \in S,$$

we deduce that $|U|$ is Σ-measurable. ∎

15. Proposition. *Assume the embedding $E \subset L(F,G)$ is an isometry. If $U : S \to E \subset L(F,G)$ is simply Σ-measurable and separably valued, then U is Σ-measurable.*

Proof. Assume U is simply Σ-measurable with separable range. Let B be a closed sphere in E with center a and radius r and show that $U^{-1}(B) \in \Sigma$.

Let (a_n) be a sequence dense in the range of U, with $a_1 = a$. Since $E \subset L(F,G)$ isometrically, for each n there is an $x_{nm} \in F$ with $|x_{nm}| = 1$ and $|a_n x_{nm}| > |a_n| - \frac{1}{m}$. Then

$$|a_n| = \sup_m |a_n x_{nm}|, \text{ for each } n.$$

Let V be the closed vector space in E generated by the sequence (a_n). For each $v \in V$ we have

$$|v| = \sup_{n,m} |v x_{nm}|.$$

For each n and m, the function $Ux_{nm} - ax_{nm}$ is Σ-measurable, hence $|Ux_{nm} - ax_{nm}|$ is Σ-measurable; therefore

$$A_{nm} = \{s : |U(s)x_{nm} - ax_{nm}| \le r\} \in \Sigma.$$

Since

$$|U(s) - a| = \sup_{nm} |(U(s) - a)x_{nm}|, \text{ for } s \in S,$$

we deduce that

$$U^{-1}(B) = \bigcap_{nm} A_{nm} \in \Sigma,$$

hence U is Σ-measurable. ■

E. Weak measurability

Particular cases of simple measurability are weak measurability and weak star measurability.

16. Definition. *We say that a function $f : S \to F$ is weakly Σ-measurable, if for every $x^* \in F^*$, the real function $\langle f, x^* \rangle$ is Σ-measurable.*

A function $g : S \to F^*$ is said to be weak star Σ-measurable, if for every $x \in F$, the real function $\langle x, g \rangle$ is Σ-measurable.

Weak star measurable functions are also called weak * measurable.

If we want to emphasize the difference between different kinds of measurability, the functions that are Σ-measurable in the usual sense are called *strongly* Σ-measurable.

There is a more general weak measurability, with respect to a space $Z \subset F^*$, norming for F.

17. Definition. *Let $Z \subset F^*$ be a space norming for F. We say that a function $f : S \to F$ is Z-weakly Σ-measurable, if for every $z \in Z$, the real function $\langle f, z \rangle$ is Σ-measurable.*

Taking $Z = F^*$, the F^*-weak measurability is the weak measurability of Definition 16. If $g : S \to F^*$ is a function, considering $F \subset (F^*)^*$ and taking $Z = F$, the F-weak measurability is the weak star measurability of Definition 16.

Z-weak measurability is itself a particular case of simple measurability if we consider the isometric embedding $F \subset L(Z, \mathbb{R})$.

From the properties of simple measurability we deduce then the properties of Z-weak measurability, where $Z \subset F^*$ is a space norming for F.

18. Proposition. *If $f : S \to F$ is Z-weakly Σ-measurable and $g : S \to Z$ is Σ-measurable, then the real function $\langle f, g \rangle$ is Σ-measurable.*

19. Proposition. *If $f : S \to F$ is Z-weakly Σ-measurable and Z is separable, then $|f|$ is Σ-measurable.*

20. Proposition. *If $f : S \to F$ is Z-weakly Σ-measurable and has separable range, then f is Σ-measurable.*

In particular, the above properties are valid for weakly measurable and for weak* measurable functions.

21. Proposition. *If $f : S \to F$ is weakly Σ-measurable and has separable range, then f is strongly Σ-measurable.*

If F is separable, then for functions $f : S \to F$ weak measurability and strong measurability are equivalent.

22. Proposition. *If $g : S \to F^*$ is weak star Σ-measurable and has separable range, then g is strongly Σ-measurable.*

If F^* is separable, then for functions $g : S \to F^*$, weak measurability, weak star measurability, and strong measurability are equivalent.

F. Integral of step functions

23. A set function $m : \mathcal{R} \to E$ defined on a *ring* $\mathcal{R}$ is called an *additive measure*, if, for every pair (A, B) of disjoint sets from $\mathcal{R}$ we have

$$m(A \cup B) = m(A) + m(B).$$

An additive measure is *finitely additive*, that is,

$$m(\bigcup_{1 \le i \le n} A_i) = \sum_{1 \le i \le n} m(A_i)$$

for any finite family $(A_i)_{1 \le i \le n}$ of mutually disjoint sets from $\mathcal{R}$.

A set function $m : \mathcal{R} \to E$ is called a *σ-additive measure*, if, for any sequence (A_n) of mutually disjoint sets from $\mathcal{R}$ with union in $\mathcal{R}$, we have

$$m(\bigcup_n A_n) = \sum_n m(A_n).$$

If $m : \mathcal{R} \to E \subset L(F, G)$ is an additive measure and $f = \sum_{i \in I} \varphi_{A_i} x_i$ is an $\mathcal{R}$-step function from $\mathcal{S}_F(\mathcal{R})$, the integral $\int f dm$ is an element of G defined by the equality

$$\int f dm = \sum_{i \in I} m(A_i) x_i.$$

Since m is additive, the integral $\int f dm$ is independent of the particular representation of f as an $\mathcal{R}$-step function.

If we want to define the integral for a larger class of functions, we should impose additional conditions on $\mathcal{R}$ and m.

G. Totally measurable functions and the immediate integral

An immediate extension of the integral $\int f dm$ is for totally measurable functions f, provided that the measure m has *finite semivariation.* This immediate integral is used in the Riesz representation theorem.

Let $\mathcal{R}$ be a ring of subsets of S.

24. Definition. *A function $f : S \to F$ is said to be totally $\mathcal{R}$-measurable, if it vanishes outside a set $A \in \mathcal{R}$ and if it is the uniform limit of a sequence (f_n) of F-valued, $\mathcal{R}$-step functions.*

The set of totally $\mathcal{R}$-measurable functions $f : S \to E$ is denoted by $TM_F(\mathcal{R})$.

If $F = \mathbb{R}$ we write $TM(\mathcal{R})$ instead of $TM_{\mathbb{R}}(\mathcal{R})$. Any totally measurable function is bounded. We consider on $TM_F(\mathcal{R})$ the topology of uniform convergence, defined by the sup norm:

$$\|f\| = \sup_{s \in S} |f(s)|.$$

Then the set $\mathcal{S}_F(\mathcal{R})$ is dense in $TM_F(\mathcal{R})$.

According to Theorem 4, if Σ is a σ-algebra, then a *real-valued,* measurable function f is totally Σ-measurable iff f is bounded. But a vector-valued, bounded, Σ-measurable function need not be totally measurable.

We remark also that a totally Σ-measurable function is Σ-measurable.

If S is a locally compact, Hausdorff space and $\mathcal{B}$ is the δ-ring of the relatively compact, Borel subsets of S, then $TM_F(\mathcal{B})$ contains the space $\mathcal{K}_F(S)$ of continuous functions $f : S \to F$ with compact support.

25. Definition. *Let $m : \mathcal{R} \to E \subset L(F, G)$ be an additive measure defined on a ring $\mathcal{R}$. The semivariation of m on a set $A \in \mathcal{R}$, relative to the pair (F, G), is a number denoted by $\tilde{m}_{F,G}(A)$ and defined by the following equality:*

$$\tilde{m}_{F,G}(A) = \sup\{|\int f dm| : f \in \mathcal{S}_F(\mathcal{R}),\ |f| \le \varphi_A\}.$$

We say m has finite semivariation relative to (F, G) if $\tilde{m}_{F,G}(A) < \infty$ for every $A \in \mathcal{R}$.

The semivariation will be studied in detail in §4.

26. Proposition. *Let $m : \mathcal{R} \to E \subset L(F, G)$ be an additive measure with finite semivariation $\tilde{m}_{F,G}$. Then for every $\mathcal{R}$-simple function $f \in \mathcal{S}_F(\mathcal{R})$ with support $A \in \mathcal{R}$ we have*

$$|\int f dm| \le \|f\| \tilde{m}_{F,G}(A).$$

Proof. The inequality is evident, if $\|f\| = 0$. If $\|f\| \ne 0$, then $\left|\frac{1}{\|f\|} f\right| \le \varphi_A$ and the inequality follows from the definition of the integral of step functions. ∎

27. It follows that the linear mapping $f \to \int f dm$ is continuous on $\mathcal{S}_F(\mathcal{R})$ for the sup norm. Since $\mathcal{S}_F(\mathcal{R})$ is dense in $TM_F(\mathcal{R})$, this linear mapping can be extended by continuity to the whole space $TM_F(\mathcal{R})$. The value of the extension for a function $f \in TM_F(\mathcal{R})$ is still denoted by $\int f dm$ and is called the *immediate integral* of f with respect to m. We still have

$$|\int f dm| \leq \|f\| \tilde{m}_{F,G}(A),$$

for $f \in TM_F(\mathcal{R})$ with support in A.

H. The Riesz representation theorem

The immediate integral is easily defined, but it does not have too many properties. For example, the Lebesgue convergence theorem cannot be proved in this context. But it is good enough to represent continuous linear operators:

28. Theorem. *Let $\mathcal{A}$ be an algebra of subsets of S and $U: TM_F(\mathcal{A}) \to G$ a continuous linear operator. Then there is an additive measure $m : \mathcal{A} \to L(F, G)$ with finite semivariation $\tilde{m}_{F,G}(S)$ such that*

$$U(f) = \int f dm, \text{ for } f \in TM_F(\mathcal{A})$$

and

$$\|U\| = \tilde{m}_{F,G}(S).$$

The proof of this theorem is immediate.

The measure m is defined by $m(A) = U(\varphi_A)$, for $A \in \mathcal{A}$. For more details see [D.1].

But not so immediate is the proof of the following Riesz-type representation theorem.

29. Theorem. *Let K be a compact Hausdorff space and $C_F(K)$ the space of continuous functions $f : K \to F$, endowed with the sup norm.*

Let $U : C_F(K) \to G$ be a continuous linear operation.

Then there is an additive Borel measure $m : \mathcal{B}(K) \to L(F, G^{**})$ with finite semivariation $\tilde{m}_{F,G^{**}}$ such that

$$U(f) = \int f dm, \text{ for } f \in C_F(K)$$

and

$$\|U\| = \tilde{m}_{F,G^{**}}(K).$$

Moreover, for each $z \in G^*$, the measure $m_z : \mathcal{B}(K) \to F^*$ defined by

$$\langle x, m_z(A)\rangle = \langle m(A)x, z\rangle, \text{ for } x \in F \text{ and } A \in \mathcal{B}(K),$$

is regular, σ-additive and with finite variation.

For the proof see [D.1] and [D–U].

30. An additive measure $m : \mathcal{B}(K) \to L(F, G^{**})$ with finite semivariation $\tilde{m}_{F,G}$ and such that m_z is regular, σ-additive, and with finite variation for every $z \in G^*$, is sometimes called a *representing measure.*

31. There are cases when the measure m in the above theorem is neither σ-additive, nor regular, nor with values in $L(F, G)$.

It is an open problem to give a characterization of the operations U for which the corresponding measure m has one or more of the above-mentioned properties.

There are partial answers to this problem:

a) *If $U : C_{\mathbb{R}}(K) \to G$ is weakly compact, then the corresponding measure m is σ-additive, regular, and has values in G.*

For a complete presentation of this case see [D–U].

b) An operation $U : C_F(K) \to G$ is said to be *dominated* if there is a positive, regular Borel measure μ such that

$$|U(f)| \leq \int |f| d\mu, \text{ for } f \in C_F(K).$$

If $U : C_F(K) \to G$ is dominated, then the corresponding measure m is σ-additive, regular, with values in $L(F, G)$, and with finite variation $|m|$.

For a complete presentation of this case see [D.1].

A continuous linear functional $U : C_F(K) \to \mathbb{R}$ is dominated; therefore the corresponding measure $m : \mathcal{B}(K) \to F^*$ is σ-additive, regular, and with finite variation.

We give one more case which answers the above problem.

c) *If $G = D^*$ is a dual of a Banach space and $U : C_F(K) \to D^*$ is a continuous linear operation, then the corresponding measure m has values in $L(F, D^*)$, is σ-additive, and regular.*

32. If an additive measure $m : \mathcal{R} \to E \subset L(F, G)$ defined on a ring $\mathcal{R}$ has finite semivariation $\tilde{m}_{F,G}$, it might not be possible to extend the integral $\int f dm$ beyond the space $TM_F(\mathcal{R})$ of totally measurable functions.

In order to define the integral for a larger class of functions we need an additive measure $m : \mathcal{D} \to E \subset L(F, G)$ defined on a δ-ring, with finite semivariation $\tilde{m}_{F,G}$ such that the measures $m_z : \mathcal{D} \to F^*$ are σ-additive for every z in a subspace $Z \subset G^*$ norming for G. This integral will be presented in §5 and is an extension of the immediate integral. It follows that this integral can be used in the Riesz representation theorem instead of the immediate integral.

33. There are 4 stages in the development of the integral $\int f dm$:

I) The classical integral, with $m \geq 0$ and f *real-valued* ;

II) The Bochner integral, with $m \geq 0$ and f *vector-valued* ;